K-groups of rings of algebraic integers and Iwasawa theory

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$F/\mathbb{Q}$: finite abelian extension

$\mathcal{O}_F$: the ring of algebraic integers of F

- $K_n(\mathcal{O}_F) \quad (n \geq 0)$
- $H^i(\mathcal{O}_F, \mathbb{Z}_p(n)) := H^i_{\acute{e}t}(Spec(\mathcal{O}_F[1/p]), \mathbb{Z}_p(n))$

$$(i = 0, 1, 2, \quad n \in \mathbb{Z})$$

$$\simeq H^i(G_\Sigma(F), \mathbb{Z}_p(n)) \ (:= \varprojlim H^i(G_\Sigma(F), \mathbb{Z}/p^m\mathbb{Z}(n)))$$

Σ : all Archimedean primes & primes above p

F_Σ/F : the maximal algebraic Σ-ramified extension

$G_\Sigma(F)$: $=\mathrm{Gal}(F_\Sigma/F)$

p: odd prime number

Conjecture (Quillen-Lichtenbaum) $n \geq 1$

p-adic Chern class maps :

$$K_{2n+1}(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow H^1(\mathcal{O}_F, \mathbb{Z}_p(n+1))$$

$$K_{2n}(\mathcal{O}_F) \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow H^2(\mathcal{O}_F, \mathbb{Z}_p(n+1))$$

are isomorphisms.

Remarks

- surjectivity & finite kernel
  [Soulé($1 \leq n \leq p-1$), Dwyer and Friedlander ($n \geq 1$)]

- K: totally imaginary field & $p = 2$ [Kahn]

§ Lemmas on H^i $(i = 1, 2)$

$$\mathrm{fr}_{\mathbb{Z}_p} H^i(\mathcal{O}_F, \mathbb{Z}_p(n)) := H^i(\mathcal{O}_F, \mathbb{Z}_p(n)) / \mathrm{tor}_{\mathbb{Z}_p} H^i(\mathcal{O}_F, \mathbb{Z}_p(n))$$

$F_\infty := F(\mu_{p^\infty})$

$G_\infty :=\mathrm{Gal}(F_\infty/F)$

$w_n(F) :=\max\{N \mid \mathrm{Gal}(F(\mu_N)/F)^n = \{1\}\}$ $(n \neq 0)$

$w_n(F) = \prod_p w_n^{(p)}(F)$

$A'_m :=$ p-part of the Σ-class groups of $F_m := F(\mu_{p^{m+1}})$

$X' := \varprojlim A'_m$

Lemma (H^1) $n \neq 0$

(1) $\mathrm{tor}_{\mathbb{Z}_p} H^1(\mathcal{O}_F, \mathbb{Z}_p(n)) \simeq \mathbb{Q}_p/\mathbb{Z}_p(n)^{G_\infty} \simeq \mu_{w_n^{(p)}(F)}^{\otimes n}$

(2) $\mathrm{fr}_{\mathbb{Z}_p} H^1(\mathcal{O}_F, \mathbb{Z}_p(n)) \simeq \mathrm{Hom}(G_\Sigma(F_\infty)^{\mathrm{ab}} \otimes \mathbb{Z}_p, \mathbb{Z}_p(n))^{G_\infty}$

Lemma (H^2) $n \neq 1$

$$0 \to X'(n-1)_{G_\infty} \to H^2(\mathcal{O}_F, \mathbb{Z}_p(n)) \xrightarrow{\mathrm{loc}_2} \left(\frac{\prod_{\mathfrak{p}|p} \mu_{w_{1-n}^{(p)}(F_\mathfrak{p})}^{\otimes 1-n}}{\mu_{w_{1-n}^{(p)}(F)}^{\otimes 1-n}} \right)^\vee \to 0$$

- $(*)^\vee := \mathrm{Hom}(*, \mathbb{Q}_p/\mathbb{Z}_p)$
- $\mathrm{loc}_2 : H^2(G_\Sigma(F), \mathbb{Z}_p(n)) \to \prod_{\mathfrak{p}|p} H^2(G_{F_\mathfrak{p}}, \mathbb{Z}_p(n))$

<u>Keys of the Proofs</u>

Duality & Long exact sequence of Poitou-Tate. □

§ K-groups and cyclotomic $\mathbb{Z}_p$-extensions

F : totally real number field

$F_m := F(\mu_{p^{m+1}})$, $\qquad F_\infty := F(\mu_{p^\infty})$

$\Delta :=\mathrm{Gal}(F_0/F)$, $\qquad \Gamma_m :=\mathrm{Gal}(F_m/F_0)$, $\qquad \Gamma :=\mathrm{Gal}(F_\infty/F_0)$

$G_m :=\mathrm{Gal}(F_m/F) \simeq \Delta \times \Gamma_m$

$G_\infty :=\mathrm{Gal}(F_\infty/F) \simeq \Delta \times \Gamma$

- $\widetilde{\kappa} : G_\infty \to \mathbb{Z}_p^\times$ cyclotomic character

  i.e. $\zeta^\tau = \zeta^{\widetilde{\kappa}(\tau)}$ for ${}^\forall\tau \in G_\infty$, ${}^\forall\zeta \in \mu_{p^\infty}$

- $$\widehat{G}_\infty \simeq \widehat{\Delta} \times \widehat{\Gamma}$$
  $$\widetilde{\kappa} = \omega \times \kappa$$

- For even character $\chi \in \widehat{G}_m$, $\quad {}^{\exists} L_p(\chi, s)$

s.t. $\quad L_p(\chi, 1-n) = L(\chi\omega^{-n}, 1-n) \prod_{\mathfrak{p}|p}(1-\chi\omega^{-n}(\mathfrak{p})N(\mathfrak{p})^{n-1}))$

for all integers $n \geq 1$

$\neq 0$

Conjecture (L_p) $\quad L_p(\chi, 1-n) \neq 0$

for all even cheracters $\chi \in \widehat{G}_m$ and integers $n \leq 0$.

- $d_i(n) := \mathrm{rank}_{\mathbb{Z}_p} H^i(\mathcal{O}_{F_m}, \mathbb{Z}_p(n)) = \mathrm{corank}_{\mathbb{Z}_p} H^i(\mathcal{O}_{F_m}, \mathbb{Q}_p/\mathbb{Z}_p(n))$

Conjecture $(d_2))$ $\quad d_2(n) = 0 \quad (n \neq 1)$

Conjecture (d_1) $\quad d_1(n) = \begin{cases} [F_m : \mathbb{Q}]/2 & (n \neq 0, 1) \\ [F_m : \mathbb{Q}]/2 + 1 & (n = 0) \end{cases}$

Remarks

- $d_2(1) = \sharp\{\mathfrak{p} \mid \mathfrak{p}|p\} - 1, \quad d_1(1) = d_2(1) + [F_m : \mathbb{Q}]/2$

- Finiteness of even-numbered K-groups $\Rightarrow$ $d_2(n) = 0$ $(n \geq 2)$

- $\boxed{\text{Conjecture } (L_p)} + \boxed{\mu = 0}$

$$\Rightarrow \boxed{\text{Conjecture } (d_2)} \Leftrightarrow \boxed{\text{Conjecture } (d_1)}$$

$\boxed{\text{Conjecture}}$ $m \geq 0$, $n \geq 2$: integers.

For all integers j,

$$(K_{2n-2}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j} \simeq \begin{cases} 0 & \text{if } n \not\equiv j \pmod 2 \text{ or } n \equiv j \pmod{p-1}, \\ \left(\Lambda/(f_{\omega^{n-j}}, g_{1-n}^{(m)})e_{\omega^{j-n+1}}\right)(n-1) & \text{otherwise.} \end{cases}$$

$$(K_{2n-1}(\mathbb{Z}[\mu_{p^{m+1}}]) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\omega^j} \simeq \begin{cases} \mu_{D_{p,n,j}}^{\otimes n} & \text{if } n \equiv j \pmod 2, \\ \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda/(g_n^{(m)})e_{\omega^{n-j}}, \mathbb{Z}_p(n)) & \text{otherwise.} \end{cases}$$

- $g_n^{(m)} = \gamma^{p^m} - \kappa^n(\gamma^{p^m}) \quad \in \Lambda := \mathbb{Z}_p[\Gamma]$
- $N_{p,n,j}$, $D_{p,n,j}$: the power of p such that

$$\prod_{\psi \in \widehat{\Gamma}_m} L_p(1-n, \omega^{n-j}\psi) = \frac{\mathcal{N}_{p,n,j}}{\mathcal{D}_{p,n,j}} \quad \in \mathbb{Q}_p$$

$(\mathcal{N}_{p,n,j}, \mathcal{D}_{p,n,j} \in \mathbb{Z}_p,\ v_p(\mathcal{N}_{p,n,j}) = 0$ or $v_p(\mathcal{D}_{p,n,j}) = 0,$

$N_{p,n,j} \sim_p \mathcal{N}_{p,n,j}$ and $D_{p,n,j} \sim_p \mathcal{D}_{p,n,j}$ $)$

- $f_{\omega^{n-j}} \in \Lambda \simeq \mathbb{Z}_p[[T]]$ such that

$$f_{\omega^{n-j}}((1+p)^s - 1) = L_p(s, \omega^{n-j})$$

§ Brumer conjecture and Coates-Sinnott conjecture

F : totally real number field

K/F : finite abelian extension

G :=Gal(K/F)

S : finite set of primes of F containing
all Archimedean primes & primes which ramify in K

Definition (Partial zeta function) For $\sigma \in G$,

$$\zeta_{F,S}(\sigma, s) = \sum_{\substack{(\mathfrak{a},K/F)=\sigma \\ \mathfrak{a} \text{ is prime to } S}} N\mathfrak{a}^{-s} \quad (\mathrm{Re}(s) > 1).$$

Theorem [Klingen and Siegel] For all integers $n \geq 0$,

$$\zeta_{F,S}(\sigma, -n) \in \mathbb{Q}.$$

Definition (Higer Stickelberger elements)

For an integer $n \geq 0$,

$$\theta_S(n) = \sum_{\sigma \in G} \zeta_{F,S}(\sigma, -n)\sigma^{-1} \quad \in \mathbb{Q}[G].$$

Theorem (Integrality)

[Iwasawa, Coates, Sinnott,
Cassou-Noguès, Deligne and Ribet]

For all integers $n \geq 0$,

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu_{w_{n+1}(K)}^{\otimes n+1})\ \theta_S(n) \quad \subseteq \mathbb{Z}[G].$$

Lemma $n \geq 0$

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu_{w_{n+1}(K)}^{\otimes n+1}) =$$

$$\langle N\mathfrak{a}^{n+1} - (\mathfrak{a}, K/F) \mid \text{ integral ideals } \mathfrak{a} \text{ prime to } S \text{ and } w_{n+1}(K)\rangle_{\mathbb{Z}[G]}.$$

- $(N\mathfrak{a}^{n+1} - (\mathfrak{a}, K/F))\theta_S(n) = \sum_{\sigma \in G} \delta_{n+1}(\sigma, \mathfrak{a})\sigma^{-1}$,

where $\delta_{n+1}(\sigma, \mathfrak{a}) = N\mathfrak{a}^{n+1}\zeta_{F,S}(\sigma, -n) - \zeta_{F,S}(\sigma(\mathfrak{a}, K/F), -n)$.

Then Theorem (Integrality) $\Rightarrow \delta_{n+1}(\sigma, \mathfrak{a}) \in \mathbb{Z}$.

Theorem (Congruence)

[Iwasawa, Coates, Sinnott, Doligne and Ribet]

Assume S contains all primes above p.

$$\delta_{n+1}(\sigma, \mathfrak{a}) \equiv (N\mathfrak{b}_\sigma\mathfrak{a})^n \delta_1(\sigma, \mathfrak{a}) \mod w_n(K)\mathbb{Z}_p$$

where $\mathfrak{b}_\sigma$ is an integral ideal of F s.t. $(\mathfrak{b}_\sigma, K/F) = \sigma$.

Theorem [Coates]

Theorem (Integrality) + Theorem (Congruence)

$\Rightarrow$ p-adic L-function exsists.

Conjecture (Brumer)

$$\mathrm{ann}_{\mathbb{Z}[G]}(\mu_K)\ \theta_S(0) \quad \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K).$$

Conjecture (Brumer-p)) p : prime number.

$$\mathrm{ann}_{\mathbb{Z}_p[G]}(\mu_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)\ \theta_S(0) \quad \subseteq \mathrm{ann}_{\mathbb{Z}_p[G]}(\mathrm{Cl}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

Remarks

- Conjecture (Brumer) $F = \mathbb{Q}$ [Stickelberger].
- Conjecture (Brumer-p) CM fields K such that $p \nmid [K : F]$ [Wiles]
- Conjecture (Brumer-p) CM fields K under some assumptions including a condition on the μ-invariant. [Greither]

Conjecture (Coates-Sinnott) $n \geq 1$: integer

(1) $\mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{tor}_{\mathbb{Z}}(K_{2n+1}(\mathcal{O}_K)))\,\theta_S(n) \quad \subseteq \mathbb{Z}[G]$

(2) $\mathrm{ann}_{\mathbb{Z}[G]}(\mathrm{tor}_{\mathbb{Z}}(K_{2n+1}(\mathcal{O}_K)))\,\theta_S(n) \quad \subseteq \mathrm{ann}_{\mathbb{Z}[G]}(K_{2n}(\mathcal{O}_K))$

Conjecuture (Coates-Sinnott-p)

p: prime number, $n \geq 1$: integer

(1) $\mathrm{ann}_{\mathbb{Z}_p[G]}(\mathrm{tor}_{\mathbb{Z}_p}(K_{2n+1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p))\,\theta_S(n) \subseteq \mathbb{Z}_p[G]$

(2) $\mathrm{ann}_{\mathbb{Z}_p[G]}(\mathrm{tor}_{\mathbb{Z}_p}(K_{2n+1}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p))\,\theta_S(n)$
$\subseteq \mathrm{ann}_{\mathbb{Z}_p[G]}(K_{2n}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$

Remarks

- Conjecture (Quillen-Lichtenbaum)

  $+$

  Lemma (H^1)

  $+$

  Theorem (Integrality)

  $\Downarrow$

  Conjecture (Coates-Sinnott-p) (1)

• In cases $F = \mathbb{Q}$ and finite abelian extensions $K/\mathbb{Q}$

For $n = 1$ and odd prime numbers p [Coates and Sinnott]

For integers n and odd prime numbers p
(under Conjecture(Q-L)) [Nguyen Quang Do]

<u>For all results below, assume Conjecture(Q-L)</u>

• In cases finite abelian extensions K/F of totally real fields both F and K

For even integers n and odd prime numbers p under some assumption on the μ-invariant [Nguyen Quang Do]

• In cases finite abelian extensions K/F of totally real fields F and CM fields K

For odd prime numbers p under some assumptions [Burns and Greither]

For odd integers n and odd prime numbers p
such that $p \nmid n[K:F] \dfrac{\prod_{v|p} w_n^{(p)}(K_v)}{w_n^{(p)}(K)}$ [Banaszak]

Theorem p : odd prime number

Assume:

I. Conjecture (Quillen-Lichtenbaum).
II. S contains all primes above p.
III. Conjecture (Brumer-p) for K_m/F ($K_m = K(\mu_{p^{m+1}})$) with sufficiently large m.
IV. $\dfrac{\prod_{v|p} w_n^{(p)}(K_v)}{w_n^{(p)}(K)} = 1.$

Then, Conjecture (Coates-Sinnott-p) is true.

Remarks on the condition IV

- In particular, the condition IV holds in the following cases
  - p does not ramify in K and $(p-1) \nmid n$.
  - p does not split in K_∞.

- The condition yields that the localization map :

$$H^2_{ét}(Spec(\mathcal{O}_K[1/p]), \mathbb{Z}_p(n+1)) \to \oplus_{v|p} H^2(K_v, \mathbb{Z}_p(n+1))$$

is a zero map.

Outline of the proof

Proposition Assume condition II. For all integers $m, n \geq 0$ and integral ideals $\mathfrak{a}$ which are coprime to S,

$$(N\mathfrak{a}^{n+1} - (\mathfrak{a}, K_m/F)) \sum_{\sigma \in G_m} \zeta_{F,S}(\sigma, -n)\sigma^{-1}$$

$$\equiv (N\mathfrak{a}^{n+1} - (\mathfrak{a}, K_m/F)) \sum_{\sigma \in G_m} \zeta_{F,S}(\sigma, 0)\kappa_m(\sigma)^n \sigma^{-1} \pmod{p^{m+1}\mathbb{Z}_p}.$$

(Proof: Use Theorem (Congruence))

By Condition I and IV,

$$\begin{aligned} K_{2n}(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{Z}_p &\simeq H^2(\mathcal{O}_K, \mathbb{Z}_p(n+1)) \\ &\simeq X'(n)_{\mathrm{Gal}(K_\infty/K)} \\ &\simeq X'/IX'(n) \end{aligned}$$

where $I = \langle \sigma - \kappa^{-n}(\sigma) \mid \sigma \in \mathrm{Gal}(K_\infty/K) \rangle$.

Coose $m \geq 0$ satisfying the following conditions:

- $|X'/IX'| \leq p^{m+1}$
- All ramified primes in K_∞/K_m are totally ramified.
- $\nu_{m,e}(X'/IX') = 0 \quad (\nu_{m,e} = 1 + \gamma^{p^e} + \gamma^{2p^e} + \cdots + \gamma^{p^m - p^e})$

Put $\zeta = \zeta_{p^{m+1}}$.

For $a \otimes \zeta^{\otimes n} \in (X'/IX')(n) = (X'/IX') \otimes \mu_{p^{m+1}}^{\otimes n}$,

$(N\mathfrak{a}^{n+1} - (\mathfrak{a}, K/F))\theta_S(n)\ (a \otimes \zeta^{\otimes n})$

$= (N\mathfrak{a}^{n+1} - (\mathfrak{a}, K_m/F)) \sum_{\sigma \in G_m} \zeta_{F,S}(\sigma, -n)\sigma^{-1}\ (a \otimes \zeta^{\otimes n})$

$= (N\mathfrak{a}^{n+1} - (\mathfrak{a}, K_m/F)) \sum_{\sigma \in G_m} \zeta_{F,S}(\sigma, 0)\kappa_m(\sigma)^n\sigma^{-1}\ (a \otimes \zeta^{\otimes n})$

$= \{N\mathfrak{a}^n(N\mathfrak{a} - (\mathfrak{a}, K_m/F)) \sum_{\sigma \in G_m} \zeta_{F,S}(\sigma, 0)\sigma^{-1}a\} \otimes \zeta^{\otimes n}$

$= 0.$ $\square$

講演後日付記

(1) p8 の

$\boxed{\text{Conjecture } (L_p)} + \boxed{\mu = 0}$

$\Rightarrow \boxed{\text{Conjecture } (d_2)} \Leftrightarrow \boxed{\text{Conjecture } (d_1)}$

で, $\boxed{\mu = 0}$ (岩澤主予想が解かれているので, 代数的サイド・解析的サイド, 同一の μ) が当日の講演で抜けていました. お詫び申し上げます.

(2) p5 Lemma $(H^1)(H^2)$　(参考文献：[S] +少しの計算)

(3) p7 Conjecture (L_p) 参考文献：
・ [KNF] p687 Conjecture (D_k) + $\boxed{\mu = 0}$ $\Rightarrow$ Conjecture (L_p).

(4) p7 Conjecture (d_2) 参考文献：
・ [KNF] p683 p683 Conjecture (C_m)
・ [CL] p521 Conjecture 3.1

(5) p8 $\boxed{\text{Conjecture}(d_2)} \Leftrightarrow \boxed{\text{Conjecture}(d_1)}$ (参考文献：[S] p192 Satz 6)

(6) p9 $\boxed{\text{Conjecture}}$ (参考文献 [A1])

(7) p12 $\boxed{\text{Theorem}}$ [Coates] (参考文献：[C] p319 Theorem 4.4)

(8) § Brumer conjecture and Coates-Sinnott conjecture 全般 (参考文献：[A2])

[A1] M. Aoki, On some exact sequences in the theory of cyclotomic fields, Journal of Algebra, 2008, **320**, 4156-4177.

[A2] M.Aoki, A note on the Coates-Sinnott conjecture, to appear in Bull. Lond. Math. Soc.

[C] J. Coates, 'p-adic L-functions and Iwasawa's theory', Algebraic Number Fields : L-functions and Galois properties (ed. A. Fröhlich), Academic Press, London (1977) 269-353.

[CL] J. Coates and S. Lichtenbaum, On ℓ-adic zeta functions, Ann. of Math. (2), **98** (1973)498-550.

[KNF] M. Kolster, T. Nguyen Quang Do and V. Fleckinger, Twisted S-units, p-adic class number formulas, and the Lichtenbaum conjectures, Duke Math. J. **84** (1996), 679-717.

[S] P. Schneider, 'Über gewisse Galoiscohomologiegruppen', Math. Z. 168 (1979) 181-205.