ON THE EQUIVARIANT TAMAGAWA NUMBER CONJECTURE FOR HECKE CHARACTERS OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. In this article we consider the equivariant Tamagawa number conjecture for Hecke characters over imaginary quadratic fields. Assuming weak Leopoldt conjecture and the equivariant main conjecture for imaginary quadratic fields, we give a proof of a weak version of the equivariant Tamagawa number conjecture for Hecke characters.

1. INTRODUCTION

In this paper we prove a weak version of the equivariant Tamagawa number conjecture for Hecke characters over imaginary quadratic fields under certain assumptions. The Tamagawa number conjecture by Bloch and Kato [3] describes the special values of L-functions of pure Chow motives over number fields in terms of regulator (or Chern class) maps of motivic cohomologies into Deligne's and étale cohomologies. Fontaine and Perrin-Riou [10] reformulated the Tamagawa number conjecture using the determinant functor. This conjecture is a generalization of Dirichlet's class number formulas and Birch and Swinnerton-Dyer conjecture. Moreover Burns and Flach [4] formulated the equivariant version of Tamagawa number conjecture for a Chow motives over number fields with the action of a semisimple finite dimensional $\mathbb{Q}$-algebra. For the (equivariant) Tamagawa number conjecture there are some results. Burns and Greither [5] proved the ℓ-part of ETNC for all abelian extension of $\mathbb{Q}$ for the L-values at any integer points and all odd primes ℓ. A survey paper by Flach [8] gives a proof of ETNC for all abelian fields including the case of $\ell = 2$. Huber and Kings gave independently a proof of a weaker version of the same case. Recently Bley [2] and Johnson [11] considered the case of abelian extension of imaginary quadratic fields. For another case, Kings proves a weak varsion of TNC for CM elliptic curves in non-critical cases, and Bars [1] extended this result to Hecke characters with higher weights. In this paper we consider the equivariant version of their results.

2. DETERMINANT FUNCTOR

In this section we review some facts on the determinant functor of Knudsen and Mumford [15]. For any commutative ring R, let $P(R)$ denote the category of finitely generated projective R-modules and $(P(R), is)$ its subcategory of isomorphisms. A graded invertible R-module is a pair (L, α) consisting of an invertible (that is, projective rank 1) R-module L and a local constant function $\alpha : \mathrm{Spec}\,(R) \to \mathbb{Z}$.

A homomorphism $h : (L, \alpha) \to (M, \beta)$ of graded invertible modules is a homomorphism of R-modules $h : L \to M$ such that the localization $h_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$ for which $\alpha(\mathfrak{p}) \neq \beta(\mathfrak{p})$. Let $Inv(R)$ denote the category of graded invertible modules and isomorphisms. Then the category $Inv(R)$ is a symmetric monoidal category with tensor product

$$(L,\alpha)\otimes(M,\beta)=(L\otimes_R M,\alpha+\beta),$$

the associativity constraint, the unit object $(R,0)$, and the commutativity constraint

$$(L,\alpha)\otimes(M,\beta)\cong(M,\beta)\otimes(L,\alpha).$$

We define

$$(L,\alpha)^{-1}=(\mathrm{Hom}(L,R),-\alpha).$$

For a finitely generated projective R-module P we define

$$\mathrm{Det}_R P=\left(\bigwedge_R^{\mathrm{rank}_R P} P,\mathrm{rank}_R P\right).$$

This is a graded invertible R-module, so Det_R gives a functor $(P(R),is)\to Inv(R)$. For a bounded complex of finitely generated R-modules $P^\bullet$ we define

$$\mathrm{Det}_R(P^\bullet)=\bigotimes_{i\in\mathbb{Z}}\mathrm{Det}_R^{(-1)^{i+1}}(P^i).$$

Furthermore we set $\mathrm{Det}_R^{-1}(P^\bullet)=\mathrm{Det}_R(P^\bullet)^{-1}$.

We write $\mathcal{D}(R)$ for the derived category of the homotopy category of bounded complexes of R-modules, and $\mathcal{D}^p(R)$ for the full triangulated subcategory of perfect complexes of R-modules. Let $\mathcal{D}^{pis}(R)$ be the subcategory of $\mathcal{D}^p(R)$ in which the objects are the same but the morphisms are restricted to quasi-isomorphisms. We assume that R is reduced. Then the functor Det_R can be extended to a functor $\mathcal{D}^{pis}(R)\to Inv(R)$ in such way that for every distinguished triangle $C_1\to C_2\to C_3$ in $\mathcal{D}^{pis}(R)$ there is an isomorphism in $Inv(R)$

$$(\mathrm{Det}_R C_1)^{-1}\otimes\mathrm{Det}_R C_2\xrightarrow{\cong}\mathrm{Det}_R C_3$$

which is functorial in the triangle.

For R-module X, if $X[-1]$ belong to $\mathcal{D}^p(R)$, then we say that X is perfect. And we define $\mathrm{Det}_R(X):=\mathrm{Det}_R(X[-1])$ for any such X. If a complex C is bounded and each cohomology module is perfect, then C belongs to $\mathcal{D}^p(R)$ and there is a canonical isomorphism

$$\mathrm{Det}_R C\xrightarrow{\cong}\bigotimes_{i\in\mathbb{Z}}\mathrm{Det}_R^{(-1)^{i+1}}(H^i(C)).$$

If C is acyclic, then there is a canonical isomorphism

$$\mathrm{Det}_R C\xrightarrow{\cong}(R,0).$$

Let G be any finite abelian group. For any commutative ring Z we write $x\mapsto x^\#$ for the Z-linear involution of the group ring $Z[G]$ which satisfies $g^\#=g^{-1}$ for each $g\in G$. If X is any complex of $Z[G]$-modules, then we write $X^\#$ for the scalar extension with respect to the morphism $x\mapsto x^\#$.

For any finitely generated projective $Z[G]$-module X (resp. object X of $\mathcal{D}^p(Z[G])$), we set $X^*:=\mathrm{Hom}_Z(X,Z)$ (resp. $X^*:=R\mathrm{Hom}_Z(X,Z)$), which we regard as endowed with the contragradient G-action. We see that if X is a finite generated projective

$Z[G]$-module (resp. object of $\mathcal{D}^p(Z[X])$), then also X^* is. We recall that for any $Z[G]$-module X one has a canonical isomorphism $X^* \cong \mathrm{Hom}_{Z[G]}(X, Z[G])^{\#}$, and that this induces that for each object X of $\mathcal{D}^p(Z[G])$ a canonical isomorphism in $Inv(Z[G])$

$$\mathrm{Det}_{Z[G]} X^* \cong \mathrm{Det}^{-1}_{Z[G]}(X^{\#}).$$

3. Equivariant Tamagawa number conjecture for Hecke characters

Let K be an imaginary quadratic fields. Denote by $\mathcal{O}_K$ the integer ring of K. Let E be an elliptic curve over K with complex multiplication by $\mathcal{O}_K$ (i.e. $\mathrm{End}_K(E) \cong \mathcal{O}_K$). We fix an abelian extension F/K. Then there is an ideal $\mathfrak{m}$ of $\mathcal{O}_K$ such that $F \subset K(\mathfrak{m})$, where $K(\mathfrak{m})$ is the ray class field mod $\mathfrak{m}$. We denote

$$\begin{aligned} \mathfrak{A} &:= \mathcal{O}_K[\mathrm{Gal}(F/K)] \\ A &:= K[\mathrm{Gal}(F/K)]. \end{aligned}$$

Also we write

$$E' := E \times_K \mathrm{Spec}\, F$$

for the base change of E and

$$\mathcal{A} := \mathrm{Res}_{F/K}\, E'$$

for the Weil restriction of E' to K. Then the abelian variety $\mathcal{A}$ has complex multiplication by A. Set $T = \mathrm{End}(\mathcal{A}) \otimes \mathbb{Q} \cong A$. Let $\varphi_{\mathcal{A}} : \mathbb{A}_K^\times \to A^\times$ be the Serre-Tate character associated to the abelian variety $\mathcal{A}$. Moreover we denote by $M_1 = \mathrm{Res}_{F/K}\, h^1(E')$ the Grothendieck restriction of the motive $h^1(E')$. This motive has multiplication by A. We set $M_w = M_1 \otimes \cdots \otimes M_1$($w$-times) and $T_w = T \otimes \cdots \otimes T$($w$-times). Then we have a decomposition

$$T_w = \prod_{\Theta} T_\Theta,$$

where Θ runs over the $\mathrm{Aut}\,(\mathbb{C})$-orbits of $\mathrm{Hom}(T_w, \mathbb{C})$. T_Θ is a field. For $\theta = (\lambda_1, \ldots, \lambda_w) \in \mathrm{Hom}(T_w, \mathbb{C})$, denote $T_\theta := \theta(T_w)$. Writing e_Θ for the projector in $\mathrm{Hom}(T_w, \mathbb{C})$ corresponding to T_Θ. Because e_Θ is an element in $T_w \subset \mathrm{End}(M_w)$, we can define the motive $M_\Theta := e_\Theta M_w$ as a Chow motive over K. Also we define the CM character $\varphi_\Theta : \mathbb{A}_K^\times \to T_\Theta^\times$ by $\varphi_\Theta = e_\Theta(\varphi_{\mathcal{A}} \otimes \cdots \otimes \varphi_{\mathcal{A}})$ and the Hecke character $\varphi_\theta : \mathbb{A}_K^\times / K^\times \to T_\theta^\times$ by $\varphi_\theta = \lambda_1(\varphi_{\mathcal{A}}) \cdots \lambda_w(\varphi_{\mathcal{A}})$. We denote by $\mathfrak{f} = \mathfrak{f}_\Theta$ the conductor of φ_Θ.

Proposition 3.1 (cf. Deninger [6][Proposition 1.3.1]). *For a Dirichlet character $\chi : \mathrm{Gal}(F/K) \to \mathbb{C}^\times$ and the grössencharacter ψ_E associated with the elliptic curve E, the Hecke character $\varphi = \chi \cdot \psi_E^a \cdot \overline{\psi}_E^b\, (a, b \geq 0, a + b = w \geq 1)$ has the form φ_θ for suitable $\theta \in T_w$.*

Let E be an elliptic curve over K with complex multiplication by O_K. Note that this situation implies that the class number of K is one. We choose a positive integer w and an $\mathrm{Aut}\,(\mathbb{C})$-orbit Θ in $\mathrm{Hom}(\otimes_{i=1}^w \mathrm{End}(E), \mathbb{C})$. Denote the infinite type of φ_Θ by (a, b). We may assume that an ideal $\mathfrak{m}$ of K satisfies $F \subset K(\mathfrak{m})$ and $\mathfrak{f}_\Theta | \mathfrak{m}$. Then we consider the motive $M_\Theta := e_\Theta(h^1(E) \otimes \cdots \otimes h^1(E))$ and $M'_\Theta := M_\Theta \otimes h^0(\mathrm{Spec}\, F)$ the base change of M_Θ to F. The A-equivariant L-function of the motive $M = M'_\Theta(r)$ $(r \in \mathbb{Z})$ is defined by

$$L({}_AM, s) := \prod_{\mathfrak{p}} \mathrm{Det}_A(1 - \mathrm{Frob}_{\mathfrak{p}}^{-1} \cdot N\mathfrak{p}^{-s} | M_\ell'^{I_\mathfrak{p}})^{-1},$$

where M'_ℓ is the ℓ-adic realization of M and $I_\mathfrak{p}$ is the inertia group at $\mathfrak{p}$. We are concerning the leading term of the Taylor expansion at $s = 0$, denoted by $L^*({}_AM)$. Let r be an integer satisfying that $-r \leq \min(a, b)$ if $a \neq b$ and $-r < a = b = w/2$ if $a = b$. Then the motive M is non-critical. We denote the motivic cohomology by

$$H^1_f(M) := H^{w+1}_{\mathcal{M}}(M'_\Theta, \mathbb{Q}(r + w + 1))$$

Deninger [6] proved that there is a constructible subspace of the motivic cohomology $H^1_f(M)^{\mathrm{constr}}$ such that the Beilinson's regulator map gives an A-equivariant isomorphism

$$\rho_\infty : H^1_f(M)^{\mathrm{constr}} \otimes \mathbb{R} \xrightarrow{\cong} (H_B(M) \otimes \mathbb{R})^+,$$

where $H_B(M)^+ = e_\Theta H^w_B(\otimes^w_{i=1} h^1(E')\otimes_K \mathbb{C}, \mathbb{Q}(w + r))$ is the Betti realization of M. Thus, defining

$$\Xi({}_AM) := \mathrm{Det}_A(H^1_f(M)^{\mathrm{constr}})^* \otimes_A \mathrm{Det}^{-1}_A(H_B(M)^+)^*,$$

we obtain an isomorphism

$${}_A\vartheta_\infty : A\otimes_{\mathbb{Q}}\mathbb{R} \to \Xi({}_AM)\otimes_{\mathbb{Q}}\mathbb{R}.$$

Fix a prime ℓ and put $A_\ell := A\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. Choose a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable projective $\mathcal{O}_K$-lattice

$$T_\ell := e_\Theta H^w_{\text{ét}}(\otimes^w_{i=1} h^1(E)\otimes_K \overline{K}, \mathbb{Z}_\ell) \subset M_\ell := e_\Theta H^w_{\text{ét}}(\otimes^w_{i=1} h^1(E)\otimes_K \overline{K}, \mathbb{Q}_\ell)$$

and a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable projective $\mathfrak{A}$-lattice

$$T'_\ell := e_\Theta H^w_{\text{ét}}(\otimes^w_{i=1} h^1(E')\otimes_K \overline{K}, \mathbb{Z}_\ell) \subset M'_\ell := e_\Theta H^w_{\text{ét}}(\otimes^w_{i=1} h^1(E')\otimes_K \overline{K}, \mathbb{Q}_\ell).$$

For the formulation of equivariant Tamagawa number conjecture, we need to assume the weak Leopoldt conjecture

$$H^2(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(w + r + 1)) \otimes \mathbb{Q}_\ell = 0.$$

Under this assumption, the ℓ-adic regulator map

$$\rho_{\text{ét},\ell} : H^1_f(M)^{\mathrm{constr}} \otimes \mathbb{Q}_\ell \to H^1(\mathcal{O}_K[1/\mathfrak{m}\ell], M'_\ell(w + r + 1))$$

induces an isomorphism

$${}_A\vartheta_\ell : \Xi({}_AM)\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \xrightarrow{\cong} \mathrm{Det}_{A_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M'_\ell).$$

Conjecture 3.2 (A weak version of equivariant Tamagawa number conjecture)**.** *For every prime ℓ,*

$$\mathfrak{A}_\ell \cdot {}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM)^{-1}) = \mathrm{Det}_{\mathfrak{A}_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell)$$

inside of $\mathrm{Det}_{A_\ell} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M'_\ell)$.

For this conjecture, we have the following result.

Theorem 3.3. *Let r be an integer satisfying that $-r \leq \min(a, b)$ if $a \neq b$ and $-r < a = b = w/2$ if $a = b$ and ℓ an odd prime. Assume the weak Leopoldt conjecture. Then the ℓ-part of the weak version of equivariant Tamagawa number conjecture holds for the motive $M = M'_\Theta(r)$ and the order $\mathfrak{A} = \mathcal{O}_K[\mathrm{Gal}(F/K)]$.*

Remark 3.4. 1. By a functoriality of equivariant Tamagawa number conjecture, it suffices to give a proof for the cases that $F = K(\mathfrak{m})$. So we will consider only the case of $F = K(\mathfrak{m})$.
2. Since $\psi_E^r(\mathfrak{p})\overline{\psi_E}^r(\mathfrak{p}) = N\mathfrak{p}^r$, it is enough to prove this conjecture for $r = 0$ by using $(a+r, b+r)$ instead of (a, b).
3. Using computations analogous to [8][Theorem 5.2] one can show that the equivariant main conjecture for imaginary quadratic fields for split primes ℓ which is implied by Rubin's work [17][Theorem 4.1].
4. One can show that the weak Leopoldt conjecture for almost all r using Rubin's result.

4. Computations on the regulator map

Let K be an imaginary quadratic field of class number one and E an elliptic curve over K with complex multiplication by $\mathcal{O}_K$. We denote $G_\mathfrak{m} = \mathrm{Gal}(K(\mathfrak{m})/K)$ for an ideal $\mathfrak{m}$ of O_K and $E' = E \times_K \mathrm{Spec}\, K(\mathfrak{m})$ for the base change of E. For χ a rational character of $G_\mathfrak{m}$ we consider a Hecke character

$$\varphi_{\theta_\chi} = \chi\psi_E^a\overline{\psi_E}^b$$

with conductor $\mathfrak{f}$, weight $w = a + b$, where a, b are non negative integers and ψ_E is the grössencharacter associated with E. We will denote the $\mathrm{Aut}\,(\mathbb{C})$-orbit of θ_χ by Θ_χ. We fix an isomorphism

$$\theta_{E'} : \mathcal{O}_K \xrightarrow{\cong} \mathrm{End}_{K(\mathfrak{m})}(E')$$

such that $\theta_{E'}^*(\alpha)\omega = \alpha\omega$ for any $\omega \in H^0(E, \Omega^1_{E'})$ and an embedding τ_0 of $K(\mathfrak{m})$ in $\mathbb{C}$ such that $j(E') = j(\mathcal{O}_K)$. Then $E' \cong \mathbb{C}/\Gamma$ where $\Gamma = \Omega\mathcal{O}_K$ for some $\Omega \in \mathbb{C}$. We let $\rho_\mathfrak{f} \in \mathbb{A}_K^\times$ be an idèle with ideal $\mathfrak{f}$ and choose $f_\mathfrak{f} \in K^\times$ with $v_\mathfrak{p}(f_\mathfrak{f}) \leq 0$ if $\mathfrak{p} \nmid \mathfrak{f}$ and $v_\mathfrak{p}(f_\mathfrak{f}^{-1} - (\rho_\mathfrak{f})_\mathfrak{p}^{-1}) \geq 0$ if $\mathfrak{p} \mid \mathfrak{f}$. By the theory of complex multiplication, there exists an unique $f \in K^\times$ satisfying that the following two properties:

1. $f\mathcal{O}_K = (\rho_\mathfrak{f})$.
2. For any fractional ideal $\mathfrak{a} \subset K$ and any analytic isomorphism $\lambda : \mathbb{C}/\mathfrak{a} \to E(\mathbb{C})$, the following diagram commutes:

$$\begin{tikzcd} K/\mathfrak{a} \arrow[r, "f\cdot\rho_\mathfrak{f}^{-1}"] \arrow[d, "\lambda"] & K/\mathfrak{a} \arrow[d, "\lambda"] \\ E(K^{\mathrm{ab}}) \arrow[r, "\left(\frac{K^{\mathrm{ab}}/K}{\rho_\mathfrak{f}}\right)"] & E(K^{\mathrm{ab}}), \end{tikzcd}$$

where K^{ab} is the maximal abelian extension of K.

Then f satisfies the condition of $f_\mathfrak{f}$, so we put $f_\mathfrak{f} = f$.

Then we define a divisor

$$_\mathfrak{f}\beta'' := ([\Omega f_\mathfrak{f}^{-1}]) \in E'[\mathfrak{f}](K(\mathfrak{m}))$$

and

$$ {}_{\mathfrak{f}}\beta := {}_{\mathfrak{f}}\beta'' - (\deg_{\mathfrak{f}}\beta'')(O) + \frac{\deg_{\mathfrak{f}}\beta''}{N^2}(1-N^{-w-2})^{-1}\left(N^2(O) - \sum_{P\in E'[N\mathfrak{f}]}(P)\right) $$

in $\mathbb{Q}[E'[\mathfrak{f}]\backslash O]$ for $N \geq 2$.Deninger constructed the motivic elements in motivic cohomology using the Eisenstein symbol

$$ \mathcal{E}_{\mathcal{M}} : \mathbb{Q}[E'[N\mathfrak{f}]\backslash O]^0 \to H^{w+1}_{\mathcal{M}}(\mathrm{Sym}^w h^1(E'), \mathbb{Q}(w+1)) $$

and the Kronecker map

$$ \mathcal{K}_{\mathcal{M}} : H^{w+1}_{\mathcal{M}}(\mathrm{Sym}^w h^1(E'), \mathbb{Q}(w+1)) \to H^{w+1}_{\mathcal{M}}(M_{\Theta_\chi}, \mathbb{Q}(w+1)). $$

Let

$$ \rho_\infty : H^{w+1}_{\mathcal{M}}(M_{\Theta_\chi}, \mathbb{Q}(w+1)) \otimes \mathbb{R} \to H^w_B(M_{\Theta_\chi}, \mathbb{R}(w)) $$

be the Beilinson's regulator map.

Theorem 4.1 (Deninger, [6][(2.11)]). *Setting $\beta = {}_{\mathfrak{f}}\beta''$ and $P = E[\mathfrak{f}]$ if $a \not\equiv b \mod |O_K^\times|$, $\beta = {}_{\mathfrak{f}}\beta$ and $P = E[N\mathfrak{f}]$ if $a \equiv b \mod |O_K^\times|$. Then*

$$ \rho_\infty(\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}(\beta)) = t_{\Theta_\chi} L'(\overline{\varphi_{\Theta_\chi}}, 0) e_{\Theta_\chi}\eta, $$

where

$$ t_{\Theta_\chi} = (-1)\frac{2^{-1}|P(\mathbb{C})|^w \Phi(\mathfrak{m})}{w! N\mathfrak{f}^w \Phi(\mathfrak{f})}\varphi_{\Theta_\chi}(\rho_{\mathfrak{f}}), $$

$\Phi(\mathfrak{m}) =| (\mathcal{O}_K/\mathfrak{m})^\times |$ and η is a A-basis of $H_B(M)^+$.

For $\mathfrak{f} \mid \mathfrak{m}$ we define the element

$$ \xi_{\mathfrak{f}} := \mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}(\beta) $$

in the motivic cohomology $H^{w+1}_{\mathcal{M}}(M_{\Theta_\chi}, \mathbb{Q}(w+1))$. From the assumption that K has class number one, $\chi(\rho_{\mathfrak{f}}) = 1$. Also one can show that $\psi_E(\rho_{\mathfrak{f}}) = f$, so we have $\varphi_{\Theta_\chi}(\rho_{\mathfrak{f}}) = f^a\overline{f}^b$. Therefore by the Deninger's theorem, we have the following formula.

$$ \rho_\infty(\xi_{\mathfrak{f}}) = (-1)\frac{2^{-1} \mid P(\mathbb{C})|^w \Phi(\mathfrak{m}) f^a \overline{f}^b}{w!\Phi(\mathfrak{f})N\mathfrak{f}^w} L'(\overline{\varphi_{\Theta_\chi}}, 0) e_{\Theta_\chi}\eta. $$

5. Elliptic units

We will use elliptic units to describe the image of L-values under the ℓ-adic regulator map. Here we give a short review of elliptic units. For details, see de Shalit's book [7][Chapter 2]. Let $L = \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ be a lattice in $\mathbb{C}$ with a basis (ω_1, ω_2) satisfying $\mathrm{Im}(\omega_1/\omega_2) > 0$. The Dedekind eta function is defined by

$$ \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^\infty (1 - q_\tau^n); \quad q_\tau := e^{2\pi i \tau} $$

and we put

$$ \eta^{(2)}(\omega_1, \omega_2) = \omega_2^{-1} 2\pi i \eta(\omega_1/\omega_2)^2. $$

This function depends on the choice of basis but the discriminant function

$$ \Delta(L) = \Delta(\tau) = (2\pi i)^{12}\eta(\tau)^{24} $$

does not. Define a theta function

$$\varphi(z,\tau)=e^{\pi iz\frac{z-\bar{z}}{\tau-\bar{\tau}}}q_\tau^{1/12}(q_z^{1/2}-q_z^{-1/2})\prod_{n=1}^{\infty}(1-q_zq_\tau^n)(1-q_z^{-1}q_\tau^n)$$

where $q_z=e^{2\pi iz}$ and

$$\varphi(z;\omega_1,\omega_2)=\varphi(z/\omega_1,\omega_1/\omega_2).$$

For any pair of lattices $L\subseteq L'$ of index prime to 6 with bases $\omega=(\omega_1,\omega_2)$ and $\omega'=(\omega_1',\omega_2')$ satisfying $\mathrm{Im}(\omega_1/\omega_2)>0$ and $\mathrm{Im}(\omega_1'/\omega_2')>0$, it is shown by Robert in [16][Theorems 1,2] that there exists a unique choice of 12-th root of unity $C(\omega,\omega')$ so that the functions

$$\delta(L,L'):=C(\omega,\omega')\eta^{(2)}(\omega)^{[L':L]}/\eta^{(2)}(\omega')$$

and

$$\psi(z,L,L'):=\delta(L,L')\prod_{u\in T}(\wp(z;L)-\wp(u;L))^{-1}$$

only depends on the lattices L,L', where the set T is any set of representatives of $(L'\backslash\{O\})/(\pm1\ltimes L)$ and $\wp$ is the Weierstrass $\wp$-function associated to L, moreover ψ satisfies the distribution relation

$$\psi(z;K,K')=\prod_{i=1}^{[L:K]}\psi(z+t_i;L,L')$$

for any lattice $L\subseteq K$ so that $K\cap L'=L$ and where $K'=K+L'$ and $t_i\in K$ are a set of representatives of K/L. Then $\psi(z,L,L')$ is an elliptic function on the elliptic curve $E=\mathbb{C}/L$ with divisor $[L':L](O)-\sum_{P\in L'/L}(P)$. Kato reproved Robert's result in a scheme theoretic context as following.

Lemma 5.1 (Kato [13] [Proposition 1.3])**.** *Let E/S be an elliptic curve over a base scheme S and $c:E\to\tilde{E}$ an S-isogeny of degree prime to* 6*. Then there is a unique function*

$${}_c\Theta_{E/S}\in\Gamma(E\backslash\ker(c),\mathcal{O}^\times)$$

satisfying that

1. $\mathrm{div}({}_c\Theta_{E/S})=\deg(c)(0)-\sum_{P\in\ker(c)}(P)$.
2. *For any morphism $g:S'\to S$ we have $g_E^*({}_c\Theta_{E/S})={}_{c'}\Theta_{E'/S}$, where $g_E:E':=E\times_SS'\to E$ and c' is the base change of c.*
3. *For any S-isogeny $b:E\to E'$ of degree prime to* $\deg(c)$ *have $b_*({}_c\Theta_{E/S})={}_{c'}\Theta_{E'/S}$ where b_* is the norm map associated to the finite flat morphism $E\backslash\ker(c)\to E'\backslash\ker(c')$. Here c' is the isogeny $E'\to E'\backslash b(\ker(c))$.*
4. *For $S=\mathrm{Spec}\,\mathbb{C},E=\mathbb{C}/L$ and $c:\mathbb{C}/L\to\mathbb{C}/\tilde{L}$ for lattices $L\subseteq\tilde{L}$ we have*

$${}_c\Theta_{E/S}(z)=\psi(z,L,\tilde{L}).$$

Let K be an imaginary quadratic field. For any integral ideal $\mathfrak{f}\neq1$ and any (auxiliary) $\mathfrak{a}$ which is prime to $6\mathfrak{f}$ we define the elliptic unit by

$${}_\mathfrak{a}z_\mathfrak{f}=\psi(z,\mathfrak{f},\mathfrak{a}^{-1}\mathfrak{f})$$

and for $\mathfrak{f}=1$ we define a family of elements indexed by all ideals $\mathfrak{a}$ of K by

$$u(\mathfrak{a})=\frac{\Delta(\mathcal{O}_K)}{\Delta(\mathfrak{a})}.$$

Lemma 5.2 (For the proof see [7][Chapter II.2]). *The complex numbers* ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ *and* $u(\mathfrak{a})$ *satisfy the following properties:*

1. (Rationality) ${}_{\mathfrak{a}}z_{\mathfrak{f}} \in K(\mathfrak{f})$, $u(\mathfrak{a}) \in K(1)$.
2. (Integrality)
$$ {}_{\mathfrak{a}}z_{\mathfrak{f}} \in \begin{cases} \mathcal{O}^{\times}_{K(\mathfrak{f})} & \mathfrak{f} \text{ divisible by primes } \mathfrak{p} \neq \mathfrak{q} \\ \mathcal{O}^{\times}_{K(\mathfrak{f}),\{v|\mathfrak{f}\}} & \mathfrak{f} = \mathfrak{p}^n \text{ for some prime } \mathfrak{p}. \end{cases} $$
and
$$ u(\mathfrak{a}) \cdot \mathcal{O}_{K(1)} = \mathfrak{a}^{-12}\mathcal{O}_{K(1)}. $$
3. (Galois action) *For* $(\mathfrak{c}, \mathfrak{f}\mathfrak{a}) = 1$ *with Artin symbol* $\sigma_{\mathfrak{c}} \in \mathrm{Gal}(K(\mathfrak{f})/K)$ *we have*
$$ {}_{\mathfrak{a}}z_{\mathfrak{f}}^{\sigma_{\mathfrak{c}}} = \psi(1; \mathfrak{c}^{-1}\mathfrak{f}, \mathfrak{c}^{-1}\mathfrak{a}^{-1}\mathfrak{f}), \quad u(\mathfrak{a})^{\sigma_{\mathfrak{c}}} = u(\mathfrak{a}\mathfrak{c})/u(\mathfrak{c}). $$
4. (Norm compatibility) *For a prime ideal* $\mathfrak{p}$ *one has*
$$ N_{K(\mathfrak{p}\mathfrak{f})/K(\mathfrak{f})}({}_{\mathfrak{a}}z_{\mathfrak{p}\mathfrak{f}})^{w_{\mathfrak{f}}/w_{\mathfrak{p}\mathfrak{f}}} = \begin{cases} {}_{\mathfrak{a}}z_{\mathfrak{f}} & \mathfrak{p} \mid \mathfrak{f} \neq 1 \\ {}_{\mathfrak{a}}z_{\mathfrak{f}}^{1-\sigma_{\mathfrak{p}}^{-1}} & \mathfrak{p} \nmid \mathfrak{f} \neq 1 \\ u(\mathfrak{p})^{\sigma_{\mathfrak{p}} - N\mathfrak{a}/12} & \mathfrak{f} = 1. \end{cases} $$

Remark 5.3.
1. The relations in 2. show the auxiliary nature of $\mathfrak{a}$. In the group ring $\mathbb{Q}[G_{\mathfrak{f}}]$ the element $N\mathfrak{a} - \sigma_{\mathfrak{a}}$ becomes invertible and
$$ z_{\mathfrak{f}} = (N\mathfrak{a} - \sigma_{\mathfrak{a}})^{-1}{}_{\mathfrak{a}}z_{\mathfrak{f}} \in \mathcal{O}^{\times}_{K(\mathfrak{f}),\{v|\mathfrak{f}\}} \otimes_{\mathbb{Z}} \mathbb{Q} $$
is independent of choice of $\mathfrak{a}$.
2. The Galois action in 3. together with the relation
$$ \psi(\lambda z; \lambda L, \lambda L') = \psi(z, L, L') $$
for any $\lambda \in \mathbb{C}^{\times}$ show that the Galois conjugates of ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ are the numbers ${}_{\mathfrak{a}}\Theta_{E/\mathbb{C}}(\alpha)$ where (E, α) runs through all isomorphism classes of pairs with $E \cong \mathbb{C}/L$ an elliptic curve with CM by $\mathcal{O}_K$, $\mathfrak{a} : \mathbb{C}/L \to \mathbb{C}/\mathfrak{a}^{-1}L$ and $\alpha \in E(\mathbb{C})$ a primitive $\mathfrak{f}$-torsion point. In fact ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ is the value of ${}_{\mathfrak{a}}\Theta_{\mathcal{E}/K(1)}$ at a single closed point with residue field $K(\mathfrak{f})$ on an elliptic curve $\mathcal{E}/K(1)$.

6. Computations on the ℓ-adic regulator map

In this section, we compute the image of the element $\xi_{\mathfrak{f}}$ by ℓ-adic regulator map. From now on, we fix a prime $\mathfrak{l}$ of $\mathcal{O}_K$ dividing ℓ. Write $K_\ell = K \otimes \mathbb{Q}_\ell = \prod_{\mathfrak{l}|\ell} K_{\mathfrak{l}}$. In the last section, we choose an ideal $\mathfrak{m}$ of $\mathcal{O}_K$. Now we set
$$ \mathfrak{m} = \mathfrak{m}_0\mathfrak{l}^m $$
with $\mathfrak{l} \nmid \mathfrak{m}_0$. Since $M = M_\Theta \otimes_K h^0(\mathrm{Spec}\, F)$ is a direct summand of $M' = M \otimes_F h^0(\mathrm{Spec}\, F')$ for any extension F'/F, and we have $M' \otimes_{A'} A \cong M$ if F'/K is also abelian, we may always replace F by such a larger extension F'/F and prove the equivariant Tamagawa number conjecture for M with the functriality of this conjecture. By the following lemma according to Flach, we can assume $E/K(\mathfrak{m})$ has good reduction at all primes dividing $\mathfrak{l}$.

Lemma 6.1 (Flach [9][Lemma 4.2]). *After possibly replacing* $\mathfrak{m}$ *by a multiple we may assume that* $F = K(\mathfrak{m})$ *and that in addition* E *is defined over* $K(\mathfrak{m}_0)$ *and has good reduction at primes dividing* $\mathfrak{l}$.

Now we give a brief review of result of Kings [14]. Let E be an elliptic curve over a base scheme T. Denote by $\pi : E \to T$ the structural morphism. Put $U = E\backslash e$, where e is the zero section. Let $Pol_{\mathbb{Q}_\ell}$ be the elliptic polylogarithm sheaf on U (lisse pro-sheaf on U). For any divisor

$$\beta = \sum_{t\in E[N](T)\backslash e} n_t t \in \mathbb{Q}[E[N](T)\backslash e]$$

we define the ℓ-adic Eisenstein class associated to β by

$$(\beta^* Pol_{\mathbb{Q}_\ell})^m := \sum_{t\in E[N]\backslash e} n_t(\sigma^k \mathrm{pr}_t t^* Pol_{\mathbb{Q}_\ell}) \in H^1(T, \mathrm{Sym}^m \mathcal{H}_{\mathbb{Q}_\ell}(1)),$$

where $\mathcal{H}_{\mathbb{Q}_\ell} := \underline{\mathrm{Hom}}_T(R^1\pi_*\mathbb{Q}_\ell, \mathbb{Q}_\ell)$ and σ^m, pr_t are suitable projections. For an ideal $\mathfrak{a}$ with $(N\mathfrak{a}, \ell N) = 1$, consider $[\mathfrak{a}] : E \to E$ any isogeny of degree $N\mathfrak{a}$. Kings gave a explicit description of the ℓ-adic Eisenstein class associated β using the elliptic units.

Theorem 6.2 (Kings [14][Theorem 4.2.9])**.** *Let the notation as above. For any $m > 0$, the ℓ-adic Eisenstein class*

$$N\mathfrak{a}([\mathfrak{a}]^{\otimes m} N\mathfrak{a} - 1)(\beta^* Pol_{\mathbb{Q}_\ell})^m)$$

is given by

$$\pm\frac{1}{m!}(\delta \sum_{t\in E[N](T)\backslash e} n_t \sum_{[\ell^n]t_n = t} {}_{\mathfrak{a}}\Theta_E(-t_n)\tilde{t_n}^{\otimes m})_n$$

in the cohomology group $H^1(T, \mathrm{Sym}^m\mathcal{H}_{\mathbb{Q}_\ell}(1))$, where δ is the boundary map, $\tilde{t_n}$ is the projection of t_n to $E[\ell^n]$.

The following result gives the relations between the image of $\mathcal{E}_{\mathcal{M}}(\beta)$ by ℓ-adic regulator map $\rho_{\text{ét},\ell}$ and ℓ-adic Eisenstein class associated to β.

Theorem 6.3 (Kings [14][Theorem 1.2.5])**.** *For a divisor*

$$\beta = \sum_{t\in E[N](T)\backslash e} n_t t \in \mathbb{Q}[E[N](T)\backslash e],$$

we have that

$$\rho_{\text{ét},\ell}(\mathcal{E}_{\mathcal{M}}(\beta)) = -N^{2m}(\beta^* Pol_{\mathbb{Q}_\ell})^m.$$

Corollary 6.4. *For a divisor as in Theorem 6.3, we have that*

$$e_{\Theta_\chi}\rho_{\text{ét},\ell}(\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}(\beta)) = \pm N^{2w}\frac{1}{w! N\mathfrak{a}([\mathfrak{a}]N\mathfrak{a} - 1)} \times \delta\left(\sum_{t\in E[N]\backslash e} n_t \sum_{[\ell^n]t_n = t} {}_{\mathfrak{a}}\Theta_E(-t_n) \otimes e_{\Theta_\chi}(\tilde{t_n}^{\otimes w})\right)_n.$$

We will apply these results for the case $\beta = {}_{\mathfrak{f}}\beta$ or ${}_{\mathfrak{f}}\beta''$, $N = |P(\mathbb{C})|$, $m = w$, $T = \operatorname{Spec}\mathcal{O}_K\left[\frac{1}{\mathfrak{m}\ell}\right]$ and $\mathcal{H}_{\mathbb{Q}_\ell} = V_\ell E' := T_\ell E' \otimes \mathbb{Q}_\ell$. Now there is a commutative diagram

$$\begin{array}{ccc} H^{w+1}_{\mathcal{M}}(\mathrm{Sym}^{w+}h^1(E'),\mathbb{Q}(w+1)) & \xrightarrow{\mathcal{K}_{\mathcal{M}}} & H^{w+1}_{\mathcal{M}}(M_{\Theta_\chi},\mathbb{Q}(w+1)) \\ \downarrow \rho_{\text{ét},\ell} & & \downarrow \rho_{\text{ét},\ell} \\ H^1(\mathcal{O}_K[1/\mathfrak{m}],\mathrm{Sym}^w V_\ell E'(1)) & \xrightarrow{\mathcal{K}_\ell} & H^1(\mathcal{O}_K[1/\mathfrak{m}], M'_\ell(w+1)) \end{array}$$

satisfying

$$\mathcal{K}_\ell(\chi(\mathfrak{a})\psi_E \otimes \psi_E(\mathfrak{a})^{\otimes w-1}\mathrm{Sym}^w V_\ell E'(1)) = \varphi_{\Theta_\chi}(\mathfrak{a})M'_\ell(w+1).$$

Also we have that

$$\mathcal{K}_\ell((\tilde{t_n}^{\otimes w})_n) = \delta(e_{\Theta_\chi}(\tilde{t_n}^{\otimes w}))_n$$

in the cohomology group $H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}], M_\ell(w+1))$.

Lemma 6.5 (Johnson, [11][Lemma2.3])**.** *For $k>0$, there is a variation of the Eisenstein symbol*

$$\mathcal{E}_{\mathcal{M}} : \mathbb{Q}[E[\mathfrak{f}]\backslash O] \to H^{k+1}_{\mathcal{M}}(E^k, \mathbb{Q}(k+1))$$

which is defined for divisors of any degree. Moreover,

$$\mathcal{E}_{\mathcal{M}}({}_{\mathfrak{f}}\beta'') = \mathcal{E}_{\mathcal{M}}({}_{\mathfrak{f}}\beta).$$

Therefore we conclude that

$$e_{\Theta_\chi}\rho_{\text{ét},\mathfrak{l}}(\mathcal{K}_{\mathcal{M}}\circ\mathcal{E}_{\mathcal{M}}(\beta)) = \frac{\pm|P(\mathbb{C})|^w}{w!N\mathfrak{a}(\varphi_{\Theta_\chi}(\mathfrak{a})N\mathfrak{a}-1)} \times \delta\left(\sum_{[\mathfrak{l}^n]t_n=\beta} {}_{\mathfrak{a}}\Theta_E(-t_n)\otimes e_{\Theta_\chi}(\tilde{t_n}^{\otimes w})\right)_n .$$

in the cohomology

$$H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], M_{\mathfrak{l}}) = H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], M_\ell)\otimes_{K_\ell}K_{\mathfrak{l}},$$

where

$$\rho_{\text{ét},\mathfrak{l}} = \rho_{\text{ét},\ell}\otimes_{K_\ell}K_{\mathfrak{l}} : H^1_f(M)\otimes_K K_{\mathfrak{l}} \to H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], M_\ell)\otimes_{K_\ell}K_{\mathfrak{l}}.$$

is the $\mathfrak{l}$-part of the ℓ-adic regulator map.

Lemma 6.6. *Let $t\in E[\mathfrak{f}]$. Setting $w_{\mathfrak{f}} = \#\{u\in\mathcal{O}_K^\times \mid u\equiv 1 \mod \mathfrak{f}\}$, we have that*

$$\begin{aligned}(1-\mathrm{Frob}_{\mathfrak{l}}^{-1})&\left(\sum_{[\mathfrak{l}^n]t_n=t} {}_{\mathfrak{a}}\Theta_E(-t_n)\otimes e_{\Theta_\chi}(\tilde{t_n}^{\otimes w})\right)_n \\ &= w_{\mathfrak{f}}\big(\mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})}{}_{\mathfrak{a}}\Theta_E(-s_n)\otimes e_{\Theta_\chi}(\tilde{s_n}^{\otimes w})\big)_n\end{aligned}$$

where s_n is a primitive ℓ^n-th root of $t=\Omega f_{\mathfrak{f}}^{-1}$.

Proof. The proof is the same with [11] [Lemma 3.2.4] or [14] [Theorem 5.1.2]. Let $\mathfrak{l}$ be a prime of K and set $\nu = \mathrm{ord}_{\mathfrak{l}}(\mathfrak{f}), \mathfrak{f} = \mathfrak{l}^\nu\mathfrak{f}_0$. For t_n satisfying $\mathfrak{l}^n t_n = t\in E[\mathfrak{l}^{n+\nu}]$, we write

$$t_n = (\tilde{t_n}, t_{n,0}) \in E[\mathfrak{l}^{n+\nu}]\oplus E[\mathfrak{f}_0].$$

We define the filtration F_n^i for $H_{n,t}^{\mathfrak{l}} := \{t_n \in E[\mathfrak{l}^n\mathfrak{f}] \mid \mathfrak{l}^n t_n = t\}$ by

$$F_n^i = \{t_n = (\tilde{t_n}, t_{n,0}) \mid \mathfrak{l}^{n+\nu+i}\tilde{t_n} = 0\}.$$

The action of $\mathrm{Frob}_{\mathfrak{l}}$ are given by

$$(\mathrm{Frob}_{\mathfrak{l}}^{-1})(\tilde{t_n}^{\otimes w})_n = ((\psi_E(\mathfrak{l})\tilde{t_n})^{\otimes w})_n = (\tilde{t_{n-1}}^{\otimes w})_n.$$

Hence we have

$$\begin{aligned}
&(\mathrm{Frob}_{\mathfrak{l}}^{-i})\mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{l}^{n-i}\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_n)_n \otimes (\tilde{s_n}^{\otimes w})_n \\
&= \mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{l}^{n-i}\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-(\tilde{s_n}, s_{n,0}))_n \otimes (\tilde{s_{n-i}}^{\otimes w})_n \\
&= {}_{\mathfrak{a}}\Theta_E(-(\tilde{s_{n-i}}, s_{n-i,0}))_n \otimes (\tilde{s_{n-i}}^{\otimes w})_n
\end{aligned}$$

by the distribution relation of elliptic units. Since $\mathrm{Gal}(K(\mathfrak{l}^{n-i}\mathfrak{f})/K(\mathfrak{f}))$ acts transitively on F_n^i/F_n^{i+1},

$$\begin{aligned}
&(\mathrm{Frob}_{\mathfrak{l}}^{-i})\mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_n)_n \otimes (\tilde{s_n}^{\otimes w})_n \\
&= \frac{1}{w_{\mathfrak{f}}} \sum_{t_{n-i}\in F_n^i/F_n^{i+1}} {}_{\mathfrak{a}}\Theta_E(-(\tilde{t_{n-i}}, t_{n-i,0}))_n \otimes (\tilde{s_{n-i}}^{\otimes w})_n.
\end{aligned}$$

These elements are annilated by $\mathfrak{l}^n$, so taking the sum over i and limit as $i \to \infty$ we get

$$\begin{aligned}
&\sum_{\mathfrak{l}^n t_n = t} {}_{\mathfrak{a}}\Theta_E(-t_n) \otimes (\tilde{t_n}^{\otimes w}) \\
&= w_{\mathfrak{f}}(\sum_{i=1}^{n} (\mathrm{Frob}_{\mathfrak{l}}^{-1})^i \mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_n) \otimes (\tilde{s_n}^{\otimes w}))_n \\
&= (1 - \mathrm{Frob}_{\mathfrak{l}}^{-1})^{-1}(\mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})}\,{}_{\mathfrak{a}}\Theta_E(-s_n) \otimes (\tilde{s_n}^{\otimes w}))_n.
\end{aligned}$$

Finally applying the projection e_{Θ_χ}, we have the lemma. □

Combining this lemma with Theorem 6.4, we can give the computation of the image of $\xi_{\mathfrak{f}}$ by the ℓ-adic regulator map.

Theorem 6.7. *With the notation as above, we have that*

$$e_{\Theta_\chi}\rho_{\text{ét},\mathfrak{l}}(\xi_{\mathfrak{f}}) = \pm\frac{w_{\mathfrak{f}}|P(\mathbb{C})|^w}{w!}(1 - \overline{\varphi_{\Theta_\chi}}(\mathfrak{l}))\delta(\mathrm{Tr}_{K(\mathfrak{l}^n\mathfrak{f})/K(\mathfrak{f})} z_{\mathfrak{l}^n\mathfrak{f}})_n$$

in the cohomology $H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], M_{\mathfrak{l}}(w+1))$, where

$$z_{\mathfrak{l}^n\mathfrak{f}} := \frac{1}{N\mathfrak{a}(\varphi_{\Theta_\chi}(\mathfrak{a})N\mathfrak{a} - 1)}\,{}_{\mathfrak{a}}\Theta_E(-s_n)_n \otimes e_{\Theta_\chi}(\tilde{s_n}^{\otimes w})_n.$$

7. Computation of the image of L-value in the cohomology

In this section we compute the image of L-values in the cohomology with compact support. By Deninger's formula,

$$\rho_\infty(\xi_{\mathfrak{f}}) = (-1)\frac{2^{-1} \mid P(\mathbb{C})|^w \Phi(\mathfrak{m}) f^a \overline{f}^b}{w!\Phi(\mathfrak{f})N\mathfrak{f}^w} L'(\overline{\varphi_{\Theta_\chi}}, 0)e_{\Theta_\chi}\eta$$

in $e_{\Theta_\chi} H_B(M)^+ = H^w_B(M_{\Theta_\chi} \times_K \operatorname{Spec}\mathbb{C}, \mathbb{Q}(w))$. Since

$$\Xi({}_AM)^{\#} = \operatorname{Det}_A^{-1}(H^1_f(M)^{\mathrm{constr}}) \otimes_A \operatorname{Det}_A(H_B(M)),$$

we have

$$({}_A\vartheta^{\#}_\infty(L^*({}_AM)^{-1}))_\chi = \frac{2^{-1} \mid P(\mathbb{C}) \mid^w \Phi(\mathfrak{m}) f^a \overline{f}^b}{w! \Phi(\mathfrak{f}) N\mathfrak{f}^w} (\xi_\mathfrak{f})^{-1} \otimes e_{\Theta_\chi}\eta$$

where $(\xi_\mathfrak{f})^{-1}$ is the χ-part of the dual basis of $H^1_f(M)^{\mathrm{constr}}$. Now we define the complex $\Delta_\Theta(K(\mathfrak{m}))_r$ by

$$\Delta_\Theta(K(\mathfrak{m}))_r := \operatorname{RHom}_{\mathbb{Z}_\ell}(R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(r)^*), \mathbb{Z}_\ell)[-3]$$

for any integer r, then

$$\operatorname{Det}_{A_\ell} \Delta_\Theta(K(\mathfrak{m}))_r \otimes \mathbb{Q}_\ell \cong \operatorname{Det}_{A_\ell}(R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], M'_\ell(r)^*))^{\#}$$

by a property of determinant functor, where $A_\ell = A \otimes \mathbb{Q}_\ell$. By Shapiro's lemma,

$$R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(r)) \cong R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell(r)),$$

and by Artin-Verdier duality,

$$R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell(r)) \cong R\Gamma_c(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell(r+1)^*)^* \oplus (T'_\ell(r+1)^*)^+.$$

Therefore we have

$$H^1(\Delta_\Theta(K(\mathfrak{m}))_w) \cong H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell(w+1))$$

and

$$H^2(\Delta_\Theta(K(\mathfrak{m}))_w) \cong H^2(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell(w+1)) \oplus \left(\bigoplus\nolimits_{\tau \in \operatorname{Hom}(K(\mathfrak{m}),\mathbb{C})} T_\ell(w)\right)^+.$$

Assuming the weak Leopoldt conjecture, the ℓ-adic regulator map induces the isomorphism

$$\begin{aligned}
&\Xi({}_AM)^{\#} \otimes \mathbb{Q}_\ell \cong \operatorname{Det}^{-1}_{A_\ell}(H^1_f(M)^{\mathrm{constr}} \otimes \mathbb{Q}_\ell) \otimes \operatorname{Det}_{A_\ell}(H_B(M)^+ \otimes \mathbb{Q}_\ell)\\
&\xrightarrow{\cong} \operatorname{Det}^{-1}_{A_\ell}(H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], M_\ell(w+1))) \otimes \operatorname{Det}_{A_\ell}\left(\bigoplus\nolimits_{\tau \in \operatorname{Hom}(K(\mathfrak{m}),\mathbb{C})} M_\ell(w)\right)^+\\
&\overset{(*)}{\xrightarrow{\cong}} \operatorname{Det}^{-1}_{A_\ell}(H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], M_\ell(w+1))) \otimes \operatorname{Det}_{A_\ell}\left(\bigoplus\nolimits_{\tau \in \operatorname{Hom}(K(\mathfrak{m}),\mathbb{C})} M_\ell(w)\right)^+\\
&\xrightarrow{\cong} \operatorname{Det}_{A_\ell}(\Delta_\Theta(K(\mathfrak{m}))_w \otimes \mathbb{Q}_\ell),
\end{aligned}$$

where the map $(*)$ is multiplication with the Euler factors $\prod_{\mathfrak{p}|\mathfrak{m}\ell}(1 - \mathrm{Frob}_\mathfrak{p})^{-1\#}$. Hence by Theorem 6.7 and the relation $\rho_{\text{ét},\ell} = \bigoplus_{\mathfrak{l}|\ell} \rho_{\text{ét},\mathfrak{l}}^{\oplus \mathrm{ord}_\ell \mathfrak{l}}$, we have the following.

Theorem 7.1. *Assume the weak Leopoldt conjecture for M'_ℓ. The χ-part of ${}_A\vartheta_\ell \circ {}_A\vartheta_\infty(L^*({}_AM)^{-1})$ is given by*

$$\pm \prod_{\mathfrak{p}|\mathfrak{m}_0} (1 - \overline{\varphi_{\Theta_\chi}}(\mathfrak{p}))^{-1} \frac{\Phi(\mathfrak{m}) 2^{-1} f^a \overline{f}^b}{w_\mathfrak{f} \Phi(\mathfrak{f}) N\mathfrak{f}^w} (\delta(\operatorname{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})} z_{\ell^n\mathfrak{f}}))_n^{-1} \otimes (t_n \otimes \zeta_{\ell^n}^{\otimes w} \otimes e_{\Theta_\chi}\tau_0)_n.$$

8. The basis of integral lattices

We recall the statement of the equivariant main conjecture for imaginary quadratic fields which is formulated by Johnson [11]. Put

$$\Lambda = \varprojlim_n \mathbb{Z}_\ell[G_{\mathfrak{m}\ell^n}] \cong \mathbb{Z}_\ell[G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}][[S_1, S_2]]$$

where $G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}$ is the torsion subgroup of $G_{\mathfrak{m}\ell^\infty} = \varprojlim_n G_{\mathfrak{m}\ell^n}$. Define a rank one free Λ-module

$$\mathbb{T} = \varprojlim_n H^0(\mathrm{Spec}\,(K(\mathfrak{m}\ell^n))\otimes_K \overline{\mathbb{Q}}, \mathbb{Z}_\ell).$$

and a perfect complex of Λ-modules

$$\Delta^\infty := R\mathrm{Hom}_\Lambda(R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], \mathbb{T}), \Lambda)^\#[-3].$$

Then $H^i(\Delta^\infty) = 0$ for $i \neq 1, 2$ and there is a canonical isomorphism

$$H^1(\Delta^\infty) \cong U^\infty_{\{v|\mathfrak{m}\ell\}} := \varprojlim_n \mathcal{O}_{K(\mathfrak{m}_0\ell^n)}[1/\mathfrak{m}\ell]^\times$$

and a short exact sequence

$$0 \to P^\infty_{\{v|\mathfrak{m}\ell\}} \to H^2(\Delta^\infty) \to X^\infty_{\{v|\mathfrak{m}\ell\infty\}} \to 0$$

where

$$P^\infty_{\{v|\mathfrak{m}\ell\}} := \varprojlim_n \mathrm{Pic}(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[1/\mathfrak{m}\ell])\otimes_{\mathbb{Z}}\mathbb{Z}_\ell$$

$$X^\infty_{\{v|\mathfrak{m}\ell\infty\}} := \varprojlim_n \mathrm{Ker}\left[\bigoplus_{v|\mathfrak{m}\ell\infty \text{ in } K(\mathfrak{m}\ell^n)} \mathbb{Z} \xrightarrow{\Sigma} \mathbb{Z}\right] \otimes_{\mathbb{Z}}\mathbb{Z}_\ell.$$

These limits are taken with respect to Norm maps.

Let $\mathfrak{m}_0$ be the prime to ℓ-part of $\mathfrak{m}$. Put

$${}_{\mathfrak{a}}z_{\mathfrak{m}_0\ell^\infty} = ({}_{\mathfrak{a}}z_{\mathfrak{m}_0\ell^n})_n \in H^1(\Delta^\infty \otimes Q(\psi)),$$

$$\tau = (\tau_{\mathfrak{m}_0\ell^n})_n \in Y^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} := \varprojlim_n Y_{\{v|\mathfrak{m}_0\ell\infty\}}(K(\mathfrak{m}_0\ell^n)) \otimes \mathbb{Z}_\ell,$$

where ${}_{\mathfrak{a}}z_{\mathfrak{f}}$ is the elliptic unit defined by

$${}_{\mathfrak{a}}z_{\mathfrak{f}} = \psi(1; \mathfrak{f}, \mathfrak{a}^{-1}\mathfrak{f})$$

and

$$Y_{\{v|\mathfrak{m}_0\ell\infty\}}(K(\mathfrak{m})) := \bigoplus_{v|\mathfrak{m}\ell\infty \text{ in } K(\mathfrak{m})} \mathbb{Z}.$$

We fix an embedding $\overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and identify $\widehat{G}$ with the set of $\overline{\mathbb{Q}}_\ell$-valued characters. The total ring of fractions

$$Q(\Lambda) \cong \prod_{\psi \in (\widehat{G}^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty})^{\mathbb{Q}_\ell}} Q(\psi)$$

of Λ is a product of fields indexed by the $\mathbb{Q}_\ell$-rational characters of $G^{\mathrm{tor}}_{\mathfrak{m}\ell^\infty}$. Then one can show

$$\dim_{Q(\psi)} U^\infty_{\{v|\mathfrak{m}\ell\}}\otimes_\Lambda Q(\psi) = \dim_{Q(\psi)} Y^\infty_{\{v|\mathfrak{m}_0\ell\infty\}}\otimes_\Lambda Q(\psi) = 1$$

for all characters ψ. Note that the inclusion $X^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} \subseteq Y^\infty_{\{v|\mathfrak{m}_0\ell\infty\}}$ becomes an isomorphism after tensoring with $Q(\psi)$ and that $e_\psi({}_\mathfrak{a}z_{\mathfrak{m}_0\ell^\infty} \otimes \tau)$ is a basis of

$$\mathrm{Det}^{-1}_{Q(\psi)}(U^\infty_{\{v|\mathfrak{m}\ell\}} \otimes_\Lambda Q(\psi)) \otimes \mathrm{Det}_{Q(\psi)}(X^\infty_{\{v|\mathfrak{m}_0\ell\infty\}} \otimes_\Lambda Q(\psi)) \cong \mathrm{Det}_{Q(\psi)}(\Delta^\infty \otimes_\Lambda Q(\psi)).$$

Hence we obtain a $Q(\psi)$-basis

$$\mathcal{L} := (N\mathfrak{a} - \sigma_\mathfrak{a})({}_\mathfrak{a}z_{\mathfrak{m}_0\ell^\infty})^{-1} \otimes \tau$$

of $\mathrm{Det}_{Q(\psi)}(\Delta^\infty \otimes_\Lambda Q(\psi))$.

Conjecture 8.1 (Equivariant main conjecture, [11][Conjecture 3])**.** *There is an identity of invertible Λ-submodules*

$$\Lambda \cdot \mathcal{L} = \mathrm{Det}_\Lambda \Delta^\infty$$

of $\mathrm{Det}_{Q(\Lambda)}(\Delta^\infty \otimes_\Lambda Q(\Lambda))$.

Remark 8.2. Johnson and Kings [12] announced that this conjecture is proved for all primes $\ell \neq 2$.

Now $T'_\ell(w)^*$ is a G_K-stable $\mathfrak{A}_\ell$-lattice in $M'_\ell(w)^*$. The action of G_K on $T'_\ell(w)^*$ factors through a character

$$\kappa : G_{\mathfrak{m}_0\ell^\infty} \longrightarrow \mathfrak{A}_\ell^\times$$

and we also denote by $\kappa : \Lambda \to \mathfrak{A}_\ell$ the corresponding ring homomorphism. Let $(\xi_n)_{n\geq 0}$ be a $\mathfrak{A}_\ell$-basis of $T'_\ell(w)$ with the uniformization $E \cong \mathbb{C}/f\Omega\mathcal{O}_K$.

Lemma 8.3. (a) *There is a natural isomorphism*

$$\Delta^\infty \otimes^{\mathbb{L}}_{\Lambda,\kappa} \mathfrak{A}_\ell \cong \mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(w)^*)^*[-3] = \Delta_\Theta(K(\mathfrak{m}))_w.$$

(b) *The image of an element*

$$u = (u_n)_{n\geq 0} \in \varprojlim_n H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], \mathbb{Z}/\ell^n\mathbb{Z}(1)) \cong U^\infty_{\{v|\mathfrak{m}\ell\}} = H^1(\Delta^\infty)$$

under the induced isomorphism

$$H^1(\Delta^\infty) \otimes^{\mathbb{L}}_{\Lambda,\kappa} \mathfrak{A}_\ell \cong H^1(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(w+1)) \cong H^1(\mathcal{O}_{K(\mathfrak{m})}[1/\mathfrak{m}\ell], T_\ell(w+1))$$

is given by

$$\mathrm{Tr}_{K(\mathfrak{m}\ell^n)/K(\mathfrak{m})}(u_n \cup \xi_n)_{n\geq 0}.$$

(c) *The image of an element*

$$s = (s_n)_{n\geq 0} \in \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}[G_{\mathfrak{m}_0\ell^n}] \cdot \tau = Y^\infty_{\{v|\infty\}}$$

under the isomorphism $Y^\infty_{\{v|\infty\}} \otimes_{\Lambda,\kappa} \mathfrak{A}_\ell \cong H^0(\mathrm{Spec}\,(K \otimes_\mathbb{Q} \mathbb{R}), M'_\ell(w)) = M'_\ell(w)$ *is given by*

$$(s_n \cup \xi_n)_{n\geq 0}.$$

Proof. The proof is just similar to [9][Lemma 4.3]. We set $\mathfrak{A}_n = \mathcal{O}_K/\ell^n\mathcal{O}_K$, $T'_{\ell,n} = T'_\ell(w)^*/\ell^n T'_\ell(w)^*$, $\Lambda_{\mathfrak{A},n} = \mathfrak{A}_n[G_{\mathfrak{m}_0\ell^n}]$ and denote by $\kappa_n : G_{\mathfrak{m}_0\ell^n} \to \mathfrak{A}_n^\times$ the action on $T'_{\ell,n}$. We also denote by κ_n the automorphism of $\Lambda_{\mathfrak{A},n}$ induced by the character $g \mapsto \kappa_n(g)g$ of $G_{\mathfrak{m}_0\ell^n}$. Then $\kappa_\infty = \varprojlim_n \kappa_n$ is an automorphism of

$$\Lambda_\mathfrak{A} := \mathfrak{A}_\ell[[G_{\mathfrak{m}_0\ell^\infty}]].$$

For the natural map $f_n : \mathrm{Spec}\,(\mathcal{O}_{K(\mathfrak{m}_0\ell)}[1/\mathfrak{m}\ell])$ and the constant sheaf $\mathfrak{A}_n$, the sheaf $\mathcal{F}_n := f_{n,*}f_n^*\mathfrak{A}_n$ is free of rank one over $\Lambda_{\mathfrak{A},n}$ with G_K-action given by the inverse of the natural projection $G_K \to G_{\mathfrak{m}_0\ell^n} \subset \Lambda_{\mathfrak{A}_n}^\times$. There is a $\Lambda_{\mathfrak{A},n} - \kappa_n^{-1}$-semilinear isomorphism $\mathrm{tw} : \mathcal{F}_n \to \mathcal{F}_n \otimes_{\mathfrak{A}_n} T'_{\ell,n}$ sending 1 to $1 \otimes \xi_n$. Shapiro's lemma gives a commutative diagram of isomorphisms

$$\begin{array}{ccc} R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], \mathcal{F}_n) & \xrightarrow{\mathrm{tw}} & R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], \mathcal{F}_n \otimes_{\mathfrak{A}_n} T'_{\ell,n}) \\ \downarrow & & \downarrow \\ R\Gamma_c(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[1/\mathfrak{m}\ell], \mathcal{F}_n) & \xrightarrow{\cup\xi_n} & R\Gamma_c(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[1/\mathfrak{m}\ell], T'_{\ell,n}) \end{array} \tag{8.1}$$

where the horizontal arrows are $\Lambda_{\mathfrak{A},n} - \kappa_n^{-1}$-semilinear. Taking the $\mathcal{O}_K/\ell^n$-dual (with conragradient $G_{\mathfrak{m}_0\ell^n}$-action) we obtain a $\# \circ \kappa_n^{-1} \circ \# = \kappa_n$-semilinear isomorphism

$$R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], \mathcal{F}_n \otimes_{\mathfrak{A}_n} T'_{\ell,n})^*[-3] \to R\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], \mathcal{F}_n)^*[-3].$$

where $\mathcal{F}_\infty = \varprojlim_n \mathcal{F}_n \cong \mathbb{T} \otimes_\Lambda \Lambda_{\mathfrak{A}}$. Hence a Λ-linear isomorphism

$$(\Delta^\infty \otimes_\Lambda \Lambda_{\mathfrak{A}}) \otimes_{\Lambda_{\mathfrak{A}},\kappa_\infty} \Lambda_{\mathfrak{A}} \cong \mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}_0\ell], \mathcal{F}_\infty \otimes_{\mathfrak{A}_\ell} T'_\ell)^*[-3].$$

Part (a) follows by noting that κ coinsides with the composite map

$$\Lambda \to \Lambda_{\mathfrak{A}} \overset{\kappa_\infty}{\to} \Lambda_{\mathfrak{A}} \to \mathfrak{A}_\ell$$

where the last map is the augmentation map, and that $\mathcal{F}_\infty \otimes_{\Lambda_{\mathfrak{A}}} \cong \mathfrak{A}_\ell$ with trivial G_K-action. The $\mathcal{O}_K/\ell^n$-dual of the H^2 of the inverse map in the lower row in 8.1 coinsides with

$$H^1(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[1/\mathfrak{m}\ell], {T'_{\ell,n}}^*(1)) \overset{\cup\xi_n}{\leftarrow} H^1(\mathcal{O}_{K(\mathfrak{m}\ell^n)}[1/\mathfrak{m}\ell], \mathfrak{A}_n(1))$$

by Poitou-Tate duality. This gives part (b). Similarly to the lower row in 8.1 we have a κ^{-1}-semilinear isomorphism

$$\mathcal{F}_n = H^0(K(\mathfrak{m}\ell^n) \otimes \mathbb{R}, \mathfrak{A}_n) \overset{\cup\xi_n^{-1}}{\to} H^0(K(\mathfrak{m}\ell^n) \otimes \mathbb{R}, \mathcal{F}_n \otimes_{\mathfrak{A}_n} T'_{\ell,n}) = \mathcal{F}_n \otimes_{\mathfrak{A}_n} T'_{\ell,n}$$

the $\mathcal{O}_K/\ell^n$-dual of the inverse of which is the κ-semilinear isomorphism given by the cup product with ξ_n. Passing to the limit and tensoring over $\Lambda_{\mathfrak{A}}$ with A_ℓ we deduce part (c). □

Now the main theorem follows from Theorem 7.1 and the following theorem.

Theorem 8.4. *Let R be a direct factor of A_ℓ which is a field, $\mathfrak{q}$ the kernel of the map $\kappa : \Lambda \to A_\ell \to R$ and $\Lambda_{\mathfrak{q}}$ the localization of Λ at $\mathfrak{q}$. Then $\mathcal{L}$ is a basis of $\mathrm{Det}_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^\infty$. Denote by $\mathcal{L} \otimes 1$ the image of $\mathcal{L}$ under the determinant*

$$\mathrm{Det}_{\Lambda_{\mathfrak{q}}} \Delta_{\mathfrak{q}}^\infty \otimes_{\Lambda_{\mathfrak{q}},\kappa} A_\ell \cong \mathrm{Det}_{A_\ell} \mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(w)^*)^*[-3]$$

of the isomorphism Lemma 8.3 (a) . Moreover assume the weak Leopoldt conjecture for M'_ℓ. Then the χ-part of the image of $\mathcal{L} \otimes 1$ under

$$\mathrm{Det}_{A_\ell} \mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(w)^*)^*[-3] \cong \mathrm{Det}_{A_\ell} \mathrm{R}\Gamma_c(\mathcal{O}_K[1/\mathfrak{m}\ell], T'_\ell(w)^*)^\# \cong \Xi({}_A M)^\# \otimes_A A_\ell$$

is given by

$$\prod_{\mathfrak{p}|\mathfrak{m}_0} (1 - \overline{\varphi_{\Theta_\chi}}(\mathfrak{p}))^{-1} \frac{\Phi(\mathfrak{m}) f^a \overline{f}^b}{w_{\mathfrak{f}} \Phi(\mathfrak{f}) N\mathfrak{f}^w} (\delta(\mathrm{Tr}_{K(\ell^n\mathfrak{f})/K(\mathfrak{f})} z_{\ell^n\mathfrak{f}}))_n^{-1} \otimes (t_n \otimes \zeta_{\ell^n}^{\otimes w+r})_n \otimes e_{\Theta_\chi} \tau_0.$$

Proof. By [11][Lemma 5.5], we have

$$H^i(\Delta_{\mathfrak{q}}^{\infty} \otimes_R^{\mathbb{L}} k) \cong H^i(\Delta_{\mathfrak{q}}^{\infty}) \otimes_R k.$$

Moreover Lemma 8.3 gives an isomorphism of complexes of R-modules $\Delta_{\mathfrak{q}}^{\infty} \otimes_R^{\mathbb{L}} k \cong \Delta_\Theta(K(\mathfrak{m}))_w \otimes_{\mathfrak{A}_\ell} k$. Therefore the isomorphism of determinants

$$\phi : \mathrm{Det}_k(\Delta_{\mathfrak{q}}^{\infty} \otimes_R^{\mathbb{L}} k) \xrightarrow{\cong} \mathrm{Det}_k(\Delta_\Theta(K(\mathfrak{m}))_w \otimes_{\mathfrak{A}_\ell} k)$$

can be computed as a map on the cohomology groups

$$\begin{aligned}\phi : \bigotimes_{i=1}^{2} H^i(\Delta_{\mathfrak{q}}^{\infty}) \otimes k &\xrightarrow{\cong} \bigotimes_{i=1}^{2} H^i(\Delta_{\mathfrak{q}}^{\infty} \otimes_R^{\mathbb{L}} k) \\ &\xrightarrow{\cong} \bigotimes_{i=1}^{2} H^i(\Delta_\Theta(K(\mathfrak{m}))_w) \otimes_{\mathfrak{A}_\ell} k.\end{aligned}$$

To compute $\phi(\mathcal{L} \otimes 1)$, we consider the elements ${}_{\mathfrak{a}}z_{\mathfrak{m}_0\ell^\infty}$ and τ. By the norm compatible properties of elliptic units and the similar argument of Johnson [11][(5.7)], we can compute these element as

$$\frac{\phi({}_{\mathfrak{a}}z_{\mathfrak{m}_0\ell^\infty})}{N\mathfrak{a}(\varphi_{\Theta_\chi} N\mathfrak{a} - 1)} = [K(\mathfrak{m}) : K(\mathfrak{f})]^{-1} \prod_{\mathfrak{p} | \mathfrak{m}_0, \mathfrak{p} \nmid \mathfrak{f}} (1 - \overline{\varphi_{\Theta_\chi}}(\mathfrak{p}))(\delta(\mathrm{Tr}_{K(\ell^n \mathfrak{f})/K(\mathfrak{f})} z_{\ell^n \mathfrak{f}}))_n$$

and

$$\phi(\tau) = \frac{f^a \overline{f}^b}{N\mathfrak{f}^w} (t_n)_n \otimes (\zeta_n^{\otimes w})_n \otimes e_{\Theta_\chi} \tau_{\mathfrak{m}}$$

using Lemma 8.3, where the factor

$$\frac{f^a \overline{f}^b}{N\mathfrak{f}^w} = (\overline{f}^a f^b)^{-1}$$

comes from the difference of lattices $\mathbb{C}/\Omega\mathcal{O}_K$ and $\mathbb{C}/f\Omega\mathcal{O}_K = \mathbb{C}/\Omega\mathfrak{f}$ relating to the choice of elliptic units. Note that our choice of $\tau_{\mathfrak{m}}$ is τ_0. So the image of $\mathcal{L}$ is given by

$$[K(\mathfrak{m}) : K(\mathfrak{f})] \frac{f^a \overline{f}^b}{N\mathfrak{f}^w} \prod_{\mathfrak{p} | \mathfrak{m}_0, \mathfrak{p} \nmid \mathfrak{f}} (1 - \overline{\varphi_{\Theta_\chi}}(\mathfrak{p}))^{-1} (\delta(\mathrm{Tr}_{K(\ell^n \mathfrak{f})/K(\mathfrak{f})} z_{\ell^n \mathfrak{f}}))_n^{-1} \otimes (t_n \otimes \zeta_{\ell^n}^{\otimes w})_n \otimes e_{\Theta_\chi} \tau_0$$

$$= \prod_{\mathfrak{p} | \mathfrak{m}_0} (1 - \overline{\varphi_{\Theta_\chi}}(\mathfrak{p}))^{-1} \frac{\Phi(\mathfrak{m}) f^a \overline{f}^b}{w_{\mathfrak{f}} \Phi(\mathfrak{f}) N\mathfrak{f}^w} (\delta(\mathrm{Tr}_{K(\ell^n \mathfrak{f})/K(\mathfrak{f})} z_{\ell^n \mathfrak{f}}))_n^{-1} \otimes (t_n \otimes \zeta_{\ell^n}^{\otimes w})_n \otimes e_{\Theta_\chi} \tau_0.$$

The last equality follows from the formula

$$\Phi(\mathfrak{m})/\Phi(\mathfrak{f}) = w_{\mathfrak{f}}[K(\mathfrak{m}) : K(\mathfrak{f})]$$

and $\varphi_\Theta(\mathfrak{p}) = 0$ for $\mathfrak{p} \mid \mathfrak{f}$. □

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