Endoscopy and Shimura Varieties

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OUTLINE

– Preliminaries: notation / reminder of L-groups

– I. What is endoscopy?

– II. Langlands correspondence and Shimura varieties

– Summary

(Warning: we sacrificed mathematical precision (e.g. the similitude factor of unitary groups) for the sake of simpler exposition.)

Our notation

F – local or global field (of characteristic 0)
W_F – Weil group for F

H, G – connected reductive groups over F
G^* – quasi-split inner form of G over F
$\widehat{G}$ – Langlands dual group of G
${}^L G := \widehat{G} \rtimes W_F$ – L-group of G

Rep(G) – the set of isom. classes of autom. (resp. irred. adm.) repns of $G(\mathbb{A})$ when F is global (resp. local)
$LP(G)$ – the set of L-packets of G (local or global).

(We may also write Rep($G(\mathbb{A})$), Rep($G(F_v)$), $LP(G(\mathbb{A}))$, etc to make clear whether we are in local or global situation.)

Quick reminder on L-groups

– $\widehat{G}$ is a $\mathbb{C}$-Lie group whose based root datum is dual to that of $G \times_F \overline{F}$.

– W_F acts on $\widehat{G}$ as outer automorphisms, via a finite quotient.

– If G (or some of its inner forms) is split over F, then ${}^LG = \widehat{G} \times W_F$.

– If G^* is an inner form of G over F, then $\widehat{G} = \widehat{G^*}$ and ${}^LG = {}^LG^*$.

– If $G = GL_n$, then $\widehat{G} = GL_n(\mathbb{C})$ on which W_F acts trivially.

– If $G = U_n$, then $\widehat{G} = GL_n(\mathbb{C})$, but W_F acts on $\widehat{G}$ via a quotient group of order 2. (We will see more later.)

Part I - What is endoscopy? $\cdots$ outline

1. L-indistinguishability

2. Special case of Langlands functoriality

3. Study of repns of G via those of endoscopic groups of G

$*$ Examples will be given. Emphasis on the case of unitary groups.

Endoscopy 1 - L-indistinguishability

- Failure of strong multiplicity one (in global case)

- Non-isomorphic autom. repns may have the same L-function. (global)

- Non-isomorphic irred. adm. repns may have the same local L-factor. (local)

$\rightsquigarrow$ notion of L-packets

L-packets

• Conjecturally we should have a partition of Rep(G) into L-packets so that each L-packet consists of L-indistinguishable repns. (local or global)

• (Local Langlands Conjecture - coarse form)
If F is a p-adic field, then there is a "natural" bijection

$$LP(G(F)) \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{local } L\text{-parameters} \\ W_F \times SL_2(\mathbb{C}) \to {}^LG \end{array} \right\}$$

(If $F = \mathbb{R}$ or $F = \mathbb{C}$, use W_F instead of $W_F \times SL_2(\mathbb{C})$.)

Endoscopy 2 - instance of functoriality

• The famous Langlands functoriality conjecture says:

If $\eta : {}^L H \to {}^L G$ is an L-group hom then there exists a "natural" map $\eta_* : LP(H) \to LP(G)$. If G is not quasi-split over F then η_* is only partially defined.
(If you don't like L-packets, use $\eta_* : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$ instead.)

• Endoscopy (or endoscopic transfer) refers to the special case where $\widehat{H} \overset{\eta}{\simeq} \widehat{G}^{\langle\sigma\rangle}$ for some $\sigma \in \mathrm{Aut}(\widehat{G})$.

Examples of endoscopic transfer

1. $\eta_* : LP(H(F)) \to LP(H^*(F))$ is Jacquet-Langlands corr.
   $\cdots$ when G is a quasi-split inner form H^* of H
   $\rightsquigarrow \sigma = \mathrm{id}$, $\eta : {}^L H = {}^L G$.

2. $\eta_* : LP(H(F)) \to LP(H(K))$ is cyclic base change
   $\cdots$ when $G = \mathrm{Res}_{K/F} H$ and $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ is finite
   (Assume that H is split over F for simplicity.)

$$\begin{array}{rcl} \eta : {}^L H = \widehat{H} \times \mathrm{Gal}(K/F) & \to & {}^L G = (\widehat{H} \times \cdots \widehat{H}) \rtimes \mathrm{Gal}(K/F) \\ h \times \sigma & \mapsto & (h, \ldots, h) \rtimes \sigma \end{array}$$

Remark. #1 is known for $G = GL_n$. #2 is known for $H = GL_n$. Some other special cases are known. In general, these transfers are conjectural.

Endoscopy 3 - study of $LP(G)$ via endos. groups of G

$\mathscr{E}(G) :=$ set of elliptic endoscopic groups of G (up to isom.)
(Note that always $G^* \in \mathscr{E}(G)$.)

For each $H \in \mathscr{E}(G)$, we choose $\eta_H : {}^L H \to {}^L G$, which induces an endoscopic transfer $\eta_{H,*}$. Consider

$$\begin{array}{ccc} \{(H, \Pi_H) : H \in \mathscr{E}(G),\ \Pi_H \in LP(H)\} & \overset{\text{Trans}}{\twoheadrightarrow} & LP(G) \\ (H, \Pi_H) & \mapsto & \eta_{H,*}(\Pi_H) \end{array}$$

$*$ Trans is only partially defined if G is not quasi-split.
$*$ (When F is global) we call $\Pi \in LP(G)$ stable if it has a unique inverse image (for $H = G^*$). Otherwise call it endoscopic.

Examples of elliptic endoscopic groups

1. $G = GL_m(D)$, D/F: central div alg of deg d^2, F: local or global
   $\Rightarrow \mathscr{E}(G) = \{GL_{md}(F)\}$

   $LP(G) = \mathrm{Rep}(G)$ (L-packets are singletons.)

2. $G = U_n$ over F^+, w.r.t. a quad extn F/F^+
   $\Rightarrow \mathscr{E}(G) = \{U_a^* \times U_{n-a}^*\}_{0 \le a \le [n/2]}$
   where U_a^*, U_{n-a}^* are quasi-split unitary groups wrt F/F^+.

   (if F is p-adic) size of an L-packet is a power of 2. (expected)
   (if $F^+ = \mathbb{R}$) size of each d.s. L-packet of $U(p,q)$ is $\binom{p+q}{p}$ (known)

Interesting endoscopic problems for unitary groups

Define local (and global) L-packets for U_n and confirm the following two endoscopic transfers (among other instances). In fact, these transfers should force the definition of L-packets.

1. Base change: $LP(U_n(F^+)) \to LP(U_n(F)) = LP(GL_n(F))$

2. Elliptic endoscopy: $LP(U_a^* \times U_{n-a}^*) \to LP(U_n)$

Remark. These are known when $n \leq 3$. Partial results of base change are available when $n > 3$. Base change is expected to be injective on L-packets in this case. The endoscopy is properly understood in the context of the stable trace formula.

- Unitary groups are particularly interesting because...

Part II - Langlands corr and Shimura var - outline

– Statement of the Langlands correspondence for GL_n

– Approach via cohomology of unitary PEL Shimura varieties

– Technical difficulties and assumptions

– Strategy and expected answer for coh. of Shimura var

Prelude - class field theory

We would like to generalize the following correspondence given by class field theory, which is GL_1-case.

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{“algebraic” Hecke char.} \\ \chi : F^\times \backslash \mathbb{A}_F^\times \to \overline{\mathbb{Q}}_l^\times \end{array} \right\} & \overset{\text{global}}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{“algebraic” Galois char.} \\ \sigma : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}_l^\times \end{array} \right\} \\
\downarrow & & \downarrow \\
\left\{ \begin{array}{c} \text{local char.} \\ \chi_v : F_v^\times \to \overline{\mathbb{Q}}_l^\times \end{array} \right\} & \overset{\text{local}}{\longleftrightarrow} & \left\{ \begin{array}{c} \text{local Galois char.} \\ \sigma_v : W_{F_v} \to \overline{\mathbb{Q}}_l^\times \end{array} \right\}
\end{array}$$

Two horizontal rows are given by

$$\sigma = \chi \circ \mathrm{Art}_F^{-1}, \qquad \sigma_v = \chi_v \circ \mathrm{Art}_{F_v}^{-1}.$$

Dream - Langlands correspondence for GL_n

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{algebraic cuspidal}\\ \text{automorphic repns}\\ \text{of } GL_n(\mathbb{A}_F)\end{array}\right\} & \overset{\text{global}}{\longleftrightarrow} & \left\{\begin{array}{c}\text{irred. } n\text{-dim. } l\text{-adic}\\ \text{repns of Gal}(\overline{F}/F)\\ \text{(unram a.e., pst at } l)\end{array}\right\} \\
\downarrow & & \downarrow \\
\left\{\begin{array}{c}\text{irred. admissible}\\ \text{repns of } GL_n(F_v)\end{array}\right\} & \overset{\text{local}}{\longleftrightarrow} & \left\{\begin{array}{c}\text{Frob-semisimple}\\ n\text{-dim. Weil-Deligne}\\ \text{repns of } W_{F_v}\end{array}\right\}
\end{array}$$

- Top arrow = conjectural <u>global Langlands corresp.</u>
- Bottom arrow = <u>local Langlands corresp.</u> (Thm. by Harris-Taylor and Henniart.)

Cohomological realization of Langlands correspondence

We may prove some instances of the global Langlands correspondence by realizing it in the cohomology of Shimura varieties.

For GL_2, use modular curves ($F = \mathbb{Q}$) or Shimura curves ($F =$tot. real).

For GL_n with $n > 2$, we use PEL Shimura varieties of type (A), which are associated to unitary (similitude) groups. For this to work, we need to assume that

- F is a CM field,
- $\pi \in \mathrm{Rep}(GL_n(\mathbb{A}_F))$ satisfies $\pi^\vee \simeq \pi \circ c$, and
- π is cohomological.

Ex. cuspidal repns for holomorphic modular forms are cohomological if wt$\geq$2 and algebraic if wt$\geq$1.

Construction via unitary Shimura varieties

- F/F^+ – an imag. quad. extn of a tot. real field.
- U_n – unitary group over F^+ s.t. $U_n \times_{F^+} F \simeq GL_n$.

The following is the ideal picture.

$$\begin{tikzcd}
\mathrm{Rep}_{CSD}(GL_n(\mathbb{A}_F)) \arrow[d, leftrightarrow, "BC"] \arrow[dr, dashed, "\heartsuit"] & \\
\mathrm{Rep}(U_n(\mathbb{A}_{F^+})) \arrow[r, leftrightarrow, "(\star)"] & \{\text{irr. } n\text{-dim. Gal. repns}\}
\end{tikzcd}$$

where the correspondence $(\star)$ is seen in the cohomology of unitary Shimura varieties (assoc. to $\mathrm{Res}_{F^+/\mathbb{Q}} GU_n$). Note that $\heartsuit$ is the arrow that we are seeking for.

Here BC denotes the base change. We write Rep_{CSD} for the set of those π s.t. $\pi^\vee \simeq \pi \circ c$. To be precise, we should have written $LP(U_n(\mathbb{A}_{F^+}))$. We quietly assume all repns are cohomological.

Actions on the cohomology of Shimura varieites

• G – unitary group (almost $\mathrm{Res}_{F^+/\mathbb{Q}} GU_n$)
• X – Shimura variety for G, defined over the reflex field E.
(X is a proj system $\{X_U\}$ for open cpt subgroups $U \subset G(\mathbb{A}^\infty)$.)

$$H(X) := \sum_i (-1)^i \varinjlim_U H^i_{et}(X_U \times_E \overline{E}, \overline{\mathbb{Q}}_l) \in \mathrm{Groth}(G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{E}/E))$$

$\rightsquigarrow$ Write (in the Groth. group)

$$H(X) = \sum_\pi \pi \otimes R(\pi)$$

where $\pi \in \mathrm{Rep}(G(\mathbb{A}^\infty))$, $R(\pi) \in \mathrm{Groth}(\mathrm{Gal}(\overline{E}/E))$.

The correspondence $\pi \leftrightarrow R(\pi)$ is essentially what we meant by $(\star)$ in the previous slide.
(Warning: if π lies in an endoscopic packet of G, then it is more subtle than this. We'll come back to this point.)

Dreams hardly come true - technical difficulty

Two sources of technical difficulty in this game:

• boundary • endoscopy

There seems to be three degrees of generality:

1) Use "twisted" unitary group U_n which is isom. to $D^\times$ at some place v (D: cent div alg over F_v^+) to "kill boundary and endoscopy".

$\rightsquigarrow$ Price to pay: a certain restriction on π_v

2) Use U_n which is quasi-split at all finite places, but isom to $U(n, 0)$ at some infinite place. This kills boundary but retains endoscopy. This is the case that I will focus on.

$\rightsquigarrow$ Remove the restriction in 1)!

3) Work in complete generality. (Deal with boundary!)

Rem. When $n > 3$, only 1) has been worked out.

(So-far) The best result for GL_n (when $n > 2$)

Proof of the following theorem uses the unitary groups in case 1) of the previous slide. Condition (c) is the local restriction on π that we mentioned.

Theorem (Kottwitz, Clozel, Harris-Taylor, Taylor-Yoshida)
• F: CM field
• π: cuspidal autom. repn of $GL_n(\mathbb{A}_F)$ satisfying:
(a) $\pi^\vee \simeq \pi^c$ (conjugate self-dual)
(b) π is regular algebraic (=cohomological)
(c) π is a discrete series at a finite prime
Then, (up to isom.) $\exists!$ $\rho(\pi) : \mathrm{Gal}(\overline{F}/F) \to GL_n(\overline{\mathbb{Q}}_l)$ such that

$$\forall v \nmid l, \qquad \pi_v \;\leftrightarrow\; \rho(\pi)_v \qquad \text{via local Langlands.}$$

Our strategy (when endoscopy is present) 1

The advantage of the approach initiated by Harris and Taylor
= one can deal with "bad" primes of Shimura varieties
= one can deal with autom repns and Galois repns at <u>ramified</u> primes.

With some effort, much of their work, originally in trivial endoscopy case, can be extended to the case where endoscopy is non-trivial.

We show a very incomplete outline of this approach in the next slide.

Our strategy (when endoscopy is present) 2

- b: isog class of BT-groups with additional str.
- $T_b(\mathbb{Q}_p) = \mathrm{QIsog}(\Sigma_b)$, where Σ_b belongs to b.
- $\mathscr{M}_b$: Rapoport-Zink space for b (rigid space over $\mathrm{Frac}\, W(\overline{\mathbb{F}}_p)$).
- J_b: Igusa variety for b (smooth variety over $\overline{\mathbb{F}}_p$).

In case 1) and 2), the following holds in $\mathrm{Groth}(W_{E_v} \times G(\mathbb{A}^\infty))$:
(in case 3), we should include contribution from boundaries...)

$$\text{(Mantovan)} \qquad H(X) = \sum_b \mathrm{Ext}_{T_b(\mathbb{Q}_p)-\mathrm{smooth}}(H_c(\mathscr{M}_b), H_c(J_b)).$$

Problem 1. Study $H_c(M_b)$. $\cdots$ known for $U(1,n-1) \times U(0,n)^{[F^+:\mathbb{Q}]-1}$
Problem 2. Study $H_c(J_b)$ via "counting points". $\cdots$ done by S.
Problem 3. Use the trace formula method and more. $\cdots$ future.
$\rightsquigarrow$ description of $H(X)$.

Expected shape of $H(X)$

In case 2), write $G = \mathrm{Res}_{F^+/\mathbb{Q}} U_n$. Assume $F^+ \neq \mathbb{Q}$ and

$$G(\mathbb{R}) \simeq U(1, n-1) \times U(0, n)^{[F^+:\mathbb{Q}]-1}.$$

Note that the reflex field "is" F.

Any $\pi \in \mathrm{Rep}(G(\mathbb{A}^\infty)) = \mathrm{Rep}(U_n(\mathbb{A}^\infty))$ should arise from endoscopic transfer from a stable packet $\otimes_{i=1}^r \Pi_r \in LP(\prod_{i=1}^r U^*_{n_i})$.

Then in the expression $H(X) = \sum_\pi \pi \otimes R(\pi)$, we expect that there exists i, depending on endoscopic information of π, s.t.
$R(\pi)$ is the n_i-dim repn of $\mathrm{Gal}(\overline{F}/F)$ assoc. to $BC_{F/F^+}(\Pi_i)$.

$*$ The answer is up to sign. We ignored the similitude factor of G.
$*$ We see only those π s.t. $\pi \otimes \pi_\infty$ is automorphic for some $\pi_\infty \in \Pi_\infty$.
Here $\Pi_\infty \in LP(G(\mathbb{R}))$ is determined by the coeff. sheaf of $H(X)$.

Sample consequence of studying $H(X)$ in endoscopic case

Can construct n-dim Gal repns from autom repns of GL_n using endoscopy for $U(n+1)$ Shimura variety. (cf. $n = 2$ studied by Blasius-Rogawski.)

Eventually,

Will be able to construct Galois repns from autom repns in some new cases, removing the condition (c) on p.20.

Summary

1. What was endoscopy?

– L-indistinguishability
– Special case of Langlands functoriality
– Study of repns of G via those of endoscopic groups of G
– (I didn't talk about the "geometric side", especially the fundamental lemma...)

2. Construction of Galois repns from autom repns

– Use the cohomology of unitary Shimura varieties
– Why the case with non-trivial endoscopy. How we deal with it.