# CORRIGENDUM TO THE ARTICLE: ON KONTSEVICH'S CHARACTERISTIC CLASSES FOR HIGHER-DIMENSIONAL SPHERE BUNDLES II: HIGHER CLASSES

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ABSTRACT. We fix an error in the proof of Theorem 6.1 (ii) in [5], and extend the main result of [5] to all odd dimensions at least 5.

The second paragraph of the proof (p.648) of [5, Theorem 6.1] is the proof of the following.

**Claim 1.** *The map* $\tilde{f}(\pi^\Gamma) : (S^{k-1})^{\times 2n} \to \widetilde{B\mathrm{Diff}}(D_M, \partial)$ *is oriented bordant to a map from* $S^{2n(k-1)}$.

Here, $\widetilde{B\mathrm{Diff}}(W, X)$ is the abbreviation of $\widetilde{B\mathrm{Diff}}(W, X; \tau)$, the classifying space for framed $(W, X)$-bundles ([5, §2.1] for definition). In that proof, it was proved only that the composition $\tilde{f}(\pi^\Gamma) \circ \mathrm{incl}_i : S^{k-1} \to (S^{k-1})^{\times 2n} \to \widetilde{B\mathrm{Diff}}(D_M, \partial)$ is null-homotopic for each $i$, which is not sufficient for $2n \geq 4$, or equivalently if $\Gamma$ has at least 4 vertices. The argument given below not only fills the gap, but also allows us to extend the main result of [5] to all odd dimensions at least 5. The extended result is the following.

**Theorem 2.** *For* $n \geq 1$ *and for* $\dim D_M = 2k+1 \geq 5$, *the evaluation of the Kontsevich characteristic class gives an epimorphism* $\pi_{2n(k-1)}(\widetilde{B\mathrm{Diff}}(D_M, \partial)) \otimes \mathbb{R} \to \mathcal{A}_{2n,3n} \otimes \mathbb{R}$.

In [5], this was proved as part of [5, Theorem 6.1] and was proved only for $2k + 1 \geq 5$ with the restriction that $k$ is odd. To prove Claim 1, we need only to consider the case $D_M = D^{2k+1}$ since the graph surgery is performed in a small $(2k+1)$-disk in $D_M$. In this case, we shall prove the following stronger claim, instead of Claim 1.

**Claim 3.** *The map* $\tilde{f}(\pi^\Gamma) : (S^{k-1})^{\times 2n} \to \widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)$ *factors up to homotopy into a degree 1 map* $(S^{k-1})^{\times 2n} \to S^{2n(k-1)}$ *and a map* $S^{2n(k-1)} \to \widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)$.

We shall prove Claim 3 by proving two lemmas (Lemmas A and B). The first one is an improvement of [5, Proposition 4.2]. Let $Q^{2k+1} = D^k \times D^{k+1}$ and let $i : (D^k)^{\sqcup 3} \hookrightarrow Q^{2k+1}$ be the inclusion $D^k \times \{p_1, p_2, p_3\} \hookrightarrow D^k \times D^{k+1}$ for some fixed distinct points $p_1, p_2, p_3 \in \mathrm{Int}\, D^{k+1}$. Let $N_{p_\ell} \subset D^{k+1}$ $(\ell = 1, 2, 3)$ be open $(k+1)$-disks about $p_\ell$ of small radius $h < \frac{1}{100} \min\{|p_\lambda - p_\mu|; \lambda \neq \mu\}$ and let $N_{D^k} = D^k \times U_h^{k+1}$, where $U_h^{k+1}$ is the open $(k+1)$-disk of radius $h$. Let $\widetilde{i} : (N_{D^k})^{\sqcup 3} \hookrightarrow Q^{2k+1}$ be the inclusion $D^k \times (N_{p_1} \sqcup N_{p_2} \sqcup N_{p_3}) \hookrightarrow D^k \times D^{k+1}$. Let $V$ be the complement of the image of $\widetilde{i}$. This is a standard model of $V_0$ in [5, p.634]. For $\ell \in \{1, 2, 3\}$, let $i_\ell$ and $\widetilde{i}_\ell$ be the embeddings of $(D^k)^{\sqcup 2}$ and $(N_{D_k})^{\sqcup 2}$ into $Q^{2k+1}$ obtained by forgetting the $\ell$-th component from $i$ and $\widetilde{i}$, respectively. Let $V_{[\ell]}$ be the complement of the image of $\widetilde{i}_\ell$, in particular $V \subset V_{[\ell]}$.

The complements $V$ and $V_{[\ell]}$ come with standard framings 'st' induced by restricting the canonical framing of $Q^{2k+1} \subset \mathbb{R}^{2k+1}$. In these terms, the element $\delta(\alpha') \in \pi_{k-1}(B\mathrm{Diff}(V, \partial))$ of [5, p.638] arises as the image of a long Borromean rings construction $\alpha' \in \pi_{k-1}(\mathrm{Emb}_\partial^{\mathrm{f}}((D^k)^{\sqcup 3}, Q^{2k+1})_i)^\dagger$ under the map $\delta$ induced on homotopy groups by the zig-zag

$$\mathrm{Emb}_\partial^{\mathrm{f}}((D^k)^{\sqcup 3}, Q^{2k+1})_i \xleftarrow{\simeq} \mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} \xrightarrow{c} B\mathrm{Diff}(V, \partial). \tag{0.1}$$

Here,

- $\mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} \subset \mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})$ denotes the path-component of $\widetilde{i}$, where we impose a restriction on embeddings in $\mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})$ that each embedding $f$ has a smooth extension to an embedding $\bar{f} : (D^k \times \overline{U}_h^{k+1})^{\sqcup 3} \to Q^{2k+1}$ that agrees with the canonical

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†We use the notation $\mathrm{Emb}_\partial^{\mathrm{f}}$ here instead of $\mathrm{Emb}^{\mathrm{f}}$ to indicate the embeddings have fixed behaviour near the boundary.

extension of $\widetilde{i}$ near $\partial D^k \times \overline{U}_h^{k+1}$. In particular, $\bar{f}^{-1}(\partial Q^{2k+1}) = \partial D^k \times \overline{U}_h^{k+1}$. We impose this condition so that the complement of the embedding $(N_{D^k})^{\sqcup 3} \hookrightarrow Q^{2k+1}$ is a smooth manifold with corners.

- $\mathrm{Emb}^{\mathrm{f}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i$ is the path-component of the point in
$$\mathrm{hofib}_*\big(\mathrm{Emb}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i \to \Omega^k(BSO_{k+1})^3\big)$$
given by $i$ with the constant path at the base point $* \in \Omega^k(BSO_{k+1})^3$, where the map $\mathrm{Emb}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i \to \Omega^k(BSO_{k+1})^3$ is induced by taking normal bundles. The space $\mathrm{Emb}^{\mathrm{f}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i$ can be considered as a component of the space of normally framed embeddings that agree with $i$ and the standard normal framing in a neighborhood of the boundary.
- The left equivalence in the zig-zag is induced by restriction (using the standard framing of $\mathbb{R}^{2k+1}$). The right arrow $c$ takes complements. Extension of a (framed) $(V,\partial)$-bundle to a (framed) $(V_{[\ell]},\partial)$-bundle by filling a trivialized (framed) $N_{D^k}$-bundle into one of the three complementary handles gives maps

$$e_\ell : B\mathrm{Diff}(V,\partial) \to B\mathrm{Diff}(V_{[\ell]},\partial), \quad \text{and} \quad \widetilde{e}_\ell : \widetilde{B\mathrm{Diff}}(V,\,\partial;\mathrm{st}) \to \widetilde{B\mathrm{Diff}}(V_{[\ell]},\partial;\mathrm{st}). \tag{0.2}$$

**Lemma A.** *The element $\delta(\alpha') \in \pi_{k-1}(B\mathrm{Diff}(V,\partial))$ admits a lift $\beta \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st}))$, which becomes trivial under the map $\pi_{k-1}(\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})) \to \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V_{[\ell]},\partial;\mathrm{st}))$ for all $\ell \in \{1,2,3\}$.*

*Remark* 4. Lemma A also follows from [4, Proposition 6.4] by Krannich and Randal-Williams, proved by different method. Their class $\alpha \in \pi_{k-1}(\widetilde{B\mathrm{Diff}}(V,\partial)) \otimes \mathbb{Q}$ is symmetric in the three handles by definition, whereas it is not obvious for our $\beta$.

*Proof of Lemma A.* To give a framed analogue of the zig-zag (0.1) for $\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})$, we consider the spaces

$$\overline{\mathrm{Emb}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i = \mathrm{hofib}_*\big(\mathrm{Emb}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i \to \mathrm{Bun}_\partial(T(D^k)^{\sqcup 3}, TQ^{2k+1})\big)_{(i,\mathrm{const})},$$
$$\overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} = \mathrm{hofib}_*\big(\mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} \to \mathrm{Bun}_\partial(T(N_{D^k})^{\sqcup 3}, TQ^{2k+1})\big)_{(\widetilde{i},\mathrm{const})},$$

where $\mathrm{Bun}_\partial(T(D^k)^{\sqcup 3}, TQ^{2k+1}) \simeq \Omega^k(SO_{2k+1}/SO_{k+1})^3$ and $\mathrm{Bun}_\partial(T(N_{D^k})^{\sqcup 3}, TQ^{2k+1}) \simeq \Omega^k(SO_{2k+1})^3$ are the spaces of bundle monomorphisms $T(D^k)^{\sqcup 3} \to TQ^{2k+1}$ and $T(N_{D^k})^{\sqcup 3} \to TQ^{2k+1}$, respectively, with fixed behaviour on the boundary, and the identification in terms of the orthogonal groups are induced by the standard framing of the disk. We denote by $*$ the base points of $\mathrm{Bun}_\partial(T(D^k)^{\sqcup 3}, TQ^{2k+1})$ and $\mathrm{Bun}_\partial(T(N_{D^k})^{\sqcup 3}, TQ^{2k+1})$ given by the map induced by the standard inclusion and $\widetilde{i}$, respectively. The zig-zag (0.1) extends to a diagram, commutative up to homotopy,

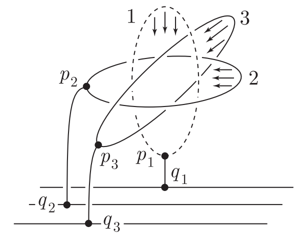
$$\begin{array}{ccccc}
\overline{\mathrm{Emb}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i & \xleftarrow{\simeq} & \overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} & \xrightarrow{\widetilde{c}} & \widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Emb}^{\mathrm{f}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i & \xleftarrow{\simeq} & \mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} & \xrightarrow{c} & B\mathrm{Diff}(V,\partial)
\end{array} \tag{0.3}$$

where the leftmost vertical map is induced by the map $\mathrm{Bun}_\partial(T(D^k)^{\sqcup 3}, TQ^{2k+1}) \to \Omega^k(BSO_{k+1})^3$ given by taking normal bundles. The left horizontal map on the top row is induced by restriction along the inclusion $(D^k)^{\sqcup 3} \hookrightarrow N_{D^k}^{\sqcup 3}$, which is a homotopy equivalence since its homotopy fiber is given by $\mathrm{hofib}_*(\Omega^k(SO_{k+1})^3 \overset{=}{\to} \Omega^k(SO_{k+1})^3) \simeq *$. Here,

(a) The map $\widetilde{c} : \overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} \to \widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})$ can be defined so that the diagram (0.3) is commutative up to homotopy.
(b) Then the zig-zag on the top row of (0.3) is compatible up to homotopy with the forgetful maps of the embedding spaces induced by forgetting components and the extension maps (0.2).

We postpone the proofs of these facts and proceed to prove the lemma using the diagram (0.3). Now what needs to be shown is that the element $\alpha' \in \pi_{k-1}(\mathrm{Emb}^{\mathrm{f}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i)$ constructed by the long Borromean rings

(1') admits a lift $\widetilde{\alpha} \in \pi_{k-1}(\overline{\mathrm{Emb}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i)$, which
(2') becomes trivial when forgetting any of the components.




To check the condition (1'), we note that each component in the standard Borromean rings has a preferred isotopy in $\mathbb{R}^{2k+1}$ to a small unknotted sphere about the base point, where the isotopy is induced by a smooth deformation retract of its spanning disk to a small disk, as in the figure on the right. This isotopy of a single component gives rise to a desired lift, as follows. From the explicit Borromean rings embedding $b' \in \mathrm{Emb}_\partial(D^{2k-1} \sqcup D^k \sqcup D^k, Q^{2k+1})$, we get an embedding $b \in \mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k))$ as in [5, p.637] (denoted $\phi_B^\square$ therein), where $\emptyset \sqcup D^k \sqcup D^k$ denotes $D^k \times \{p_2, p_3\}$. This is carried out as follows. The spanning disks of the second and third component of $b'$ gives a path in $\mathrm{Emb}_\partial(D^k \sqcup D^k, Q^{2k+1})$ from $i|_{\emptyset \sqcup D^k \sqcup D^k}$ to $b'|_{D^k \sqcup D^k}$, where the third component is deformed before the second component so that the two components do not intersect. Then we use the isotopy extension to lift this to a path $\psi_t$ in $\mathrm{Diff}(Q^{2k+1}, \partial)$ from the identity to a diffeomorphism $\psi_1$. Now the restriction of the embedding $\psi_1^{-1} \circ b'$ to the second and third component agrees with those of $i$, and $\psi_1^{-1} \circ b'$ gives rise to an embedding $b \in \mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k))$.

Now we consider the following commutative diagram

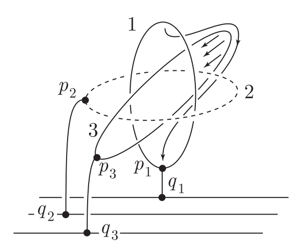
$$\begin{array}{ccc} \mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k)) & \longrightarrow & \Omega^{k-1}\mathrm{Emb}_\partial(D^k \sqcup D^k \sqcup D^k, Q^{2k+1})_i \\ \downarrow & & \downarrow \\ \mathrm{Bun}_\partial(T(D^{2k-1}), T(Q^{2k+1})) & \longrightarrow & \Omega^{k-1}\mathrm{Bun}_\partial(T(D^k \sqcup D^k \sqcup D^k), T(Q^{2k+1}))_{di} \end{array}$$

whose vertical maps are induced by taking the derivative and whose horizontal maps are induced by the identification $D^{2k-1} = D^k \times D^{k-1}$ and the isotopy constructed in [5, Lemma 4.1]. By the definition of $\alpha'$ in [5, p.638], the image of $b$ under the top horizontal map represents $\alpha'$. The derivative of the embedding $b$ is canonically homotopic in $\mathrm{Bun}_\partial(T(D^{2k-1}), T(Q^{2k+1}))$ to a constant bundle map, hence $b$ gives an element $\widetilde{b}$ in the homotopy fiber $\overline{\mathrm{Emb}}'_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k))$ of the left vertical map over the derivative of the standard inclusion. Here, we can choose the lift $\widetilde{b}$ so that its class in $\pi_0$ is canonically determined by choosing the spanning disk of the first component obtained by applying $\psi_1^{-1}$ to the standard one for $b'$. The image of $\widetilde{b}$ in the homotopy fiber of the right vertical map is a class in $\pi_0(\Omega^{k-1}\overline{\mathrm{Emb}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i) \cong \pi_{k-1}(\overline{\mathrm{Emb}}_\partial((D^k)^{\sqcup 3}, Q^{2k+1})_i)$, which gives the required lift $\widetilde{\alpha}$ by the commutativity of the diagram above.

To check the condition (2'), we consider the element of $\pi_{k-1}(\overline{\mathrm{Emb}}_\partial((D^k)^{\sqcup 2}, Q^{2k+1})_i)$ obtained from $\widetilde{\alpha}$ by forgetting the $\ell$-th component. We prove the triviality of this in terms of the Borromean rings $b \in \mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k))$ and its lift $\widetilde{b}$. If the $\ell$-th component removed is the one coming from the component of $D^{2k-1}$, the condition (2') is obvious. Now suppose that the $\ell$-th component removed is the one coming from the two fixed $D^k$ components. We consider the following diagram commutative up to homotopy:

$$\begin{array}{ccc} \overline{\mathrm{Emb}}'_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k)) & \xrightarrow{\overline{s}'_\ell} & \overline{\mathrm{Emb}}'_\partial(D^{2k-1}, Q^{2k+1} \setminus D^k) \\ \downarrow & & \downarrow \\ \mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus (\emptyset \sqcup D^k \sqcup D^k)) & \xrightarrow{s'_\ell} & \mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus D^k) \end{array}$$

where the $D^k$ in the right column is $D^k \times \{p_2\}$ or $D^k \times \{p_3\}$ depending on $\ell$, $\overline{\mathrm{Emb}}'_\partial(D^{2k-1}, Q^{2k+1} \setminus D^k)$ is the homotopy fiber of the derivative map $\mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus D^k) \to \mathrm{Bun}_\partial(T(D^{2k-1}), T(Q^{2k+1}))$ over the derivative of the standard inclusion, and the maps $s'_\ell$, $\overline{s}'_\ell$ are those obtained by filling the $\ell$-th $D^k$-component.




With the help of the explicit coordinate description of the spanning disks of the Borromean rings in ([5, p.636]), it can be seen that one of the spanning disks of the remaining two components can be deformed by isotopy relative to the boundary to one which is disjoint from the spanning disk of the first component (figure on the right). This implies that there is a smooth map $D^{2k-1} \times [0,1] \to Q^{2k+1} \setminus D^k$ that restricts to an embedding isotopic to $s'_\ell(b)$ and the standard inclusion at $D^{2k-1} \times \{0,1\}$, and induces a path in $\mathrm{Emb}_\partial(D^{2k-1}, Q^{2k+1} \setminus D^k)$, considered as a subspace of the immersion space $\mathrm{Imm}_\partial(D^{2k-1}, Q^{2k+1}) \simeq \mathrm{Bun}_\partial(TD^{2k-1}, TQ^{2k+1})$ (the last equivalence by Smale–Hirsch). This path gives a lift $\widetilde{b}_\ell$ of $s'_\ell(b)$ in $\overline{\mathrm{Emb}}'_\partial(D^{2k-1}, Q^{2k+1} \setminus D^k)$. More precisely, by an isotopy extension for the deformation through the spanning disk of the remaining $D^k$-component analogous to the path $\psi_t$, we get a path $\psi'_t$ in $\mathrm{Diff}(Q^{2k+1},\partial)$. Then by applying ${\psi'_1}^{-1}$ to the (possibly deformed for the disjunction) spanning disk of the $D^{2k-1}$-component, we obtain an embedded spanning disk in $Q^{2k+1}\setminus D^k$. Since the space of paths in $\mathrm{Emb}_\partial(D^{2k-1},Q^{2k+1}\setminus D^k)$ to the standard inclusion is contractible, $\widetilde{b}_\ell$ is equivalent to the base point in $\pi_0(\overline{\mathrm{Emb}}'_\partial(D^{2k-1},Q^{2k+1}\setminus D^k))$. Note that the disjunction isotopy of the spanning disk does not change the class of $\widetilde{b}_\ell$ in $\pi_0(\overline{\mathrm{Emb}}'_\partial(D^{2k-1},Q^{2k+1}\setminus D^k))$.

Moreover, we see that the class of $\widetilde{b}_\ell$ agrees with that of $\overline{s}'_\ell(\widetilde{b})$. Namely, there is a path in $\overline{\mathrm{Emb}}'_\partial(D^{2k-1},Q^{2k+1}\setminus D^k)$ between $\overline{s}'_\ell(\widetilde{b})$ and $\widetilde{b}_\ell$ constructed as follows. Note that the spanning disks of the $D^{2k-1}$-components used to define $\overline{s}'_\ell(\widetilde{b})$ and $\widetilde{b}_\ell$ are obtained from the standard one by different paths $\psi_t$ and $\psi'_t$ in $\mathrm{Diff}(Q^{2k+1},\partial)$, respectively. Thus it suffices to show that the path $\psi'_t$ can be deformed into the path $\psi_t$. Let $\rho_{2,t}$ and $\rho_{3,t}$ be paths in $\mathrm{Diff}(Q^{2k+1},\partial)$ from the identity that are obtained from the standard spanning disks of the second and third components, respectively, by isotopy extension. Then the path $\psi_t$ can be given by

$$\psi_t=\begin{cases}\rho_{3,2t} & (0\leq t\leq 1/2)\\ \rho_{2,2t-1}\circ\rho_{3,1} & (1/2\leq t\leq 1)\end{cases}$$

If $\ell=2$, we have $\psi'_t=\rho_{3,t}$, and we define a homotopy $\Psi_{s,t}\in\mathrm{Diff}(Q^{2k+1},\partial)$ $(s\in[0,1])$ by

$$\Psi_{s,t}=\begin{cases}\rho_{3,2t} & (0\leq t\leq 1/2)\\ \rho_{2,2t-1}\circ\rho_{3,1} & (1/2\leq t\leq 1-s/2)\\ \rho_{2,1-s}\circ\rho_{3,1} & (1-s/2\leq t\leq 1)\end{cases}$$

Then $\Psi_{0,t}=\psi_t$ and $\Psi_{1,t}\simeq\psi'_t$ relative to id by a parameter change. If $\ell=3$, we have $\psi'_t=\rho_{2,t}$, and we define a homotopy $\Psi_{s,t}\in\mathrm{Diff}(Q^{2k+1},\partial)$ $(s\in[0,1])$ by

$$\Psi_{s,t}=\begin{cases}\rho_{3,2t} & (0\leq t\leq 1/2-s/2)\\ \rho_{3,1-s} & (1/2-s/2\leq t\leq 1/2)\\ \rho_{2,2t-1}\circ\rho_{3,1-s} & (1/2\leq t\leq 1)\end{cases}$$

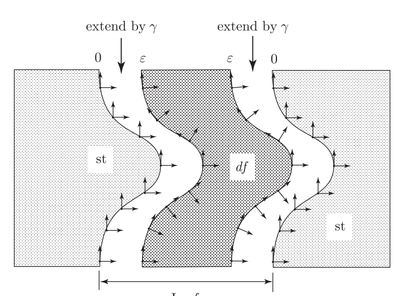
Then $\Psi_{0,t}=\psi_t$ and $\Psi_{1,t}\simeq\psi'_t$ relative to id by a parameter change. The homotopy between $\psi_t$ and $\psi'_t$ induces a path between $\overline{s}'_\ell(\widetilde{b})$ and $\widetilde{b}_\ell$. Now the (trivial) class of the image of $\widetilde{b}_\ell$ under the map

$$\overline{\mathrm{Emb}}'_\partial(D^{2k-1},Q^{2k+1}\setminus D^k)\to\Omega^{k-1}\overline{\mathrm{Emb}}_\partial(D^k\sqcup D^k,Q^{2k+1})$$

agrees with the one obtained from $\widetilde{\alpha}$ by forgetting the $\ell$-th component, which is thus trivial.

Proof of (a): The map $\widetilde{c}:\overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3},Q^{2k+1})_{\widetilde{i}}\to\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})$ is defined as follows. A point of the space $\overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3},Q^{2k+1})_{\widetilde{i}}$ is given by an embedding $f\in\mathrm{Emb}_\partial((N_{D^k})^{\sqcup 3},Q^{2k+1})_{\widetilde{i}}$ with a path $\gamma$ in $\mathrm{Bun}_\partial(T(N_{D^k})^{\sqcup 3},TQ^{2k+1})$ from $df$ to the standard inclusion. This path can be used to connect the framing induced from the standard one of $(N_{D^k})^{\sqcup 3}$ with the framing on its complement induced by the canonical framing of $Q^{2k+1}$. Namely, let $F_\ell$ be the $\ell$-th component of the faces $\overline{\mathrm{Im}\,\widetilde{i}}\setminus\mathrm{Im}\,\widetilde{i}$, where $\overline{\,\cdot\,}$ denotes the closure in $Q^{2k+1}$. We fix a real number $\varepsilon$ such that $0<\varepsilon<h/100$ and let $[0,\varepsilon]\times F_\ell\subset\overline{\mathrm{Im}\,\widetilde{i}}$ be a collar neighborhood of $F_\ell$ in $\overline{\mathrm{Im}\,\widetilde{i}}$ such that $F_\ell=\{0\}\times F_\ell$, whose product structure is induced by $f$ from the product structure of $D^k\times\overline{U}{}^{k+1}_h$. The image of the standard framing of $(N_{D^k})^{\sqcup 3}$ under $df$ restricts to a framing over $f(\{\varepsilon\}\times F_\ell)$ for each $\ell$, and the canonical framing of $Q^{2k+1}$ gives a framing over $Q^{2k+1}\setminus\mathrm{Im}\,f$. Now we would like to extend the framing given on $(Q^{2k+1}\setminus\mathrm{Im}\,f)\coprod\bigcup_\ell f(\{\varepsilon\}\times F_\ell)$ over $\bigcup_\ell f([0,\varepsilon]\times F_\ell)$.

Such an extension is possible by using the path $\gamma$, that is, at each $x\in\{\varepsilon\}\times F_\ell$, we define a framing over the arc $f([0,\varepsilon]\times\{x\})$ by restricting $\gamma$ to a path in the space of linear isomorphisms $\mathrm{Iso}(T_x(N_{D^k}),T_{f(x)}Q^{2k+1})=\mathrm{Iso}(T_x(N_{D^k}),\mathbb{R}^{2k+1})$ so that $f((1-s)\varepsilon,x)$ is framed by $\gamma(s)$. The extension obtained is continuous and gives a framing on the complement of a slightly smaller embedding $f':(N'_{D^k})^{\sqcup 3}\to Q^{2k+1}$, where $N'_{D^k}=N_{D^k}\setminus\bigcup_\ell(0,\varepsilon]\times F_\ell$, such that its restriction to the boundary $\partial\overline{N'_{D^k}}$ agrees with the framing induced from the standard one of $N_{D^k}$ by $df$. That is, the complement of the image of $f'$ is equipped with a framing that is standard near the boundary. We obtain a map $\widetilde{c}:\overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3},Q^{2k+1})_{\widetilde{i}}\longrightarrow\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})$, by restriction to $(N'_{D^k})^{\sqcup 3}$ and by



taking the framed complement as above. The homotopy commutativity of (0.3) is obvious from the construction.

Proof of (b): To check that there is a canonical homotopy from $\widetilde{e}_\ell \circ \widetilde{c}$ to $\widetilde{c}_\ell \circ s_\ell$, where $\widetilde{e}_\ell$ is the extension map in (0.2) and $s_\ell : \overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}} \to \overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 2}, Q^{2k+1})_{\widetilde{i}}$ is the forgetful map of the $\ell$-th component, we need to show that the framing on $V_{[\ell]}$ obtained by $\widetilde{e}_\ell \circ \widetilde{c}$ can be homotoped in a canonical way to the one such that the part where the $\ell$-th thickened component is filled into is entirely framed by the canonical framing of $Q^{2k+1}$. This homotopy can be constructed by giving a homotopy of framing in the $\ell$-th thickened component, as follows. Let $(f, \gamma)$ be the pair that represents an element of $\overline{\mathrm{Emb}}_\partial((N_{D^k})^{\sqcup 3}, Q^{2k+1})_{\widetilde{i}}$ as in the previous paragraph. The framing provided on the $\ell$-th component of $\overline{\mathrm{Im}\, f}$ after filling into that component corresponds to a map $(D^k \times \overline{U}_h^{k+1}, \partial(D^k \times \overline{U}_h^{k+1})) \to (GL_{2k+1}(\mathbb{R}), \mathrm{id})$, which measures the difference from the canonical framing of $Q^{2k+1}$. By construction of the framing in the definition of $\widetilde{c}$, this map can be relatively homotoped into the identity. Namely, for the given pair $(f, \gamma)$, we may consider similar construction of framed filling at $\overline{\mathrm{Im}\, f}$ by using shortened path $\gamma_\lambda$:

$$\gamma_\lambda(t) = \begin{cases} \gamma(t) & t \in [0, 1-\lambda] \\ \gamma(1-\lambda) & t \in [1-\lambda, 1] \end{cases}$$

instead of $\gamma$. By considering this simultaneously for $0 \leq \lambda \leq 1$, we obtain a 1-parameter family of framings on $\overline{\mathrm{Im}\, f}$ for the given $f$, and the corresponding family of maps $(D^k \times \overline{U}_h^{k+1}, \partial(D^k \times \overline{U}_h^{k+1})) \to (GL_{2k+1}(\mathbb{R}), \mathrm{id})$ at the $\ell$-th component gives a relative null-homotopy. This relative null-homotopy can be taken as a parameterized family of null-homotopies over all $(f, \gamma)$. After the homotopy, the framing on the $\ell$-th component of $\overline{\mathrm{Im}\, f}$ agrees with the canonical one of $Q^{2k+1}$. This shows that $\widetilde{e}_\ell \circ \widetilde{c} \simeq \widetilde{c}_\ell \circ s_\ell$. □

**Corollary 5.** *The $(D_M, \partial)$-bundle $\pi^\Gamma : E^\Gamma \to (S^{k-1})^{2k}$ ([5, Definition 1]) admits a vertical framing that is standard near the boundary.*

The second lemma is an extension of Claim 3 to connected uni-trivalent trees in the sense that surgery on a connected trivalent graph $\Gamma$ can be considered as that of a uni-trivalent tree that is a spanning tree of $\Gamma$ with some hairs (or half-edges) attached to 1- and 2-valent vertices. Surgery on a uni-trivalent graph is explained in [5, §4.5]. Let $R = \{i_1, i_2, \ldots, i_r\}$ be a subset of $\{1, 2, \ldots, 2n\}$ which is the set of vertices of some connected subtree $L$ of $\Gamma$. Let $B\{R\} = S_1 \times S_2 \times \cdots \times S_{2n}$, where $S_j = S^{k-1}$ (if $j \in R$) or $S_j = \{*\}$ (if $j \notin R$). Let $\pi^\Gamma\{R\} : (\pi^\Gamma)^{-1}B\{R\} \to B\{R\}$ be the restriction of $\pi^\Gamma$ to the subset $B\{R\}$ of $B = (S^{k-1})^{\times 2n}$. Since by construction of $\pi^\Gamma$ the restriction of the fiber in the $(D^{2k+1}, \partial)$-bundle $\pi^\Gamma\{R\}$ to the complement of a subset $\mathcal{V}_R \subset \mathrm{Int}\, D^{2k+1}$ which includes $\bigcup_{j \in R} \mathcal{V}_j$ ($\mathcal{V}_j$ is defined in [5, §4.4(Step 3)]) gives a trivial bundle, which is also trivial as a framed bundle by the framing induced by the framing on $E^\Gamma$, the framed sub $(\mathcal{V}_R, \partial)$-bundle of $\pi^\Gamma\{R\}$ is naturally and uniquely determined. Claim 3 follows immediately from the special case $R = \{1, 2, \ldots, 2n\}$ of the following lemma.

**Lemma B.** *There exists a compact connected codimension 0 submanifold $\mathcal{V}_R$ of $\mathrm{Int}\, D^{2k+1}$ with boundary that satisfies the following conditions.*

1. *$\mathcal{V}_R \supset \mathcal{V}_j$ for all $j \in R$, and $\mathcal{V}_R \cap \mathcal{V}_j = \emptyset$ for all $j \notin R$.*
2. *The classifying map $\tilde{f}(\pi^\Gamma\{R\})_{\mathcal{V}_R} : B\{R\} \to \widetilde{B\mathrm{Diff}}(\mathcal{V}_R, \partial; \tau_R)$ for the framed sub $(\mathcal{V}_R, \partial)$-bundle of $\pi^\Gamma\{R\}$ factors up to homotopy over a map $(S^{k-1})^{\times r} \to S^{r(k-1)}$ of degree 1, where $\tau_R$ is the restriction of the standard framing to $\mathcal{V}_R$ in the fiber over the base point.*

The construction of $\mathcal{V}_R$ is easy: by taking the disjoint union $\coprod_{j \in R} \mathcal{V}_j$ and then extending the "Hopf links" between the handles of $\mathcal{V}_j$ for $j \in R$ to a ball by filling the complementary handles. Thus the rather nontrivial part of Lemma B is the condition (2).

*Proof of Lemma B.* We prove this by induction on $r = |R|$. The case $r = 1$ is obvious from the definition of surgery. Suppose that the lemma holds true for some $r = |R| \geq 1$ such that $r < 2n$. We shall check that the lemma holds true for any subset $R'$ of $\{1, 2, \ldots, 2n\}$ with $r + 1$ elements which is the set of vertices of a subtree $L'$ of $\Gamma$. Removing $i_{r+1}$ from $L'$ gives a connected subtree $L$ of $L'$ with the vertex set $R = R' \setminus \{i_{r+1}\}$. The proof of the lemma for this induction step is essentially the same as for Claim 1 when $2n = 2$, $B = S^{k-1} \times S^{k-1}$, which has already been treated in [5]. We assume that the $\ell$-th $k$-handle of $\mathcal{V}_{i_m}$ for some $i_m \in R$ and the $\ell'$-th $k$-handle of $\mathcal{V}_{i_{r+1}}$ are linked together by a Hopf link in the sense of [5, §4.4 (Step 2)]. We set $\mathcal{V}_{R'}$ to be a submanifold of $\mathrm{Int}\,D^{2k+1}$ obtained from $\mathcal{V}_R \cup \mathcal{V}_{i_m[\ell]} \cup \mathcal{V}_{i_{r+1}[\ell']}$ by smoothing the corners, where $\mathcal{V}_{i_m[\ell]}$ is the submanifold of $\mathrm{Int}\,D^{2k+1}$ obtained from $\mathcal{V}_{i_m}$ by adding a small tubular neighborhood of the standard (linear) spanning disk of the component of the Hopf link at the $\ell$-th $k$-handle of $\mathcal{V}_{i_m}$, and $\mathcal{V}_{i_{r+1}[\ell']}$ is a similar one for the $\ell'$-th $k$-handle of $\mathcal{V}_{i_{r+1}}$. Then $\mathcal{V}_{R'}$ satisfies the condition (1) for $R'$. A factorisation of $\tilde{f}(\pi^\Gamma\{R\})_{\mathcal{V}_R} : B\{R\} \to \widetilde{B\mathrm{Diff}}(\mathcal{V}_R, \partial; \tau_R)$ up to homotopy over a degree 1 map $B\{R\} \to S^{r(k-1)}$ obtained by the induction hypothesis induces a factorisation of $\tilde{f}(\pi^\Gamma\{R'\})_{\mathcal{V}_{R'}} : B\{R'\} \to \widetilde{B\mathrm{Diff}}(\mathcal{V}_{R'}, \partial; \tau_{R'})$ up to homotopy over a map $B\{R'\} \to S^{r(k-1)} \times S^{k-1}$ of degree 1, since $\mathcal{V}_R \cap \mathcal{V}_{i_{r+1}} = \emptyset$, the bundle restricted to $\mathcal{V}_{i_{r+1}}$ (resp. to $\mathcal{V}_R$) is independent of the first $(r-1)$ $S^{k-1}$-factors (resp. the last $S^{k-1}$-factor) in $B\{R'\}$, and the homotopy for $\tilde{f}(\pi^\Gamma\{R\})_{\mathcal{V}_R}$ can be done independently of $\mathcal{V}_{i_{r+1}}$.

By Lemma A, the restriction of $\tilde{f}(\pi^\Gamma\{R'\})_{\mathcal{V}_{R'}}$ to $\{*\} \times S^{k-1}$ is null-homotopic as a pointed map, since the restriction to $\{*\} \times S^{k-1}$ corresponds to trivializing on $\mathcal{V}_R$, and the $\ell'$-th $k$-handle of $\mathcal{V}_{i_{r+1}}$ is free in $\mathcal{V}_{R'}$, that is, there is a framed $k$-sphere in $\partial\mathcal{V}_{i_{r+1}}$ which intersects the belt sphere of the $\ell'$-th $k$-handle transversally by one point and which spans a compatibly framed $(k+1)$-disk in $\mathrm{Int}\,\mathcal{V}_{R'} \setminus \mathrm{Int}\,\mathcal{V}_{i_{r+1}}$. Since the $\ell'$-th $k$-handle of $\mathcal{V}_{i_{r+1}}$ is free, the base diffeomorphism $V \to \mathcal{V}_{i_{r+1}}$ in [5, p.641 (Step 4)] can be taken so that it extends to a diffeomorphism $V_{[\ell']} \to \mathcal{V}_{i_{r+1}[\ell']}$, so that Lemma A induces trivialization on $\mathcal{V}_{i_{r+1}[\ell']}$.

Also, by iterated applications of Lemma A for $\bigcup_{j=1}^r \mathcal{V}_{i_j}$, we may also see that the restriction of $\tilde{f}(\pi^\Gamma\{R'\})_{\mathcal{V}_{R'}}$ to $B\{R\} \times \{*\}$ is null-homotopic as a pointed map, since the $\ell$-th $k$-handle of $\mathcal{V}_{i_m}$ is free in $\mathcal{V}_{R'}$. Namely, after trivializing $\mathcal{V}_{i_m[\ell]}$ by Lemma A, a $k$-handle on another handlebody next to $\mathcal{V}_{i_m}$ in $\mathcal{V}_{R'}$ will become free in $\mathcal{V}_{R'}$, and we can apply Lemma A again, and so on.

Now by the induction hypothesis the restricted map on $B\{R\}\times\{*\}$ can be factored up to pointed homotopy into a degree 1 map $B\{R\}\times\{*\} \to S^{r(k-1)}\times\{*\}$ and a map $S^{r(k-1)}\times\{*\} \to \widetilde{B\mathrm{Diff}}(\mathcal{V}_{R'}, \partial; \tau_{R'})$, where the latter is null-homotopic for the following reason. For simplicity, we write $Y = \widetilde{B\mathrm{Diff}}(\mathcal{V}_{R'}, \partial; \tau_{R'})$, $X = B\{R\} \times \{*\}$, $g = \tilde{f}(\pi^\Gamma\{R'\})_{\mathcal{V}_{R'}}|_{B\{R\}\times\{*\}}$, and let $A$ be the $(r-1)(k-1)$-skeleton of $X$ for the canonical cell structure for the product of spheres. We have a pointed null-homotopy of the map $g : X \to Y$ and a pointed null-homotopy of the map $g|_A : A \to Y$‡. In other words, $g$ can be extended to a map $g' : X \cup_A CA \to Y$ by defining $g'$ on the reduced cone $CA$ by the null-homotopy of $g|_A$, and $g'$ can be further extended to a map $g'' : (X \cup_A CA) \cup_X CX \to Y$ by defining $g''$ on $CX$ by the null-homotopy of $g$. Note that $X \cup_A CA \simeq X/A \approx S^{r(k-1)}$ and $(X \cup_A CA) \cup_X CX \simeq \Sigma A$. Since $\Sigma A \to \Sigma X$ splits with cofiber $\Sigma(X/A)$ ([1, p.1662]§), the map $[\Sigma A, Y] \to [X/A, Y]$ in the the long exact sequence of sets of homotopy classes of pointed maps $[\Sigma X, Y] \to [\Sigma A, Y] \to [X/A, Y] \to [X, Y] \to [A, Y]$ is trivial. This shows that the null-homotopic map $g$ can be extended to a map $g'$ that is null-homotopic.

Hence $\tilde{f}(\pi^\Gamma\{R'\})_{\mathcal{V}_{R'}}$ can be factored up to homotopy into a degree 1 map $B\{R'\} \to S^{r(k-1)} \times S^{k-1}$ and a map $(S^{r(k-1)} \times S^{k-1}, S^{r(k-1)} \vee S^{k-1}) \to (\widetilde{B\mathrm{Diff}}(\mathcal{V}_{R'}, \partial; \tau_{R'}), *)$, and the latter one factors over a map $S^{r(k-1)} \times S^{k-1} \to S^{(r+1)(k-1)}$ of degree 1. This completes the proof. □

*Remark* 6. The points in the proof of [5, Theorem 6.1] where the assumption that $k$ is odd was needed are the following:

(a) [5, Proposition 4.2].
(b) "Since we have assumed that $k$ is odd, a non-vanishing product must be a multiple of the wedge of all $6n$ different $\eta^i_\ell(t_i)$s" in the proof of [5, Lemma 6.3].
(c) "because we have fixed cyclic orders of edges around each vertex so that it is compatible with the graph orientation" in the proof of [5, Lemma 6.3].

Lemma A gives a framing of [5, Proposition 4.2] concretely. Since Lemma A does not assume $k$ is odd, the point (a) is achieved for all $k \geq 2$ by replacing [5, Proposition 4.2] with Lemma A. Moreover, we may take $r_k = 1$ there.

The point (b) can be achieved for even $k$ as well, by assuming that we take disjoint parallel copies of $S(\widetilde{a}^i_\ell)$ to normalize propagators for different edges in [5, Proposition 5.2]. This is possible since the

‡These null-homotopies are chosen independently. The null-homotopy of $g|_A$ is induced from that of the map to $Y_R = \widetilde{B\mathrm{Diff}}(\mathcal{V}_R, \partial; \tau_R)$, whereas the null-homotopy of $g$ is not induced from that of a map to $Y_R$.

§In the notation of [1], $\Sigma A = \Sigma(Z(K; (\underline{X}, \underline{A})))$ for $K = \partial\Delta[m-1]$, $X_i = S^{k-1}$, $A_i = *$.

normal bundle of $S(\widetilde{a}^i_\ell)$ in $\widetilde{\mathcal{V}}_i$ is trivial. Then the square $\eta^i_\ell(t_i)\wedge\eta^i_\ell(t_i)$ in the original product of $6n$ $\eta^i_\ell(t_i)$s becomes zero.

The point (c) should be investigated separately since if $k$ is even, $\eta^i_\ell(t_i)$ is of even degree, and the correspondence between the sign of the form and the graph orientation is not direct as for $k$ odd, and the order of the three $\eta^i_\ell(t_i)$s at each vertex does not determine the sign. For even $k$, the linking numbers $L^{ij}_{\ell m}$ between $k$-spheres are antisymmetric and thus the form $\omega(\Gamma')(t)$ on $\mathcal{V}_1\times\cdots\times\mathcal{V}_{2n}$ depends on the choice of edge orientations. Moreover, each factor in the base space $(S^{k-1})^{\times 2n}$ is odd dimensional. Then the sign of the total integral of $\omega(\Gamma')$ will be determined by the choice of edge orientations and the correspondence $1\mapsto\mathcal{V}_1, 2\mapsto\mathcal{V}_2,\ldots,2n\mapsto\mathcal{V}_{2n}$ between the configuration of $2n$ points and the set of handlebodies $\mathcal{V}_i$. This is compatible with the graph orientation in $\det[\mathbb{R}^{\{\text{vertices of }\Gamma\}}\oplus\bigoplus_e\mathbb{R}^{H(e)}]$ in the sense that a change of vertex labelling and edge orientations result in different sign of the total integral if and only if the change of the data reverses the graph orientation.

*Remark* 7. More generally, Lemma A is also valid for an element of $\pi_i\mathrm{Emb}^{\mathrm{f}}_\partial(I^p\cup I^q\cup I^r, I^N)$ corresponding to the Borromean rings, where $p,q,r,N$ are any integers such that $(p+i)+q+r=2N-3$, $1\leq p,q,r,p+i\leq N-2$, $i>0$, $N\geq 3$, $I=[0,1]$, as one can see from the proof. In that case, the corresponding bundle is a bundle over $S^i$ with fiber diffeomorphic to $(D^{p+1}\times S^{p'})\natural(D^{q+1}\times S^{q'})\natural(D^{r+1}\times S^{r'})$, where $p'+p=q'+q=r'+r=N-1$. Also, one can also get an analogous result for $i=0$ by replacing $\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})$ with the disjoint union of $\widetilde{B\mathrm{Diff}}(V,\partial;\mathrm{st})$ for relative diffeomorphism classes of some framed manifolds $V$ that have fixed behaviour on the boundary. For $i=0$, the analogue of Lemma A is that $\delta(\alpha')$ is mapped to the component of the standard manifold of the form $(D^{p+1}\times S^{p'})\natural(D^{q+1}\times S^{q'})$ with the standard framing induced from that of the $N$-disk.

*Remark* 8. We would also like to fix some typos in [5] which did not affect the result, and yet might cause confusion.

- The coefficient $\sum_{(\ell_i,m_i)\in P_{ij}}L^{ij}_{\ell_i m_j}$ in the second equation in the proof of [5, Lemma 6.3] should be $|P_{ij}|!\prod_{(\ell_i,m_i)\in P_{ij}}L^{ij}_{\ell_i m_j}$, where we set $|P_{ij}|!=0$ when $P_{ij}=\emptyset$ unlike the usual convention. The two formulas happen to give the same result if $\Gamma'$ does not have multiple edges. However, we haven't used the wrong formula to get the final formula, which is correct. We thank Peter Teichner for helping us find this.
- In the first equation of the "Case 2" in the proof of [5, Proposition 7.1], the term $(\overline{C}_1(M,\infty)\setminus\mathcal{V}_\ell[3])\times\mathcal{V}_\ell$ should be replaced by $(\overline{C}_1(M,\infty)\setminus(\mathcal{V}_\ell[3]\cup\mathcal{V}_i[1]))\times\mathcal{V}_\ell$. Otherwise, the argument after the equation is useless since the deleted part $\mathcal{V}_i[1]\times\mathcal{V}_\ell$ will be covered by that argument. Nevertheless, normalization on the part $\mathcal{V}_i[1]\times\mathcal{V}_\ell$ has already been correctly investigated and there is no problem in the proof in the end.
- In [5, Appendix A.2], the case where $\infty\in A$ is not explained, even though we were aware that such a case needs to be checked. The following sentence should be added in [5, Appendix A.2]: "When $\infty\in A$, then we have $E\mathcal{S}_A=\overline{C}^{\mathrm{local}}_j(\mathbb{R}^d)\times E\overline{C}_{n-j+1}(\pi)$, and we have $\omega(\Gamma)|_{E\mathcal{S}_A}=\mathrm{pr}_1^*\omega(\Gamma/A^c)\wedge\mathrm{pr}_2^*\omega(\Gamma_{A^c})$." Also, the following sentence should be added at the beginning of the proof of [5, Lemma A.1]: "When $\infty\notin A$, let $\Gamma_A$ be the subgraph of $\Gamma$ obtained by restricting the vertices to $A$. When $\infty\in A$, let $\Gamma_A$ be $\Gamma/A^c$, where $A^c=\{1,\ldots,n\}-A$." We refer the reader to [6] for more detail.
- An author (Prof. B. McKay) is missing from the third item in [5, References]. We apologize for this typo. Also, the link for the third item is no longer available. Currently available link is [2]. A permanent reference for the computations is Table 2 of [3].

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