# The Ribet exact sequence for quaternionic Shimura varieties

Marc-Hubert Nicole, University of Tōkyō / JSPS

University of Kyōto, May 30th 2007

# The Ribet Exact Sequence

Consider the modular curves $X_0(pq)/_{\mathbb{Z}}$ and $X_0(p)/_{\mathbb{Z}}$, and the Shimura curve $Y/_{\mathbb{Z}}$ constructed from the indefinite quaternion algebra $B_{pq}/\mathbb{Q}$. The reductions of the Jacobians of these curves are extensions of abelian varieties by tori.

**Theorem:** (Ribet) There is an exact sequence of Hecke modules:

$$0 \longrightarrow Y_q \longrightarrow X_p(pq) \longrightarrow X_p(p)^2 \longrightarrow 0,$$

where $Y_q$, $X_p$ are the character groups of tori associated to the reduction modulo $q$ (resp. $p$) of the Jacobians of $Y$, (resp. $X_0$).

**Corollary:**

$$S_2^{pq}(\mathbb{C}) \underset{\text{Hecke-equ.}}{\hookrightarrow} S_2(\Gamma_0(pq))(\mathbb{C}), \qquad \text{with image } S_2^{pq-new}(\mathbb{C})$$

# Preliminaries on $\mathrm{GL}_2$ and inner forms

Let $G_{B^\times}, G_{D^\times}$ be the reductive groups over $\mathbb{Q}$ associated to two quaternion algebras $B, D$ over a totally real field $L$ of degree $g$.

Since we care only about the holomorphic part of the Jacquet-Langlands correspondence, we can compare two inner forms $G_{B^\times}, G_{D^\times}$ that are not necessarily isomorphic at archimedean places.

- We will suppose that the (reduced) discriminant $disc(B)$ is a product of prime ideals above two prime numbers $p$ and $q$.

- Moreover, we suppose that the discriminant of $D$ is 1. We shall add $\Gamma_0(disc(B))$ structure to the Shimura variety associated to $D$.

## In this talk

*Simplifying assumption n. 1*: suppose that $B$ is a totally indefinite quaternion algebra, so that the corresponding moduli space is of PEL type.

*Simplifying assumption n. 2*: suppose that $p\mathcal{O}_L = \mathfrak{p}$ is inert.

Simplest nontrivial comparisons of Shimura varieties

| $Sh_B$ mod $q$ | $Sh_D$ mod $p$ | Reflex field | Dimension |
|---|---|---|---|
| PEL/$_{\mathbb{Q}}$ | I) PEL, compact | $\mathbb{Q}$ | $g$ |
| | II) HMV, non-compact | $\mathbb{Q}$ | $g$ |
| | III) Non-PEL compact curve | $L$ | 1 |

Difficulties for case I and II: I) (minor) the moduli space on the $B$ side does not admit a complete $q$-adic uniformization; II) the Hilbert moduli space (of level $\Gamma_0(\mathfrak{p}q)$) is not compact.

# PEL $q$-adiq. unif. variety vs Shimura curve

Let $[L:\mathbb{Q}] = g = 2k+1$. Let $p\mathcal{O}_L = \mathfrak{p}$ and $q\mathcal{O}_L = \prod_{i=1}^{2k+1} \mathfrak{q}_i$. Consider the following quaternion algebras over $L$: $B :=$ $B_{\mathfrak{p},\mathfrak{q}_1\cdots\mathfrak{q}_{2k+1}}$ and $D := D_{\infty_1,\ldots,\infty_{2k},(\infty_{2k+1})}$. We will compare character groups of reductions of :

- the $q$-adically uniformized PEL-type variety constructed from $B$: $Y(\mathfrak{p})$, defined over $\mathbb{Q}$ and

- the Shimura curve $\mathcal{M}_0(\mathfrak{p}q)$ obtained from $D$ by adding $\mathfrak{p}q$ level, defined over $L$.

- **Theorem:** (**N.**)

$$0 \longrightarrow Y_q(\mathfrak{p}) \longrightarrow X_{\mathfrak{p}}(\mathfrak{p}q) \longrightarrow \oplus_{i=1}^{g} X_{\mathfrak{p}}(\mathfrak{p}q/\mathfrak{q}_i)^2 \longrightarrow 0.$$

## Cocharacter group via the...

Let $X$ be a semistable scheme over a (local) trait: $X$ regular, with special fiber $Z$ a reduced NCD which is a sum of smooth divisors.

- The weight spectral sequence:

  $$E_1^{ij} = H^{i+j}(Z, gr_{-i}^W R\Psi\mathbb{Q}_\ell) \implies H^*(X_{\overline{\eta}}, \mathbb{Q}_\ell),$$

  where

  $$E_1^{-r,q+r} = \oplus_{k\geq 0, k\geq r} H^{q-r-2k}(Z^{(r+2k+1)}, \mathbb{Q}_\ell(-r-k)),$$

  where $Z^{(m)}$ is the disjoint sum of $m$-by-$m$ intersections of components of $Z$.

**Definition: I.** The character group is defined by:

$$W_{2g}E_2 = \mathrm{Ker}(H^0(Z^{(g+1)}, \mathbb{Q}_\ell(-g)) \xrightarrow{d_*} H^2(Z^{(g)}, \mathbb{Q}_\ell(-g+1))),$$

**II.** The cocharacter group is defined by:

$$W_0E_2 = \mathrm{Coker}(H^0(Z^{(g)}, \mathbb{Q}_\ell) \xrightarrow{d^*} H^0(Z^{(g+1)}, \mathbb{Q}_\ell)).$$

where $d_*$ (resp. $d^*$) is an alternating sum of Gysin homomorphisms (resp., pullbacks) coming from the weight spectral sequence formalism.

N.B. The spectral sequence degenerates in $E_2$, and for $q$-adically uniformized varieties, the abutment filtration $W$ coincides with the monodromy filtration $M$. We do not use the latter property in this talk.

## ... spectral sequence.

For $N \subset M \subset \{1, \ldots, m\}$ t.q. $|M| = |N| + 1 = n + 1$, let $i_{NM} : Z_M \longrightarrow Z_N$ be the closed immersion

$$Z_M := \cap_{m \in M} Z_m \overset{i_{MN}}{\hookrightarrow} \cap_{n \in N} Z_n =: Z_N.$$

Let $M = \{i_0, \ldots, i_n\}, 0 \leq i_0 < \cdots < i_n \leq m$, and $N = \{i_0, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n\}$. Put: $\epsilon(N, M) = (-1)^j$.

Then

$$d_* : H^0(Z^{(g+1)}, \mathbb{Q}_\ell) \longrightarrow H^2(Z^{(g)}, \mathbb{Q}_\ell(-g+1)),$$

is defined by the sum $\sum_{N \subset M, |M| = |N|+1 = g+1} \epsilon(N, M)(i_{NM})_*$ of Gysin homomorphisms, and

$$d^* : H^0(Z^{(g)}, \mathbb{Q}_\ell) \longrightarrow H^0(Z^{(g+1)}, \mathbb{Q}_\ell),$$

is defined by the sum $\sum_{N \subset M, |N| = |M|-1 = g+1} \epsilon(N, M) i^*_{NM}$ of the pullbacks.

## Non-PEL Shimura curve

Recall that $[L : \mathbb{Q}]$ is odd. Consider the quaternion algebra $D$ over $L$ ramified in all archimedean places except one (that we labelled $\infty_{2k+1}$), and split in all other places.

Let $G = \mathrm{Res}_{L/\mathbb{Q}}(D^{\times})$, and $\mathcal{K}$ a compact, open subgroup of $G(\mathbb{A}^{\infty})$. Define the Shimura curve of level $\mathcal{K}$:

$$M_{\mathcal{K}}(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}^{\infty}) \times (\mathbb{C} - \mathbb{R})/\mathcal{K}.$$

This Shimura curve has a canonical model $M_{\mathcal{K}}$ defined over $L$.

There exists a scheme $\mathcal{M}_{\mathcal{K}}$ defined over $\mathcal{O}_{(\mathfrak{p})}$ such that $\mathcal{M}_{\mathcal{K}} \otimes_{\mathcal{O}_{(\mathfrak{p})}} L$ is isomorphic to $M_{\mathcal{K}}$, for $\mathcal{K}$ a $\Gamma_0(\mathfrak{p}q)$ (resp. $\Gamma_0(\mathfrak{p}q/\mathfrak{q}_i)$) level structure.

# PEL moduli problem

Fix a maximal order $\mathcal{O}_B$, equipped with an involution $'$.

The quaternionic variety associated to the totally indefinite quaternion algebra is the solution to the following moduli problem:

1. $A \longrightarrow S$ is an abelian scheme of relative dimension $2g$ equipped with an action $i : \mathcal{O}_B \longrightarrow \mathrm{End}_S(A)$ (satisfying the usual condition on the trace),

2. A $\mathcal{O}_B$-polarization $\phi : A \longrightarrow A^t$, where the $\mathcal{O}_B$-structure on $A^t$ is given by $b \mapsto i(b')^t$.

+ sufficiently small auxiliary level $\mathfrak{N}$ structure (suppressed from the notation), etc.

# $q$-adic uniformization, part I.

Some quaternionic (or unitary) Shimura varieties admit a *complete* $q$-adic uniformization. We describe the general quaternionic case.

Let $B/L$ be a quaternion algebra over a totally real field $L$. Let $q\mathcal{O}_L = \prod_{i=1}^{s} \mathfrak{q}_i$. Suppose that we have embeddings $\alpha_i : L_{\mathfrak{q}_i} \hookrightarrow \overline{\mathbb{Q}}_q$. After fixing a diagram: $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \xrightarrow{\nu} \overline{\mathbb{Q}}_q$, we obtain embeddings $L \hookrightarrow \mathbb{R}$.

Suppose that $B \otimes_{L,\alpha_i} \mathbb{R} \cong M_2(\mathbb{R}), i = 1, \ldots, s$, and that $B$ is a division algebra for all other real places, and in all places over $q$ and $p = \mathfrak{p}$. Its discriminant is thus $\mathfrak{p}q$.

Consider the quaternion algebra $\overline{B}$ over $L$ obtained from $B$ by switching places $\alpha_1, \ldots, \alpha_s$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ in $\mathbf{Ram}(B)$.

## $q$-adic uniformization, part II.

Let $E_i = \alpha_i(L_{\mathfrak{q}_i})$ for $i = 1, \ldots, s$ and let $E$ be the compositum of the fields $E_i$ in $\overline{\mathbb{Q}}_q$.

Consider the morphism:

$$h : S^1 \longrightarrow \prod_{i=1}^{s} \mathsf{GL}_2(\mathbb{R}) \cong \prod_{i=1}^{s} \widetilde{G} \otimes_{L,\alpha_i} \mathbb{R} \subset G_{\mathbb{R}},$$

defined by the action of $\mathbb{C}^\times$ on $\mathbb{C}^r = \mathbb{R}^{2r}$. The quaternionic variety $Sh_G$ is defined over $E$ (i.e., over $\mathbb{Q}$ if $s = [L : \mathbb{Q}]$, as in our special case).

## Byproduct over $\overline{\mathbb{F}}_q$ of the $q$-adic uniformization

There exists an isomorphism of $\overline{\mathbb{F}}_q$-schemes:

$$\overline{B}^{\times} \backslash \prod_{i=1}^{s} \mathbb{N}_i \times \widetilde{G}(\mathbb{A}^q_{L,f})/C \xrightarrow{\cong} Sh_{G,C},$$

where the $\mathbb{N}_i$ are moduli spaces of quasi-isogenies of $q$-divisible groups (i.e., of *special formal* modules equipped with a quasi-isogeny).

# Special formal modules

Let $F$ be a finite extension of $\mathbb{Q}_q$, $\mathcal{D}$ a central division algebra of invariant $\frac{1}{d}$ over $F$. Let $\widetilde{F}$ be the unramified extension of $F$ of degree $d$, and Gal$(\widetilde{F}/F) =< \sigma >$. Then $\mathcal{D} = \widetilde{F}[\Pi]$ with $\Pi^d = \pi$, and $\Pi a = a^\sigma \Pi$ for $a \in \widetilde{F}$. Let $\mathcal{O}_\mathcal{D}$ be a maximal order of $\mathcal{D}$, and $S$ a $\mathcal{O}_F$-scheme.

**Definition:** A special formal $\mathcal{O}_\mathcal{D}$-module is a connected $q$-divisible group $X$ over a scheme $S$ equipped with an action of $\mathcal{O}_\mathcal{D}$ such that:

- The induced action of $\mathcal{O}_F$ on Lie$(X)$ coincide with the natural action induced from $\mathcal{O}_F \longrightarrow \mathcal{O}_S$.
- Lie$(X)$ is a $\mathcal{O}_S \otimes_{\mathcal{O}_F} \mathcal{O}_{\widetilde{F}}$-module locally free of rank 1.

# Stratification of quaternionic varieties

Quaternionic varieties admit a stratification:

$$Sh_{G,C} \otimes \overline{\mathbb{F}}_q = \sqcup_S \mathcal{M}_{C,S},$$

where the strata are indexed by subsets of $\mathbb{Z}/2g\mathbb{Z}$, and satisfy the following properties:

- $\mathcal{M}_{C,S} \subset \mathcal{M}_{C,S'} \iff S' \subset S$;
- $\mathcal{M}_{C,S} \cap \mathcal{M}_{C,S'} = \mathcal{M}_{C,S \cup S'}$.
- The scheme $\mathcal{M}_{C,S}$ is of dimension $2g - |S|$.
- The subschemes $\mathcal{M}_{C,S}$ intersect transversally.

## Local structure

We have étale local models i.e., we can describe the étale neighborhood of any closed point $x$. We obtain a product of double points and a smooth scheme e.g.,

$$\widehat{\mathcal{O}_x} \cong W(\mathbb{F})[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_iY_i - p, i \in I).$$

## Parametrizations of the (PEL) strata over $\overline{\mathbb{F}}_q$

- $S_0$: **dimension** 0

  The vanishing cycles sheaf $R^g\Psi$ is supported on the dimension 0 strata. The underlying set is thus finite, and its cardinality is given by the class number of an Eichler order of level $\mathfrak{p}q$ (in the totally definite quaternion algebra ramified at $\mathfrak{p}$).

- $S_1$: **dimension** 1

  There are $2g$ strata $S_1^i$ of dimension 1. We can describe their $\overline{\mathbb{F}}_q$-rational points by using the $q$-adic uniformization: they are $\mathbb{P}^1(\overline{\mathbb{F}}_q)$-fiber bundles over a finite set whose cardinality is given by the class number of an Eichler order of level $\mathfrak{p}q/\mathfrak{q}_i$.

# Description in dimension 1, following Zink

**Definition ([Zink]):**

Let $M$ be a smooth algebraic variety over $k$. A $\mathbb{P}^1$-fibration over $M$ is:

- A projective bundle of dimension 1 over a variety $T'$:

$$\pi : \mathbb{P}_{T'}(\mathcal{E}) \longrightarrow T',$$

  where $\mathcal{E}$ is a locally free sheaf of rank 2 over $T'$.

- A universal homeomorphism:

$$f : \mathbb{P}_{T'}(\mathcal{E}) \longrightarrow M,$$

  such that for all geometric points $t$ of $T'$, $f(\pi^{-1}(t))$ is a smooth rational curve.

## ...

We call:

-$f(\pi^{-1}(t))$ the *fibers* of the $\mathbb{P}^1$-fibration;

-the image of a section of $\pi$ is a *basis* of the $\mathbb{P}^1$-fibration.

E.g., if $\dim(M) = 2$, then $M$ is a ruled surface (we do not consider this case in this talk).

**Proposition:** A basis of the $\mathbb{P}^1$-fibration of $S_1^i$ has cardinality the class number of an Eichler order of level $\mathfrak{p}q/\mathfrak{q}_i$. Every fiber intersects $S_0$ in $\mathrm{Norm}(\mathfrak{q}_i) + 1$ points.

## Folklore on universal homeomorphisms

In general, it is interesting to study different strata of Shimura varieties up to universal homeomorphism i.e., up to morphisms inducing isomorphisms in étale cohomology.

**"Folklore":** (réf.: [Kollàr, Annals, 1997]) A (finite) universal homeomorphism $f : X \longrightarrow Y$ between $\mathbb{F}_p$-schemes is a factor of a power of the Frobenius morphism $X \longrightarrow X^{(p)}$:

$$X \xrightarrow{f} Y \xrightarrow{\overline{f}} X^{(p^n)}, \text{ for } n >> 0.$$

## Blowing up a product of semistable schemes

We blow up the scheme along a smooth subscheme $S_g$ of codimension 0. The local étale model allows to check that the special fiber thus obtained is a divisor with *simple* normal crossings. In terms of the model, we compute the blow up along the subscheme $(X_0, \ldots, X_n)$ of $\mathrm{Spec}(R) = \mathrm{Spec}\Big(W(\mathbb{F})[X_1, \ldots, X_n, Y_1, \ldots, Y_n]/(X_i Y_i - p)\Big)$. E.g., for $n = 2$, it goes as follows: we obtain the subscheme of $\mathbb{P}^1_R$ defined by $X_1 x_0 = X_0 x_1, Y_0 x_0 = Y_1 x_1$, covered by the following two affine open subschemes:

$$\mathrm{Spec}(W[X_0, Y_0, (x_0/x_1)]/(X_0(x_0/x_1)Y_1 - p),$$

$$\mathrm{Spec}(W[X_1, Y_0, (x_1/x_0)]/(X_1(x_1/x_0)Y_0 - p).$$

# Computing the cocharacter group

To get a semistable model, we thus need to blow up a product of $g$ double points: $\prod_i W(\mathbb{F})[X_i, Y_i]/(X_iY_i - p)$.

We then obtain a complete description of the components $Z^{(g)}, Z^{(g+1)}$ of dimension 1 and dimension 0 that are involved in the definition of the cocharacter group. Up to $\mathbb{P}^1$'s, the computations only involve the 0- and 1-dimensional strata previously described:

-$Z^{(g)}$: we get the $2g$ strata of $S_1$ plus a $\mathbb{P}^1$-fiber bundle over $S_0$;

-$Z^{(g+1)}$: we get $g$ copies of $S_0$.

After identification (with the zero-dimensional strata of the Shimura curves) and a little computation, we thus obtain:

$$0 \longrightarrow \oplus_{i=1}^{g} \widehat{X}_{\mathfrak{p}}(\mathfrak{p}q/\mathfrak{q}_i)^2 \longrightarrow \widehat{X}_{\mathfrak{p}}(\mathfrak{p}q) \longrightarrow \widehat{Y}_q(\mathfrak{p}) \longrightarrow 0.$$

## Other examples: unitary groups

- K. Fujiwara (ICM 2006) expounded a geometric Jacquet-Langlands correspondence for the Shimura varieties studied by Harris and Taylor, in terms of Hida varieties (i.e., dimension zero varieties).

- D. Helm described a similar result in detail for compact Shimura surfaces for inner forms $G, G'$ of the group $U(2)$ (but such that $G(\mathbb{A}_f) \cong G'(\mathbb{A}_f)$), and has promising computations for $U(n)$. His PEL-type $U(2)$-surfaces are very similar to quaternionic Shimura surfaces.

## What is all this good for?

For applications, this sort of computation (for various Shimura varieties) is only the first step. We need to tackle torsion by switching the sheaf from $\mathbb{Q}_\ell$ to $\mathbb{Z}_\ell$, and then the real work begins, with *possible* applications to:

- $\ell$-adic Jacquet-Langlands correspondences; $\ell$-adic interpolation à la Emerton;

- stronger level optimization results...