Endoscopy and Shimura Varieties

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OUTLINE

– Preliminaries: notation / reminder of $L$-groups

– I. What is endoscopy?

– II. Langlands correspondence and Shimura varieties

– Summary

(Warning: we sacrificed mathematical precision (e.g. the similitude factor of unitary groups) for the sake of simpler exposition.)
Our notation

$F$ – local or global field (of characteristic 0)
$W_F$ – Weil group for $F$

$H, G$ – connected reductive groups over $F$
$G^*$ – quasi-split inner form of $G$ over $F$
$\hat{G}$ – Langlands dual group of $G$
$L^G := \hat{G} \rtimes W_F$ – $L$-group of $G$

$\text{Rep}(G)$ – the set of isom. classes of autom. (resp. irred. adm.)
repns of $G(\mathbb{A})$ when $F$ is global (resp. local)
$LP(G)$ – the set of $L$-packets of $G$ (local or global).

(We may also write $\text{Rep}(G(\mathbb{A})), \text{Rep}(G(F_v)), LP(G(\mathbb{A})), \text{etc}$ to make clear whether we are in local or global situation.)
Quick reminder on $L$-groups

- $\hat{G}$ is a $\mathbb{C}$-Lie group whose based root datum is dual to that of $G \times_F \overline{F}$.

- $W_F$ acts on $\hat{G}$ as outer automorphisms, via a finite quotient.

- If $G$ (or some of its inner forms) is split over $F$, then $^L G = \hat{G} \times W_F$.

- If $G^*$ is an inner form of $G$ over $F$, then $\hat{G} = \hat{G}^*$ and $^L G = ^L G^*$.

- If $G = GL_n$, then $\hat{G} = GL_n(\mathbb{C})$ on which $W_F$ acts trivially.

- If $G = U_n$, then $\hat{G} = GL_n(\mathbb{C})$, but $W_F$ acts on $\hat{G}$ via a quotient group of order 2. (We will see more later.)
Part I - What is endoscopy? ... outline

1. $L$-indistinguishability

2. Special case of Langlands functoriality

3. Study of repns of $G$ via those of endoscopic groups of $G$

* Examples will be given. Emphasis on the case of unitary groups.
Endoscopy 1 - \( L \)-indistinguishability

- Failure of strong multiplicity one (in global case)

- Non-isomorphic autom. repns may have the same \( L \)-function. (global)

- Non-isomorphic irred. adm. repns may have the same local \( L \)-factor. (local)

\( \leadsto \) notion of \( L \)-packets
**L-packets**

- Conjecturally we should have a partition of Rep(G) into \( L \)-packets so that each \( L \)-packet consists of \( L \)-indistinguishable repns. (local or global)

- (Local Langlands Conjecture - coarse form)
If \( F \) is a \( p \)-adic field, then there is a “natural” bijection

\[
LP(G(F)) \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c}
\text{local } L\text{-parameters} \\
W_F \times SL_2(\mathbb{C}) \to L \, G
\end{array} \right\}
\]

(If \( F = \mathbb{R} \) or \( F = \mathbb{C} \), use \( W_F \) instead of \( W_F \times SL_2(\mathbb{C}) \).)
**Endoscopy 2 - instance of functoriality**

- The famous **Langlands functoriality conjecture** says:

If $\eta : L^H \to L^G$ is an $L$-group hom then there exists a “natural” map $\eta_* : LP(H) \to LP(G)$. If $G$ is not quasi-split over $F$ then $\eta_*$ is only partially defined.
(If you don’t like $L$-packets, use $\eta_* : \text{Rep}(H) \to \text{Rep}(G)$ instead.)

- **Endoscopy** (or endoscopic transfer) refers to the special case where $\hat{H} \cong \hat{G}^{\langle \sigma \rangle}$ for some $\sigma \in \text{Aut}(\hat{G})$. 
Examples of endoscopic transfer

1. $\eta_* : LP(H(F')) \to LP(H^*(F'))$ is Jacquet-Langlands corr.
   $\cdots$ when $G$ is a quasi-split inner form $H^*$ of $H$
   $\sim \; \sigma = \text{id}, \; \eta : L^H = L^G$.

2. $\eta_* : LP(H(F')) \to LP(H(K))$ is cyclic base change
   $\cdots$ when $G = \text{Res}_{K/F} H$ and $\text{Gal}(K/F) = \langle \sigma \rangle$ is finite
   (Assume that $H$ is split over $F$ for simplicity.)
   $$\eta : L^H = \hat{H} \times \text{Gal}(K/F) \to L^G = (\hat{H} \times \cdots \hat{H}) \rtimes \text{Gal}(K/F)$$
   $$h \times \sigma \quad \mapsto \quad (h, \ldots, h) \rtimes \sigma$$

Remark. #1 is known for $G = GL_n$. #2 is known for $H = GL_n$.
Some other special cases are known. In general, these transfers are conjectural.
Endoscopy 3 - study of $LP(G)$ via endos. groups of $G$

$\mathcal{E}(G) :=$ set of elliptic endoscopic groups of $G$ (up to isom.)
(Note that always $G^* \in \mathcal{E}(G)$.)

For each $H \in \mathcal{E}(G)$, we choose $\eta_H : LH \to LG$, which induces an endoscopic transfer $\eta_{H,*}$. Consider

$$\{(H, \Pi_H) : H \in \mathcal{E}(G), \ \Pi_H \in LP(H)\} \xrightarrow{\text{Trans}} LP(G) \quad \mapsto \quad \eta_{H,*}(\Pi_H)$$

* Trans is only partially defined if $G$ is not quasi-split.
* (When $F$ is global) we call $\Pi \in LP(G)$ stable if it has a unique inverse image (for $H = G^*$). Otherwise call it endoscopic.
Examples of elliptic endoscopic groups

1. $G = GL_m(D)$, $D/F$: central div alg of deg $d^2$, $F$: local or global
   $\Rightarrow E(G) = \{GL_{md}(F)\}$

   $LP(G) = \text{Rep}(G)$ (L-packets are singletons.)

2. $G = U_n$ over $F^+$, w.r.t. a quad extn $F/F^+$
   $\Rightarrow E(G) = \{U^*_a \times U^*_{n-a}\}_{0 \leq a \leq \lfloor n/2 \rfloor}$
   where $U^*_a, U^*_{n-a}$ are quasi-split unitary groups wrt $F/F^+$.

   (if $F$ is $p$-adic) size of an $L$-packet is a power of 2. (expected)
   (if $F^+ = \mathbb{R}$) size of each d.s. $L$-packet of $U(p, q)$ is $\binom{p+q}{p}$ (known)
Interesting endoscopic problems for unitary groups

Define local (and global) $L$-packets for $U_n$ and confirm the following two endoscopic transfers (among other instances). In fact, these transfers should force the definition of $L$-packets.

1. Base change: $LP(U_n(F^+)) \rightarrow LP(U_n(F)) = LP(GL_n(F))$

2. Elliptic endoscopy: $LP(U^*_a \times U^*_{n-a}) \rightarrow LP(U_n)$

Remark. These are known when $n \leq 3$. Partial results of base change are available when $n > 3$. Base change is expected to be injective on $L$-packets in this case. The endoscopy is properly understood in the context of the stable trace formula.

- Unitary groups are particularly interesting because...
Part II - Langlands corr and Shimura var - outline

- Statement of the Langlands correspondence for $GL_n$

- Approach via cohomology of unitary PEL Shimura varieties

- Technical difficulties and assumptions

- Strategy and expected answer for coh. of Shimura var
Prelude - class field theory

We would like to generalize the following correspondence given by class field theory, which is $GL_1$-case.

$$\begin{align*}
\{ \text{"algebraic" Hecke char.} \} & \overset{\text{global}}{\longleftrightarrow} \{ \text{"algebraic" Galois char.} \} \\
\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \overline{\mathbb{Q}}_l^\times & \quad \sigma : \text{Gal}(\overline{F}/F) \rightarrow \overline{\mathbb{Q}}_l^\times \\
\downarrow & \quad \downarrow \\
\{ \text{local char.} \} & \overset{\text{local}}{\longleftrightarrow} \{ \text{local Galois char.} \} \\
\chi_v : F_v^\times \rightarrow \overline{\mathbb{Q}}_l^\times & \quad \sigma_v : W_{F_v} \rightarrow \overline{\mathbb{Q}}_l^\times
\end{align*}$$

Two horizontal rows are given by

$$\sigma = \chi \circ \text{Art}^{-1}_F, \quad \sigma_v = \chi_v \circ \text{Art}^{-1}_{F_v}.$$
**Dream - Langlands correspondence for** $GL_n$

\[
\begin{align*}
\left\{ \text{algebraic cuspidal} \atop \text{automorphic repns} \right. \\
\text{of } GL_n(\mathbb{A}_F) \quad \quad & \xleftrightarrow{\text{global}} \quad \left\{ \text{irred. } n\text{-dim. } l\text{-adic} \right. \\
& \quad \quad \quad \left. \text{repns of } \text{Gal}(\overline{F}/F) \right. \\
& \quad \quad \quad \quad \quad \text{(unram a.e., pst at } l) \left\} \quad \downarrow \\
\left\{ \text{irred. admissible} \atop \text{repns of } GL_n(F_v) \right. \\
& \quad \quad \quad \xleftrightarrow{\text{local}} \quad \left\{ \text{Frob-semisimple} \right. \\
& \quad \quad \quad \quad \quad \left. n\text{-dim. Weil-Deligne} \right. \\
& \quad \quad \quad \quad \quad \left. \text{repns of } W_{F_v} \right\}
\end{align*}
\]

- Top arrow = conjectural global Langlands corresp.
- Bottom arrow = local Langlands corresp. (Thm. by Harris-Taylor and Henniart.)
Cohomological realization of Langlands correspondence

We may prove some instances of the global Langlands correspondence by realizing it in the cohomology of Shimura varieties.

For $GL_2$, use modular curves ($F = \mathbb{Q}$) or Shimura curves ($F = \text{tot. real}$).

For $GL_n$ with $n > 2$, we use PEL Shimura varieties of type (A), which are associated to unitary (similitude) groups. For this to work, we need to assume that

- $F$ is a CM field,
- $\pi \in \text{Rep}(GL_n(\mathbb{A}_F))$ satisfies $\pi^\vee \simeq \pi \circ c$, and
- $\pi$ is cohomological.

Ex. cuspidal repns for holomorphic modular forms are cohomological if $\text{wt} \geq 2$ and algebraic if $\text{wt} \geq 1$. 

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Construction via unitary Shimura varieties

- $F/F^+$ – an imag. quad. extn of a tot. real field.
- $U_n$ – unitary group over $F^+$ s.t. $U_n \times_{F^+} F \cong GL_n$.

The following is the ideal picture.

\[
\begin{array}{ccc}
\Rep_{CSD}(GL_n(\mathbb{A}_F)) & \xrightarrow{BC} & \Rep(U_n(\mathbb{A}_{F^+})) \\
\downarrow & \updownarrow \heartsuit & \downarrow \\
\{\text{irr. } n\text{-dim. Gal. repns}\} & \leftarrow & \{\text{irr. } n\text{-dim. Gal. repns}\}
\end{array}
\]

where the correspondence $(\star)$ is seen in the cohomology of unitary Shimura varieties (assoc. to $\text{Res}_{F^+/\mathbb{Q}}GU_n$). Note that $\heartsuit$ is the arrow that we are seeking for.

Here $BC$ denotes the base change. We write $\Rep_{CSD}$ for the set of those $\pi$ s.t. $\pi^\vee \cong \pi \circ c$. To be precise, we should have written $LP(U_n(\mathbb{A}_{F^+}))$. We quietly assume all repns are cohomological.
Actions on the cohomology of Shimura varieites

- $G$ – unitary group (almost $\text{Res}_{F^+/\mathbb{Q}} GU_n$)
- $X$ – Shimura variety for $G$, defined over the reflex field $E$.
  ($X$ is a proj system $\{X_U\}$ for open cpt subgroups $U \subset G(\mathbb{A}^\infty)$.)

$$H(X) := \sum (-1)^i \lim_{\to U} H^i_{et}(X_U \times_E \overline{E}, \mathbb{Q}_l) \in \text{Groth}(G(\mathbb{A}^\infty) \times \text{Gal}(\overline{E}/E))$$

$\sim$ Write (in the Groth. group)

$$H(X) = \sum \pi \otimes R(\pi)$$

where $\pi \in \text{Rep}(G(\mathbb{A}^\infty))$, $R(\pi) \in \text{Groth}(\text{Gal}(\overline{E}/E))$.

The correspondence $\pi \leftrightarrow R(\pi)$ is essentially what we meant by $(\star)$ in the previous slide.

(Warning: if $\pi$ lies in an endoscopic packet of $G$, then it is more subtle than this. We’ll come back to this point.)
Dreams hardly come true - technical difficulty

Two sources of technical difficulty in this game:
- **boundary**
- **endoscopy**

There seems to be three degrees of generality:
1) Use “twisted” unitary group $U_n$ which is isom. to $D^\times$ at some place $v$ ($D$: cent div alg over $F_v^+$) to “kill boundary and endoscopy”.
   $\Rightarrow$ Price to pay: a certain restriction on $\pi_v$

2) Use $U_n$ which is quasi-split at all finite places, but isom to $U(n,0)$ at some infinite place. This kills boundary but retains endoscopy.
   This is the case that I will focus on.
   $\Rightarrow$ Remove the restriction in 1)!

3) Work in complete generality. (Deal with boundary!)

Rem. When $n > 3$, only 1) has been worked out.
(So-far) The best result for $GL_n$ (when $n > 2$)

Proof of the following theorem uses the unitary groups in case 1) of the previous slide. Condition (c) is the local restriction on $\pi$ that we mentioned.

**Theorem** (Kottwitz, Clozel, Harris-Taylor, Taylor-Yoshida)
- $F$: CM field
- $\pi$: cuspidal autom. repn of $GL_n(\mathbb{A}_F)$ satisfying:
  1. $\pi^\vee \simeq \pi^c$ (conjugate self-dual)
  2. $\pi$ is regular algebraic ($=$cohomological)
  3. $\pi$ is a discrete series at a finite prime

Then, (up to isom.) $\exists! \rho(\pi) : \text{Gal}(\overline{F}/F) \to GL_n(\mathbb{Q}_l)$ such that

$$\forall v \nmid l, \quad \pi_v \leftrightarrow \rho(\pi)_v \quad \text{via local Langlands}.$$
Our strategy (when endoscopy is present)

The advantage of the approach initiated by Harris and Taylor:
- one can deal with “bad” primes of Shimura varieties
- one can deal with autom repns and Galois repns at ramified primes.

With some effort, much of their work, originally in trivial endoscopy case, can be extended to the case where endoscopy is non-trivial.

We show a very incomplete outline of this approach in the next slide.
Our strategy (when endoscopy is present) 2

- \( b \): isog class of BT-groups with additional str.
- \( T_b(\mathbb{Q}_p) = \text{QIsog}(\Sigma_b) \), where \( \Sigma_b \) belongs to \( b \).
- \( \mathcal{M}_b \): Rapoport-Zink space for \( b \) (rigid space over \( \text{Frac} W(\overline{\mathbb{F}_p}) \)).
- \( J_b \): Igusa variety for \( b \) (smooth variety over \( \overline{\mathbb{F}_p} \)).

In case 1) and 2), the following holds in \( \text{Groth}(W_{E_v} \times G(\mathbb{A}^\infty)) \):
(in case 3), we should include contribution from boundaries...)

\[
(Mantovan) \quad H(X) = \sum_b \text{Ext}_{T_b(\mathbb{Q}_p) - \text{smooth}}(H_c(M_b), H_c(J_b)).
\]

Problem 1. Study \( H_c(M_b) \). \( \cdots \) known for \( U(1, n-1) \times U(0, n)[F^+:\mathbb{Q}]^{-1} \)
Problem 2. Study \( H_c(J_b) \) via “counting points”. \( \cdots \) done by S.
Problem 3. Use the trace formula method and more. \( \cdots \) future.
\( \rightsquigarrow \) description of \( H(X) \).
Expected shape of $H(X)$

In case 2), write $G = \text{Res}_{F^+/\mathbb{Q}} U_n$. Assume $F^+ \neq \mathbb{Q}$ and

$$G(\mathbb{R}) \simeq U(1, n-1) \times U(0, n)[F^+:\mathbb{Q}]-1.$$ 

Note that the reflex field “is” $F$.

Any $\pi \in \text{Rep}(G(\mathbb{A}^\infty)) = \text{Rep}(U_n(\mathbb{A}^\infty))$ should arise from endoscopic transfer from a stable packet $\otimes_{i=1}^r \Pi_i \in LP(\prod_{i=1}^r U_{n_i}^*)$.

Then in the expression $H(X) = \sum \pi \otimes R(\pi)$, we expect that there exists $i$, depending on endoscopic information of $\pi$, s.t. $R(\pi)$ is the $n_i$-dim repn of $\text{Gal}(\overline{F}/F)$ assoc. to $BC_{F/F^+}(\Pi_i)$.

* The answer is up to sign. We ignored the similitude factor of $G$.
* We see only those $\pi$ s.t. $\pi \otimes \pi_\infty$ is automorphic for some $\pi_\infty \in \Pi_\infty$. Here $\Pi_\infty \in LP(G(\mathbb{R}))$ is determined by the coeff. sheaf of $H(X)$. 

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Sample consequence of studying $H(X)$ in endoscopic case

Can construct $n$-dim Gal repns from autom repns of $GL_n$ using endoscopy for $U(n + 1)$ Shimura variety. (cf. $n = 2$ studied by Blasius-Rogawski.)

Eventually,

Will be able to construct Galois repns from autom repns in some new cases, removing the condition (c) on p.20.
Summary

1. What was endoscopy?
   - $L$-indistinguishability
   - Special case of Langlands functoriality
   - Study of repns of $G$ via those of endoscopic groups of $G$
   - (I didn’t talk about the “geometric side”, especially the fundamental lemma...)

2. Construction of Galois repns from autom repns
   - Use the cohomology of unitary Shimura varieties
   - Why the case with non-trivial endoscopy. How we deal with it.