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K. Buyukboduk, Euler systems of rank r and their Kolyvagin systems

Outline

⓪ Setting - Motivation

① Selmer groups

② ES of rank r (PR) and main technical result

③ (Conjectural) example

(A) Rubin - Stark elements

(B) PR's p-adic L function

④ Technical details

⓪ k, $p>2$, $G_k = \mathrm{Gal}(\bar{k}/k)$, $T \circlearrowleft G_k$, $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$T^* = \mathrm{Hom}(T, \mu_{p^\infty})$, $\underset{\underset{r}{\|}}{d^-} = \dim_{\mathbb{Q}_p}\left(\mathrm{Ind}_{k/\mathbb{Q}} V\right)^-$

$$\Gamma \left\{ \begin{array}{l} \mathbb{Q}_\infty \\ \quad | \\ \mathbb{Q}_n \\ \quad | \ \mathbb{Z}/p^n\mathbb{Z} \\ \mathbb{Q} \end{array} \right. \qquad \Lambda = \mathbb{Z}_p[[\Gamma]]$$

Arithmetic information $\longleftrightarrow$ Analytic information

'Selmer groups'

ES machinery

① $k = \mathbb{Q}$, $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$, $\boxed{\chi : \text{even}} \to d_- = 1$

$L = \bar{\mathbb{Q}}^{\ker \chi}$

K, $\mathcal{O}_K^\times$, $\mathcal{U}_K$: local units at p, A_K : ideal class group

K ↑ abelian

$$N_{\mathbb{Q}(\zeta_{f_K \cdot p})/K}\left(1 - \zeta_{f_K \cdot p}\right) = c_K \in \mathcal{O}_K^\times$$

Thm (Mazur-Wiles)

(i) $$\left| A_L^{\chi} \right| = \left| (\mathcal{O}_L^{\times})^{\chi} / \mathbb{Z}_p \cdot c_L^{\chi} \right| \longleftrightarrow \text{L-values}$$ (Kummer's class number)

(ii) $$\mathrm{char}\left(\varprojlim A_{L_n}^{\chi}\right) = \mathrm{char}\left(\varprojlim_n (\mathcal{O}_{L_n}^{\times})^{\chi} / \Lambda \cdot \{c_{L_n}^{\chi}\}\right)$$

$\Lambda \supset \bigcup$ $\qquad\qquad \bigcup$

$$\mathrm{char}(\mathrm{Iw}(T)) = \langle L_p^{\chi} \rangle \overset{\text{Iwasawa}}{=} \mathrm{char}\left(\varprojlim U_{L_n}^{\chi} / \Lambda \cdot \{c_{L_n}^{\chi}\}\right)$$

p-adic L-funct

2. $k = \mathbb{Q}$, $E/\mathbb{Q}$, $T = T_p(E)$, $d_- = 1$

Kato : $\forall K/\mathbb{Q}$ abelian, $c_K^{\mathrm{Kato}} \in H^1(K, T) = H^1(G_K, T)$ which satisfy a distribution relation

E has good ordinary reduction at p

Thm (Kato) :

(i) $L(E,1) \neq 0$, then $\# E(\mathbb{Q}) < \infty$

$Ш_{E/\mathbb{Q}} < \infty$

$\leq$ L-value

$\overset{?}{=}$ BSD

$\rho_E : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(E[p^{\infty}])$ surjective

(ii) $\mathrm{char}(\mathrm{Iw}(T)) \mid \langle \mathcal{L}_{MTT}^{E} \rangle$

$\overset{?}{=}$ Main Conj.

Equivalent statement :

$$\# H^1_{\mathcal{F}^*_{BK}}(\mathbb{Q}, T^*) \leqslant [H^1_s(\mathbb{Q}_p, T) : \mathrm{loc}_p^s(c_{\mathbb{Q}})]$$

'Bloch-Kato Selmer group'

$$\mathrm{loc}_p^s : H^1(\mathbb{Q}, T) \xrightarrow{\mathrm{loc}_p} H^1(\mathbb{Q}_p, T) \longrightarrow \frac{H^1(\mathbb{Q}_p, T)}{H^1_f(\mathbb{Q}_p, T)} = H^1_s(\mathbb{Q}_p, T)$$

Bloch-Kato submodule

$$H^1_f(\mathbb{Q}_p, T) = \mathrm{im}\left(E(\mathbb{Q}_p) \hat{\otimes} \mathbb{Z}_p \hookrightarrow H^1(\mathbb{Q}_p, T)\right)$$

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① Selmer groups

k, $T \circlearrowleft G_k$

Local condition at ℓ : $H^1_{\mathcal{F}}(k_\ell, T) \subset H^1(k_\ell, T)$

Selmer structure : local cond at every ℓ

Selmer group attached to a Selmer str.

$$\mathrm{Ker}\left(H^1(k, T) \longrightarrow \prod_\ell \frac{H^1(k_\ell, T)}{H^1_{\mathcal{F}}(k_\ell, T)} \right)$$

$$\begin{array}{ccccc} H^1(k_\ell, T) & \times & H^1(k_\ell, T^*) & \longrightarrow & \mathbb{Q}_p/\mathbb{Z}_p \quad \left(\begin{smallmatrix}\text{local Tate}\\ \text{pairing}\end{smallmatrix}\right) \\ \cup & & \cup & & \\ H^1_{\mathcal{F}}(k_\ell, T) & & H^1_{\mathcal{F}^*}(k_\ell, T^*) := H^1_{\mathcal{F}}(k_\ell, T)^{\perp} & & \end{array}$$

$\mathcal{F}^*$: dual Selmer structure , $H^1_{\mathcal{F}^*}(k, T^*)$

Examples : Bloch-Kato Selmer groups :

at $\ell \neq p$, $\quad H^1_{\mathcal{F}_{BK}}(k_\ell, T) = \mathrm{Ker}\left(H^1(k_\ell, T) \to H^1(k_\ell^{unr}, V)\right)$

at p $\quad H^1_{\mathcal{F}_{BK}}(k_p, T) = \mathrm{Ker}\left(H^1(k_p, T) \to H^1(k_p, V \otimes B_{cris})\right)$

(B) canonical Selmer structure : Relax $\mathcal{F}_{BK}$ at p

(C) $T = \mathbb{Z}_p(1) \otimes \chi^{-1}$, L

$$H^1_{\mathcal{F}_{BK}}(k, T) = (\mathcal{O}_L^\times)^\chi, \quad H^1_{\mathcal{F}_{BK}^*}(k, T^*) = A_L^\chi$$

(D) $T = T_p(E)$

$$0 \to E(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \underbrace{H^1_{\mathcal{F}_{BK}^*}(k, T^*)} \to Ш_{E/\mathbb{Q}}[p^\infty] \to 0$$

L-values

$ES(T) \xrightarrow[\text{descent}]{\text{Kolyvagin's}} KS(\mathcal{F}_{can}, T) \rightsquigarrow$ bounds on Selmer groups

$KS(\mathcal{F}_{can}, T \otimes \Lambda) \rightsquigarrow \frac{1}{2}$ main ~~conj~~ conj

Thm (Mazur-Rubin / Howard)

(i) $r = d_- = 1$ then $KS(\mathcal{F}_{can}, T)$ is free of rank 1

(Buyukboduk) — thesis

(ii) $d_- = 1$ $\quad KS(\mathcal{F}_{can}, T \otimes \Lambda)$ is free of rank 1

$\{c_K\}$, $c_K \in H^1(K, T)$ $\quad K \in \mathcal{K}$ — big abel ext. of k

What happens when $r > 1$?

② ES of rank r (PR)

$$C^{(r)} = \{ C_K^{(r)} \}, \quad C_K^{(r)} \in \bigwedge^r_{\mathbb{Z}_p[\mathrm{Gal}(K/k)]} H^1(K, T) \quad T \otimes \Lambda$$

$\forall K \in \mathcal{K}$

satisfies a distribution relation

$$ES^{(r)}(T) \xrightarrow[\text{Rubin}]{} ES(T) \xrightarrow[\text{Kolyvagin}]{MR} KS(\mathcal{F}_{can}, T)$$

many choices

$ES(T)$ = ES of rank 1

$\boxed{\text{produces a big Selmer group}}$ when $r = d_- > 1$

(2A) Consider a "finer" Selmer structure $\mathcal{F}_{\mathcal{L}}$:

$\boxed{\text{when } T = T^\vee(1)}$

$$\mathcal{F}_{\mathcal{L}} = \mathcal{F}_{BK} = \mathcal{F}_{can} \quad \text{when } \ell \neq p$$

$$H^1_{\mathcal{F}_{\mathcal{L}}}(k_p, T) = H^1_{\mathcal{F}_{BK}}(k_p, T) \oplus \mathcal{L} \subset H^1_{\mathcal{F}_{can}}(k_p, T)$$

rank 1 direct summand

(2B) $$ES^{(r)}(T) \xrightarrow{\text{Rubin}} ES(T) \supset ES_{\mathcal{L}}(T)$$ '$\mathcal{L}$-restricted ES'

How to make a good choice ?

$$\begin{array}{ccc} ES(T) & \supset & ES_{\mathcal{L}}(T) \\ \downarrow{\scriptstyle \text{Kolyvagin}} & & \downarrow \\ KS(\mathcal{F}_{can}, T) & \supset & KS(\mathcal{F}_{\mathcal{L}}, T) \end{array}$$

Thm 1 (B) T : Self-dual,

suppose $\exists\, c^{(r)} \in ES^{(r)}(T)$

under certain hypothesis then $\# H^1_{\mathcal{F}^*_{BK}}(k, T^*) \leq \left[\bigwedge^r_{\mathbb{Z}_p} H^1_s(k_p, T) : \mathbb{Z}_p \cdot loc^s_p(c_k) \right]$

— Iwasawa theoretic version when T is 'p-ordinary' 'non-exceptional'

singular quotient: $\dfrac{H^1(k_p, T)}{H^1_{\mathcal{F}_{BK}}(k_p, T)}$

③ (Conjectural) Example

(A) Rubin-Stark elements :

$\left(H^1(K, T) = (LK^\times)^\chi \right.$ (Kummer theory) $\left. \right)$

$$T = \mathbb{Z}_p(1) \otimes \chi^{-1}$$

totally even Dirichlet char

k : tot. real #-field

L

$r = [k : \mathbb{Q}]$

Conj (Rubin/Stark) $\forall K/k$ abelian, $\exists\, c_K^{(r)} \in \bigwedge^r (\mathcal{O}^\times_{LK,S})^\chi$ such that

$$\mathrm{Reg}(c_K^{(r)}) \doteqdot L_S(0, \chi^{-1})$$

$\rightsquigarrow \{c_K^{(r)}\}$ is an ES of rank r

Thm (B):

(i) $$\left| A_L^{\chi} \right| = \left| \bigwedge^r (\mathcal{O}_L^{\times})^{\chi} \Big/ \mathbb{Z}_p \cdot c_k^{(r)} \right|$$

(ii) $$\mathrm{char}\big(\mathrm{Iw}(T)\big) = \mathrm{char}\Big(\bigwedge^r H^1(k_p, T \otimes \Lambda) \Big/ \Lambda \cdot \{ c_{k_n}^{\chi} \} \Big)$$

$r = 1$
Gras conj.
(Mazur-Wiles)

"inverse limit of local units"

Wiles

$$\langle \mathcal{L}_{DR} \rangle$$

$r = 1$
(Iwasawa)

(B) PR's (conjectural) p-adic L-functions:

$k = \mathbb{Q}$, T, V: p-adic realization of a motive crystalline at p

B_{cris} : Fontaine's period ring

$$D_{cris}(V) = (V \otimes B_{cris})^{G_{\mathbb{Q}_p}}$$

$$\mathcal{H}_{\infty} \subset \mathbb{Q}_p[\![\Gamma]\!] = \mathbb{Q}_p[\![T]\!]$$ "growth condition"

$$\begin{array}{ccc} H^1(\mathbb{Q}_p, T\otimes\Lambda) = \varprojlim H^1(\mathbb{Q}_{p,n}, T) & \xrightarrow{\mathcal{L}og = \mathcal{L}} & \mathcal{H}_\infty \otimes D_{cris}(V) \\ \Big\downarrow \Xi_V & & \Big\downarrow \\ H^1(\mathbb{Q}_p, V) & \xrightarrow[\exp^*]{} & D_{cris}(V) \end{array}$$

$$\begin{array}{ccc} \mathcal{L}^{(r)} : \bigwedge^r H^1(\mathbb{Q}_p, T\otimes\Lambda) & \longrightarrow & \bigwedge^r D_{cris}(V) \otimes \mathcal{H}_\infty \\ & \searrow^{\mathcal{L}^{(r)}_{\bar\eta}} & \Big\downarrow \bar\eta = \eta_1\wedge\cdots\wedge\eta_r \\ & & \mathcal{H}_\infty \quad \eta_i \in D_{cris}(V^*) \end{array}$$

<u>Conj</u> (PR / extended by Rubin) tame direction

$\exists$ $c^{(r)}_{\mathbb{Q}(\mu_n)^+} \in \bigwedge^r_{\Lambda[Gal(\mathbb{Q}(\mu_n)^+/\mathbb{Q})]} H^1(\mathbb{Q}(\mu_n)^+, T\otimes\Lambda)$

and $\bar\eta$ such that

$$\mathcal{L}^{(r)}_{\bar\eta}\left(c^{(r)}_{\mathbb{Q}(\mu_n)^+}\right) = \mathbb{L}_p^{\{n\}}(V) \in \mathcal{H}_\infty$$

$\updownarrow$

conjectural p-adic L-function

<u>Thm (B)</u> V is self-dual, assume conjecture for V

$$\mathbb{1}(\mathbb{L}_p(V)) \neq 0 \iff L(M, 0) \neq 0$$

no-exceptional zero $\longleftrightarrow$ $D_{cris}(V)^{\varphi=1} = 0$

$\# H^1_{\mathcal{F}^*_{BK}}(k, T^*) < \infty$

$$
\begin{tikzcd}
ES^{(r)}(T) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2, "A"'] \arrow[rrd, bend left, "B"] & ES(T) \arrow[d] & \\
& KS(\mathcal{F}_{can}, T) \supset & KS(\mathcal{F}_{\mathcal{L}}, T)
\end{tikzcd}
$$

$$c_K^{(r)} \in \bigwedge^r H^1(K,T) \qquad \forall K/k : \text{abelian}$$

$$\Delta_K = Gal(K/k)$$

$$\varphi \in \mathrm{Hom}_{\mathcal{O}[\Delta_K]}\left(H^1(K,T), \mathcal{O}[\Delta_K]\right)$$

$$\varphi : \bigwedge^s H^1(K,T) \longrightarrow \bigwedge^{s-1} H^1(K,T)$$

$$c_1 \wedge \cdots \wedge c_s \longmapsto \sum_{i=1}^{s} (-1)^i \varphi(c_i)\, c_1 \wedge \cdots \wedge c_{i-1} \wedge c_{i+1} \wedge \cdots \wedge c_s$$

By iteration

$$\bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}\left(H^1(K,T), \mathcal{O}[\Delta_K]\right) \longrightarrow \mathrm{Hom}\left(\bigwedge^r H^1(K,T), H^1(K,T)\right)$$

$$\bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}\left(H^1(K_p,T), \mathcal{O}[\Delta_K]\right)$$

$$\longrightarrow \bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}\left(H^1(K,T), \mathcal{O}[\Delta_K]\right)$$

<u>Lemma</u> (Rubin) :

$$\Phi \in \varprojlim_{K \in \mathcal{K}} \bigwedge^{r-1} \mathrm{Hom}_{\mathcal{O}[\Delta_K]}\left(\text{———} \right)$$

$$\Phi = \{\phi_K\}$$

$$H^1(K,T) \xrightarrow{res} H^1(K',T) \quad \text{then} \quad \{\phi_K(c_K^{(r)})\} \in ES(T)$$

$$K \subset K' \quad \mathcal{O}[\Delta_K] \xrightarrow{\sim} \mathcal{O}[\Delta_{K'}]^{Gal(K'/K)}$$

(B) How to choose Φ in a good way?

Choose $\mathbb{L}^{(r)} \subset \varprojlim_{K \in \mathcal{K}} H^1(K_p, T)$ which maps to $H^1_f(k_p, T)$

rank r

rank 1 — $\mathbb{L} \subset \overline{\qquad\qquad}$ $\mathcal{L}$

$$ES_{\mathbb{L}}(T) = \{ c = (c_K) \mid \mathrm{loc}_p(c_K) \in L_K^{(r)} \oplus \mathcal{L}_K \}$$

$\mathbb{L}^{(r)} \longmapsto L_K^{(r)}$ $\qquad$ $\mathbb{L}$-modified ES

$\mathcal{L} \longmapsto \mathcal{L}_K$

Thm A (B)

(i) $\exists\, \Phi$ s.t. $\Phi(ES^{(r)}(T)) \subset ES_{\mathbb{L}}(T)$

(ii) $ES_{\mathbb{L}}(T) \longrightarrow KS(\mathcal{F}_{\mathcal{L}}, T)$

$\mathrm{im}(\mathrm{loc}_p^s) \neq 0$

$$0 \to H^1_{\mathcal{F}_{BK}}(k,T) \to H^1_{\mathcal{F}_{\mathcal{L}}}(k,T) \xrightarrow{\mathrm{loc}_p^s} \mathcal{L}$$

$\neq 0$ $\qquad$ rank 1 $\qquad$ $\mathrm{im}(\mathrm{loc}_p^s)^{\perp} = \mathrm{im}(\mathrm{loc}_p^*)$

finite

$$0 \to H^1_{\mathcal{F}_{\mathcal{L}}^*}(k,T^*) \to H^1_{\mathcal{F}_{BK}^*}(k,T^*) \xrightarrow{\mathrm{loc}_p^*} \boxed{}$$

Poitou-Tate

$\mathrm{loc}_p^s(c_k^{(r)}) \neq 0$ $\qquad$ ($\mathcal{F}_{\mathcal{L}}$ is chosen so that MR applies)

$$\frac{\wedge^r H^1_s(k_p, T)}{\mathbb{Z}_p \cdot c_k^{(r)}} \overset{\Phi}{\cong} \frac{\boxed{\#\left(\mathcal{L}/\cancel{\Phi}(\mathrm{loc}_p^s(\Phi(c_k^{(r)})))\right)}}{\#\left(H^1_{\mathcal{F}_\mathcal{L}}(k, T) / \Phi(c_k^{(r)})\right)}$$

$$\geq$$

$$= \frac{\boxed{\# H^1_{\mathcal{F}^*_{BK}}(k, T^*)}}{\# H^1_{\mathcal{F}^*_{\mathcal{L}}}(k, T^*)} \quad \geq_{MR}$$

$$\begin{array}{ccc} \wedge^r H^1_s(k_p, T) & \xrightarrow{\Phi} & \mathcal{L} \\ \cup & & \\ \mathrm{loc}_p^s(c_k^{(r)}) & \longmapsto & \Phi(c_k^{(r)}) \end{array}$$