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# UNSTABLE PSEUDO-ISOTOPIES OF SPHERICAL 3-MANIFOLDS

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ABSTRACT. In our previous works, we constructed diffeomorphisms of compact 4-manifolds $X$ by surgeries on theta-graphs embedded in $X$. In this paper, we consider the case $X = M \times I$, where $M$ is a spherical 3-manifold. For some of such $X$, we compute lower bounds of the ranks of the abelian groups $\pi_0\mathrm{Diff}(X, \partial)$. We study the behavior of the elements constructed by theta-graph surgery under the suspension functor in stable pseudo-isotopy theory, and their triviality in the space of block diffeomorphisms.

## 1. Introduction

For a finite group $\pi$, let $(\pi\times\pi)\rtimes\mathbb{Z}_2$ be the semidirect prodct of $\pi\times\pi$ and $\mathbb{Z}_2$ with respect to the homomorphism $\psi\colon \mathbb{Z}_2 = \{1, \tau\} \to \mathrm{Aut}(\pi \times \pi)$; $\tau \mapsto ((x, y) \mapsto (y, x))$. For a left $(\pi \times \pi) \rtimes \mathbb{Z}_2$-module $W$, let

$$\mathscr{A}_\Theta^{\mathrm{even}}(W) := H^0((\pi \times \pi) \rtimes \mathbb{Z}_2; \textstyle\bigwedge^3 W) = \left(\textstyle\bigwedge^3 W\right)^{(\pi\times\pi)\rtimes\mathbb{Z}_2},$$
$$\mathscr{A}_\Theta^{\mathrm{odd}}(W) := H^0((\pi \times \pi) \rtimes \mathbb{Z}_2; \mathrm{Sym}^3 W) = \left(\mathrm{Sym}^3 W\right)^{(\pi\times\pi)\rtimes\mathbb{Z}_2},$$

where we take the invariant parts. One can take $W = \mathbb{C}[\pi]$, which is both a left $\pi\times\pi$-module by $(g, h)\cdot x \mapsto gxh^{-1}$, and a $\mathbb{Z}_2$-module by the involution $x\otimes y\otimes z \mapsto x^{-1} \otimes y^{-1} \otimes z^{-1}$ $(x, y, z \in \pi)$. Also, one can take $W = \mathrm{Ker}\,\varepsilon$ for the augmentation map $\varepsilon : \mathbb{C}[\pi] \to \mathbb{C}$, which is a $(\pi \times \pi) \rtimes \mathbb{Z}_2$-submodule of $\mathbb{C}[\pi]$. The definitions of the $\mathscr{A}_\Theta$-spaces are motivated by the space of $\pi$-decorated 2-loop graphs in [GL]. Let $\mathrm{Diff}_0(X, \partial)$ denote the subgroup of $\mathrm{Diff}(X, \partial)$ of diffeomorphisms homotopic to the identity. In this paper, we consider the 4-manifold $X = M \times I$, where $M$ is a spherical (or elliptic) 3-manifold, i.e. $M = S^3/\Gamma$ for a finite subgroup $\Gamma \subset SO(4)$ acting freely on $S^3$ (e.g., [Sa, §1.2.1]). The following theorem is essentially given in [Wa1, Proposition 7.1 and Remark 7.2].

**Theorem 1.1.** *Let $M$ be a spherical 3-manifold, $\pi = \pi_1 M$, and let $X = M\times I$. Let $W$ be a $\mathbb{C}[(\pi \times \pi) \rtimes \mathbb{Z}_2]$-module satisfying Assumption 2.2. Then a homomorphism*

$$Z_\Theta^{\mathrm{even}}\colon \Gamma_0(X) \otimes \mathbb{C} \to \mathscr{A}_\Theta^{\mathrm{even}}(W)$$

*from a certain subgroup $\Gamma_0(X)$ of $\pi_1 B\mathrm{Diff}_0(X, \partial)$ is defined, and if $W = \mathrm{Ker}\,\varepsilon$, it is surjective. Hence we have*

$$\dim \pi_0\mathrm{Diff}_0(X, \partial) \otimes \mathbb{Q} \geq \dim \mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon).$$

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Note that for $X = M\times I$, the group $\mathrm{Diff}(X,\partial)$ is homotopy commutative ([ABK, 2.6.1 Lemma]) and hence $\pi_0\mathrm{Diff}_0(X,\partial)$ is abelian. Theorem 1.1 can be obtained by modifying a few points (§2 and §3 below) in [Wa1, Proposition 7.1 and Remark 7.2], where we defined an invariant of diffeomorphisms of $X$ with certain structures by a method similar to [Mar, Les1, Les2] ("equivariant triple intersection" in a configuration space). The aim of this paper is to give some examples of computations of the lower bound in Theorem 1.1.

The lower bound $\dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon)$ of Theorem 1.1 can be computed by simple calculations. We give a few examples: the Poincaré homology sphere $\Sigma(2,3,5)$ and the lens spaces $L(n,q)$. It is known that $\pi_1\Sigma(2,3,5)$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_5)$, which is also known as the binary icosahedral group, and $\pi_1 L(n,q)\cong\mathbb{Z}_n$ (e.g., [Sa, §1.1.2, §1.2.1]).

**Proposition 1.2** (Poincaré homology sphere, Proposition 4.4)**.** *When $\pi = \mathrm{SL}_2(\mathbb{F}_5)$, we have the following.*

1. $\dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon) = \dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi]) = 27$*.*
2. $\dim\mathscr{A}_\Theta^{\mathrm{odd}}(\mathrm{Ker}\,\varepsilon) = 56$*,* $\dim\mathscr{A}_\Theta^{\mathrm{odd}}(\mathbb{C}[\pi]) = 65$*.*

The case of lens spaces can be computed by a more elementary method.

**Proposition 1.3** (Lens spaces, Proposition 5.2)**.** *Let $\pi = \mathbb{Z}_n$ ($n\geq 1$), and for an integer $m\geq 0$, let $p_3(m)$ be the number of partitions of $m$ into at most three parts, namely, the number of integer solutions of the equation $x+y+z=m$ ($0\leq x\leq y\leq z$). We set $p_3(m)=0$ for $m<0$. Then we have the following.*

1. $\dim\mathscr{A}_\Theta^{\mathrm{odd}}(\mathbb{C}[\pi]) = p_3(n)$*.*
2. $\dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi]) = p_3(n-6)$*.*
3. $\dim\mathscr{A}_\Theta^{\mathrm{odd}}(\mathrm{Ker}\,\varepsilon) = p_3(n-3)$*.*
4. $\dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon) = \dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi]) = p_3(n-6)$*.*

The following is a table of the values of the dimensions of $\mathscr{A}_\Theta^{\mathrm{even/odd}}(\mathbb{C}[\pi])$ and $\mathscr{A}_\Theta^{\mathrm{even/odd}}(\mathrm{Ker}\,\varepsilon)$ for $n\leq 15$. It is known that $p_3(m)$ ($m\geq 0$) is the nearest integer to $\frac{(m+3)^2}{12}$ (e.g. [AK, Ch. 6]).

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dim\mathscr{A}_\Theta^{\mathrm{odd}}(\mathbb{C}[\pi])$ | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 | 14 | 16 | 19 | 21 | 24 | 27 |
| $\dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi])$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 |
| $\dim\mathscr{A}_\Theta^{\mathrm{odd}}(\mathrm{Ker}\,\varepsilon)$ | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 | 14 | 16 | 19 |
| $\dim\mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon)$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 10 | 12 |

Although the results for $\mathscr{A}_\Theta^{\mathrm{odd}}$ are not used in this paper, we think these would be useful to study finite type invariants of 3-manifolds in [GL].

Let $C(M) = \mathrm{Diff}(M\times I, M\times\{0\}\cup\partial M\times I)$, the group of diffeomorphisms of $M\times I$ pointwise fixing $M\times\{0\}\cup\partial M\times I$, equiped with the $C^\infty$-topology. An element of $C(M)$ is called a *pseudo-isotopy* or a *concordance* of $M$. Pseudo-isotopy theory (e.g., [Ce, HW, Ig1]) studies the topology of $C(M)$, which is related to the diffeomorphism groups via the following fiber sequence

$$\mathrm{Diff}(M\times I,\partial)\to C(M)\xrightarrow{r}\mathrm{Diff}(M), \tag{1.1}$$

where $r$ is the restriction to $M \times \{1\}$. For most 3-manifolds $M$, the left map in this sequence induces a map between $\pi_0$ which is close to an isomorphism, in the sense that $\pi_i \mathrm{Diff}(M)$ is small in many cases (generalized Smale conjecture, e.g., [Hat, Ga, HKMR, BK]).

**Corollary 1.4.** *We have the following inequalities.*

1. $\dim \pi_0 C(\Sigma(2,3,5)) \otimes \mathbb{Q} \geq 27$.
2. $\dim \pi_0 C(\Sigma(2,3,5) \times I) \otimes \mathbb{Q} \geq 27$.
3. $\dim \pi_0 C(L(n,q) \times I) \otimes \mathbb{Q} \geq p_3(n-6)$.

*Proof.* For (1), since $\mathrm{Diff}(\Sigma(2,3,5)) \simeq \mathrm{Isom}(\Sigma(2,3,5)) = SO(3)$ by [BK], and $\pi_0 SO(3) = 0$, the natural map $\pi_0 \mathrm{Diff}(\Sigma(2,3,5) \times I, \partial) \to \pi_0 C(\Sigma(2,3,5))$ from (1.1) is surjective, and the image is abelian, too. Moreover, its kernel is the image from $\pi_1 SO(3) = \mathbb{Z}_2$. Hence we have an isomorphism $\pi_0 \mathrm{Diff}(\Sigma(2,3,5) \times I, \partial) \otimes \mathbb{Q} \cong \pi_0 C(\Sigma(2,3,5)) \otimes \mathbb{Q}$. Then the result follows by Theorem 1.1 and Proposition 4.4.

For (2) and (3), we use the fact that theta-graph surgeries that are detected by $Z_\Theta^{\mathrm{even}}$ lifts to $\pi_1 BC(M \times I)$ ([BW, Theorem 1.3]). Then the results follow by Theorem 1.1 and Propositions 4.4, 5.2. $\square$

We compare our nontrivial subgroup in $\pi_0 C(M \times I)$ with Hatcher–Wagoner's stable pseudo-isotopy theory $P(M \times I) = \mathrm{colim}\, C(M \times I^N)$ ([HW], see also [Ig1]), where the colimit is taken with respect to the "suspension functor" $\sigma \colon C(X) \to C(X \times I)$ ([HW, Ch I-§5]). In [Ig2], the following commutative diagram is considered for $\dim X = 4$:

$$\begin{array}{ccccccc}
 & & \pi_0 \mathscr{D}(X) & \xrightarrow{\widetilde{\theta}} & \pi_0 C(X) & & \\
 & & \Big\downarrow {\scriptstyle \overline{\lambda}} & & \Big\downarrow & & \\
 & & Wh_1^+(\pi_1 X; \mathbb{Z}_2 \oplus \pi_2 X) & \longrightarrow & \pi_0 P(X) & \longrightarrow & Wh_2(\pi_1 X) \longrightarrow 0
\end{array}$$

where the bottom horizontal line is exact (by [HW]) and $\mathscr{D}(X)$ is the space of "lens-space models" for pseudo-isotopies of $X$ ([Ig2, Definition 1.4]). In recent works of K. Igusa ([Ig1, Ig2]) and O. Singh ([Si]), many nontrivial elements of $\pi_0 C(X)$ for some 4-manifolds $X$ with $\pi_2 X \neq 0$ are found, by realizing elements of $Wh_1^+(\pi_1 X; \mathbb{Z}_2 \oplus \pi_2 X)$ by explicit 1-parameter families in $\mathscr{D}(X)$ of 2,3-handle pairs. Their nontrivial elements are also nontrivial in $\pi_0 P(X)$.

We see that our theta-graph surgery behaves differently.

**Theorem 1.5.** *Let $M$ be a spherical 3-manifold.*

1. *Theta-graph surgery gives elements of $\pi_0 C(M \times I)$ that lift to $\pi_0 \mathscr{D}(M \times I)$.*
2. *$\pi_0 C(M \times I)$ includes a free abelian subgroup of rank $\dim \mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\, \varepsilon)$ that is included in the kernel of $\pi_0 C(M \times I) \to \pi_0 P(M \times I)$.*

*Proof.* (1) follows from [Wa1, §8], which uses a result of [BW]. Then (2) follows since $\pi_2(M \times I) = \pi_2 M = 0$, $\pi_1(M \times I) = \pi_1 M$, and

$$Wh_1^+(\pi_1 M; \mathbb{Z}_2) = \bigoplus_{c-1} \mathbb{Z}_2,$$

where $c$ is the number of conjugacy classes of elements of $\pi_1 M$ ([Ig2, p.3]). $\square$

*Remark* 1.6. The subgroup of Theorem 1.5 (2) is of finite index ($=2^{c-1}$) in the free abelian subgroup of rank $\dim \mathscr{A}^{\mathrm{even}}_\Theta(\mathrm{Ker}\,\varepsilon)$ generated by theta-graph surgery. This restriction to the smaller subgroup would be unnecessary since Theorem 1.5 (2) could also be proved by directly evaluating the homomorphism $\overline{\lambda}\colon \pi_0\mathscr{D}(X) \to Wh_1^+(\pi_1 X;\mathbb{Z}_2)$ for the 1-parameter family of the attaching 2-spheres of the 3-handles in [Wa1, §8] obtained from theta-graph surgery.

There is a similar result for the space $C^{\mathrm{Top}}(M)$ of topological pseudo-isotopies by Kwasik and Schultz [KS, Theorem 1] giving nontrivial unstable elements of $\pi_0 C^{\mathrm{Top}}(M)$ for certain 3-manifolds $M$.

Let $\widetilde{\mathrm{Diff}}(M)$ be the space of block diffeomorphisms of $M$ (see e.g., [HLLRW]). In [Hat3, Propopsition 2.1], Hatcher constructed a spectral sequence with $E^1_{pq} = \pi_q C(M\times I^p)$ converging to $\pi_{p+q+1}(\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M))$. In particular,

$$\pi_1(\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M)) = E^2_{00} = E^1_{00}/\delta_*(E^1_{10}) = \pi_0 C(M)/\delta_*(\pi_0 C(M\times I)),$$

where $\delta\colon C(M\times I)\to C(M)$ is defined by $\delta(f) = f|_{M\times I\times\{1\}}$. Similar identity for the topological case was considered in [KS, p.874]. Since the elements of $\pi_0 C(M)$ constructed by theta-graph surgery are in the image of $\delta_*$ ([BW, Theorem 1.3]), we have the following.

**Proposition 1.7.** *Let $M$ be a spherical 3-manifold. The elements of $\pi_0 C(M)$ constructed by theta-graph surgery are trivial in $\pi_1(\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M))$.*

This shows that theta-graph surgery is not like the unstable elements of $\pi_0 C^{\mathrm{Top}}(M)$ detected in [KS], where the nontrivial elements are detected in $\pi_1(\widetilde{\mathrm{Top}}(M)/\mathrm{Top}(M))$.

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## 2. Twisted homology of configuration space

In this and the next section, we check Theorem 1.1. We use a slightly generalized version of the invariant $Z^{\mathrm{even}}_\Theta$ of [Wa1] for more general local coefficient system $W$ (Lemma 2.3). To define $Z^{\mathrm{even}}_\Theta$, we need "propagators" in a family of configuration spaces of two points on $X = M\times I$. Now we check that a propagator exists for more general coefficients $W$.

2.1. **Acyclic complex.** Suppose that $M$ is a spherical 3-manifold. Let $\pi = \pi_1 M$, and let $A$ be a non-trivial irreducible $\mathbb{C}\pi$-module. The homology of $M$ with twisted coefficient $A$ is defined by

$$H_*(M;A) := H(S_*(\widetilde{M})\otimes_{\mathbb{C}\pi} A),$$

where $\widetilde{M}$ is the universal cover of $M$, $S_*(\cdot)$ is the $\mathbb{C}$-complex of singular chains.

**Lemma 2.1.** $H_*(M;A) = 0$.

*Proof.* Since $\pi$ is finite, $\mathbb{C}\pi$ is semisimple in the sense of [CE, §I.4] by Maschke's theorem and we have $H^1(\pi; A) = 0$ (Theorem VI.16.6 and Lemma VI.16.7 of [HS]). By the universal coefficient theorem, which is valid if the ring is hereditary (e.g., $\mathbb{C}\pi$ for $\pi$ finite), the sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{C}\pi}(H_{i-1}(C), A) \to H^i(\mathrm{Hom}_{\mathbb{C}\pi}(C, A)) \to \mathrm{Hom}_{\mathbb{C}\pi}(H_i(C), A) \to 0$$

is exact for $C = S_*(S^3; \mathbb{C})$ (as a $\mathbb{C}\pi$-module) and any $\mathbb{C}\pi$-module $A$ (e.g., [CE, Theorem VI.3.3]). Hence we have

$$\begin{aligned} &H^3(M; A) \cong \mathrm{Hom}_{\mathbb{C}\pi}(\mathbb{C}, A) \cong H^0(\pi; A) = A^\pi, \\ &H^2(M; A) = 0, \\ &H^1(M; A) \cong \mathrm{Ext}^1_{\mathbb{C}\pi}(\mathbb{C}, A) = H^1(\pi; A) = 0, \\ &H^0(M; A) \cong \mathrm{Hom}_{\mathbb{C}\pi}(\mathbb{C}, A) \cong H^0(\pi; A) = A^\pi. \end{aligned}$$

Since $A$ is a non-trivial irreducible $\mathbb{C}\pi$-module, we have $A^\pi = 0$. Then by Poincaré duality (e.g., [Hat2, §3.H], [Hatt, Theorem 7.17] etc.), we also have $H_*(M; A) = 0$. □

2.2. **Propagator for spherical 3-manifolds** $M$**.** Let $\Delta_M$ be the diagonal of $M \times M$. The configuration space of two points of $M$ is

$$\mathrm{Conf}_2(M) = M \times M - \Delta_M.$$

The Fulton–MacPherson compactification of $\mathrm{Conf}_2(M)$ is

$$\overline{\mathrm{Conf}}_2(M) = B\ell_{\Delta_M}(M \times M).$$

We idenfity the boundary $\partial\overline{\mathrm{Conf}}_2(M)$, which is the unit normal sphere bundle of $\Delta_M \subset M \times M$, with $ST(M)$, the unit tangent sphere bundle. We make the following assumption.

**Assumption 2.2.** A $\pi \times \pi$-module $W$ satisfies the following conditions:

(1) $H_*(M \times M; W) = 0$.
(2) There are elements $e^1_W, \dots, e^r_W \in W$ on which $\pi \times \pi$ acts trivially such that $H_*(\Delta_M; W) \cong H_*(M; \mathbb{C})^{\oplus r}$, which is generated over $\mathbb{C}$ by $[* \otimes e^i_W]$ and $[M \otimes e^i_W]$ $(i = 1, \dots, r)$.

For a non-trivial irreducible $\pi$-module $A$, let $A \boxtimes A^*$ denote the pullback of the local coefficient system $A \boxtimes_{\mathbb{C}} A^*$ on $M \times M$ to $\overline{\mathrm{Conf}}_2(M)$. We denote by $A \otimes A^*$ the restriction of $A \boxtimes A^*$ to $\partial\overline{\mathrm{Conf}}_2(M)$, on which $\pi$ acts diagonally.

**Lemma 2.3.** *The following $\pi \times \pi$-modules $W$ satisfy Assumption 2.2.*

(1) $W = A \boxtimes A^*$.
(2) $W = \bigoplus_i (A_i \boxtimes A_i^*)$, *where $A_i$ is a non-trivial irreducible $\pi$-module.*
(3) $W = \mathrm{Ker}\,\varepsilon$, *where $\varepsilon\colon \mathbb{C}\pi \to \mathbb{C}$ is the augmentation map.*

*Proof.* That (1) satisfies Assumption 2.2 follows from Lemma 2.1 and the Künneth formula for $\mathbb{C}\pi$-modules. The case (2) follows from (1) and

$$\begin{aligned} &H_*(M \times M; \textstyle\bigoplus_i (A_i \boxtimes A_i^*)) = \bigoplus_i H_*(M \times M; A_i \boxtimes A_i^*), \\ &H_*(\Delta_M; \textstyle\bigoplus_i (A_i \otimes A_i^*)) = \bigoplus_i H_*(\Delta_M; A_i \otimes A_i^*). \end{aligned}$$

The case (3) follows from the $\pi\times\pi$-module decomposition

$$\mathbb{C}\pi \cong \bigoplus_i \mathrm{End}(A_i), \quad \mathrm{Ker}\,\varepsilon \cong \bigoplus_{i\neq[1]} \mathrm{End}(A_i) \cong \bigoplus_{i\neq[1]} (A_i \boxtimes A_i^*), \tag{2.1}$$

(e.g., [FH, Proposition 3.29]) where $i$ is taken over the conjugacy classes in $\pi$, and from (2). □

**Lemma 2.4.** *Let $W$ be a $\pi\times\pi$-module satisfying Assumption 2.2. Then we have*

$$H_k(\overline{\mathrm{Conf}}_2(M);W) \cong \begin{cases} \mathbb{C}\{[ST(*)\otimes e_W^i] \mid i=1,\ldots,r\} & (k=2),\\ \mathbb{C}\{[ST(M)\otimes e_W^i] \mid i=1,\ldots,r\} & (k=5),\\ 0 & (\textit{otherwise}),\end{cases}$$

*where for an oriented submanifold $\sigma$ of $M$, we denote by $ST(\sigma)$ the restriction of the unit tangent 2-sphere bundle $ST(M)$ to $\sigma$.*

*Proof.* We consider the exact sequence

$$H_{i+1}(M^{\times 2};W) \to H_{i+1}(M^{\times 2},\mathrm{Conf}_2(M);W) \to H_i(\mathrm{Conf}_2(M);W) \to H_i(M^{\times 2};W),$$

where $H_*(M^{\times 2};W)=0$ by Assumption 2.2 (1). Letting $N(\Delta_M)$ be a closed tubular neighborhood of $\Delta_M$, we have

$$H_{i+1}(M^{\times 2},\mathrm{Conf}_2(M);W) \cong H_{i+1}(N(\Delta_M),\partial N(\Delta_M);W)$$

by excision. Since $M$ is parallelizable and the normal bundle of $\Delta_M$ can be canonically identified with $TM$, the normal bundle of $\Delta_M$ is trivial. By Assumption 2.2 (2), we have

$$\begin{aligned} H_{i+1}(N(\Delta_M),\partial N(\Delta_M);W) &= H_3(D^3,\partial D^3;\mathbb{C})\otimes_{\mathbb{C}} H_{i-2}(\Delta_M;W)\\ &\cong H_3(D^3,\partial D^3;\mathbb{C})\otimes_{\mathbb{C}} H_{i-2}(M;\mathbb{C})^{\oplus r} \cong H_{i-2}(M;\mathbb{C})^{\oplus r}.\end{aligned}$$

Here, $H_{i-2}(M;\mathbb{C})$ is rank 1 for $i-2=0,3$, and its generator is $*, M$, respectively. □

**Lemma 2.5.** *Let $W$ be a $\pi\times\pi$-module satisfying Assumption 2.2. Let $s_{\tau_0}\colon M\to ST(M)$ be the section given by the normalization of the first vector of a framing $\tau_0$ of $M$. Then we have*

$$H_3(\partial\overline{\mathrm{Conf}}_2(M);W) = \mathbb{C}\{[s_{\tau_0}(M)\otimes e_W^i] \mid i=1,\ldots,r\}.$$

*Proof.* This follows from the trivialization $\partial\overline{\mathrm{Conf}}_2(M)\cong S^2\times M$ induced by $\tau_0$, Assumption 2.2 (2), and the Künneth formula for $\mathbb{C}$-modules. □

**Lemma 2.6** (Propagator)**.** *Let $W$ be a $\pi\times\pi$-module satisfying Assumption 2.2.*

(1) *There exists a 4-chain $\omega^i$ of $\overline{\mathrm{Conf}}_2(M)$ with coefficients in $W$ that is transversal to the boundary and that satisfies*

$$\partial_W\omega^i = s_{\tau_0}(M)\otimes e_W^i.$$

(2) *For a fixed framing $\tau_0$ and $i$, the extension $\omega^i$ is unique in the sense that for two such extensions $\omega^i$ and $\omega'^i$ that agree near the boundary, there is a 5-chain $\eta$ of $\mathrm{Int}\,\overline{\mathrm{Conf}}_2(M)$ with coefficients in $W$ such that*

$$\omega'^i - \omega^i = \partial_W\eta.$$

*We call the direct sum of such extensions $\omega=\sum_i\omega^i$ a* propagator *for $\tau_0$.*

*Proof.* The assertion (1) follows immediately from the long exact sequence

$$H_4(\overline{C};W) \to H_4(\overline{C},\partial\overline{C};W) \to H_3(\partial\overline{C};W) \to H_3(\overline{C};W),$$

where we abbreviate as $\overline{C} = \overline{\mathrm{Conf}}_2(M)$, and $H_4(\overline{C};W) = H_3(\overline{C};W) = 0$ by Lemma 2.4. Here, both $[\omega^i]$ and $[s_{\tau_0}(M)\otimes e^i_W]$ restrict to the same generator of the homology of $*\times S^2 \subset SN\Delta_M$, their homology classes agree. The assertion (2) follows since the difference $\omega'^i - \omega^i$ vanishes near $\partial\overline{C}$ and represents 0 of the twisted homology $H_4(\overline{C};W)$. □

2.3. **Twisted homologies of the configuration space of $M\times I$ and its family.** Let $\overline{\mathrm{Conf}}_2(M\times I) = B\ell_{\Delta_{M\times I}}((M\times I)\times(M\times I))$, which is not a smooth manifold with corners, but satisfies the Poincaré–Lefschetz duality ([Wa2, C.3]). For an $(M\times I)$-bundle $p: E\to S^1$ over $S^1$ with structure group $\mathrm{Diff}(M\times I,\partial)$, we denote by

$$\overline{\mathrm{Conf}}_2(p)\colon E\overline{\mathrm{Conf}}_2(p) \to S^1$$

the associated $\overline{\mathrm{Conf}}_2(M\times I)$-bundle with structure group $\mathrm{Diff}(M\times I,\partial)$.

To define the invariant $Z^{\mathrm{even}}_\Theta$ as in [Wa1, Proposition 7.1 and Remark 7.2], we need a *propagator in family*, which is a 6-chain of $E\overline{\mathrm{Conf}}_2(p)$ with coefficients in $W$ satisfying some boundary condition similar to Lemma 2.6, and is implicitly defined in the proof of [Wa1, Proposition 7.1]. In [Wa1], the existence of such a 6-chain was guaranteed by the lemmas [Wa1, Lemmas 7.3 and 7.4]. The analogues of the lemmas for the $W$ in this paper are the following, whose proofs are the same except that the invariant element $c_A$ ([Wa1, Assumption 3.6]) is replaced with $\sum_i e^i_W$.

**Lemma 2.7** ([Wa1, Lemma 7.3])**.** *Let $W$ be a $\pi\times\pi$-module satisfying Assumption 2.2. Then we have*

$$H_k(\overline{\mathrm{Conf}}_2(M\times I);W) \cong \begin{cases} \mathbb{C}\{[ST(*)\otimes e^i_W] \mid i=1,\ldots,r\} & (k=3),\\ \mathbb{C}\{[ST(M)\otimes e^i_W] \mid i=1,\ldots,r\} & (k=6),\\ 0 & (\textit{otherwise}),\end{cases}$$

*where we identify $M$ with $M\times\{\frac{1}{2}\}$ in $M\times I$, and for an oriented submanifold $\sigma$ of $M\times I$, we denote by $ST(\sigma)$ the restriction of the unit tangent 3-sphere bundle $ST(M\times I)$ to $\sigma$.*

**Lemma 2.8** ([Wa1, Lemma 7.4])**.** *Let $W$ be a $\pi\times\pi$-module satisfying Assumption 2.2. Then we have $H_5(E\overline{\mathrm{Conf}}_2(p);W)=0$ and the natural map*

$$H_6(E\overline{\mathrm{Conf}}_2(p);W) \to H_6(E\overline{\mathrm{Conf}}_2(p),\partial E\overline{\mathrm{Conf}}_2(p);W)$$

*is zero. Thus the connecting homomorphism*

$$H_6(E\overline{\mathrm{Conf}}_2(p),\partial E\overline{\mathrm{Conf}}_2(p);W) \to H_5(\partial E\overline{\mathrm{Conf}}_2(p);W)$$

*is an isomorphism.*

Roughly, a propagator in family $E\overline{\mathrm{Conf}}_2(p)$ is constructed as follows. The boundary of $\overline{\mathrm{Conf}}_2(X)$ is $p^{-1}_{B\ell}(\Delta_X\cup(\partial X\times X)\cup(X\times\partial X))$, where $p_{B\ell}\colon \overline{\mathrm{Conf}}_2(X)\to X\times X$ is the canonical blow-down projection.

- On the stratum of $\partial E\overline{\mathrm{Conf}}_2(p)$ corresponding to the part $p_{B\ell}^{-1}(\Delta_X)$, we take the 5-chain $s_\tau(E)\otimes e_W^i$, where $s_\tau\colon E\to ST^vE$ is the section given by the normalization of the first vector of a vertical framing $\tau\colon T^vE=\mathrm{Ker}\,dp\overset{\cong}{\to}\mathbb{R}^4\times E$.
- On the stratum of $\partial E\overline{\mathrm{Conf}}_2(p)$ corresponding to the part $p_{B\ell}^{-1}((\partial X\times X)\cup(X\times\partial X))$, we take the pullbacks of the copies of $\omega^i$ of Lemma 2.6 in the subspace $p_{B\ell}^{-1}((M\times\{0\})\times(M\times\{0\}))\cup p_{B\ell}^{-1}((M\times\{1\})\times(M\times\{1\}))\cong\overline{\mathrm{Conf}}_2(M)\coprod\overline{\mathrm{Conf}}_2(M)$. This makes sense since the bundle $p$ is trivialized over $\partial X$.

The sum of these 5-chains of $\partial E\overline{\mathrm{Conf}}_2(p)$ is a cycle. Then by Lemma 2.8, it has an extension to a 6-chain $\widetilde{\omega}^i$ of $E\overline{\mathrm{Conf}}_2(p)$. We call $\widetilde{\omega}=\sum_i\widetilde{\omega}^i$ a propagator in family.

## 3. Properties of the invariant $Z_\Theta^{\mathrm{even}}$

Roughly, the invariant $Z_\Theta^{\mathrm{even}}$ is defined by choosing three propagators $\widetilde{\omega}_1,\widetilde{\omega}_2,\widetilde{\omega}_3$ in family $E\overline{\mathrm{Conf}}_2(p)$ with some boundary conditions and then by

$$Z_\Theta^{\mathrm{even}}(\widetilde{\omega}_1,\widetilde{\omega}_2,\widetilde{\omega}_3)=\frac{1}{6}\mathrm{Tr}_\Theta\langle\widetilde{\omega}_1,\widetilde{\omega}_2,\widetilde{\omega}_3\rangle_\Theta\in\mathscr{A}_\Theta^{\mathrm{even}}(W),$$

where $\mathrm{Tr}_\Theta\colon W^{\otimes 3}\to\mathscr{A}_\Theta^{\mathrm{even}}(W)$ is the projection, $\langle -,-,-\rangle_\Theta$ is the triple intersection among chains with twisted coefficients ([Wa1, §5]). We will not repeat the detailed definition here. The only difference in the proof of the well-definedness of $Z_\Theta^{\mathrm{even}}$ is to replace the invariant element $c_A$ in [Wa1, Proofs of Theorem 5.3 and Propositon 7.1] with $\sum_i e_W^i$. The property we need is the following.

**Proposition 3.1** ([Wa1, Theorem 6.2])**.** *Let $X,M,\pi,W$ be as in Lemma 2.3. Then for any $g_1,g_2,g_3\in\pi$, an element $\Psi_1(\Theta(g_1,g_2,g_3))$ of $\Omega_1^{SO}(B\mathrm{Diff}_0(X,\partial))$ is defined by surgery on an embedded theta-graph associated to $(g_1,g_2,g_3)$, which belongs to the image from $\pi_1B\mathrm{Diff}_0(X,\partial)$, and the following identity holds.*

$$Z_\Theta^{\mathrm{even}}(\Psi_1(\Theta(g_1,g_2,g_3)))=2\,[\rho_W(g_1)\wedge\rho_W(g_2)\wedge\rho_W(g_3)],$$

*where $\rho_W\colon\mathbb{C}[\pi]\to W=\bigoplus_i\mathrm{End}(A_i)$ is the representation of $\pi$.*

Note that the invariant $Z_\Theta^{\mathrm{even}}$ in [Wa1] was defined on a slightly different group* $\pi_1\widetilde{B\mathrm{Diff}}_{\mathrm{deg}}(M\times I,\partial)$ than $\pi_1B\mathrm{Diff}_0(X,\partial)$, however, it can be shown that $Z_\Theta^{\mathrm{even}}$ descends to a map from the image of $\pi_1\widetilde{B\mathrm{Diff}}_{\mathrm{deg}}(M\times I,\partial)\to\pi_1B\mathrm{Diff}_0(M\times I,\partial)$ (Lemma 3.3 below). Namely, the homotopy fiber of the natural map $\widetilde{B\mathrm{Diff}}_{\mathrm{deg}}(M\times I,\partial)\to B\mathrm{Diff}_0(M\times I,\partial)$ is given by

$$\mathrm{Map}((M\times I,\partial),(SO_4,1))\times\mathrm{Map}((M\times I,\partial),(M\times I,\partial))_{\mathrm{id}},$$

where $\partial=\partial(M\times I)$, $\mathrm{Map}((A,\partial A),(C,D))$ denotes the space of continuous maps $(A,\partial A)\to(C,D)$ with the $C^0$-topology that agree with the base map on $\partial A$, $(-)_{\mathrm{id}}$ denotes the component of the identity. The first factor $\mathrm{Map}((M\times I,\partial),(SO_4,1))$ is identified with the space of framings on $M\times I$, the second factor $\mathrm{Map}((M\times I,\partial),(M\times I,\partial))_{\mathrm{id}}$ is used to give a fiberwise (relative) degree one map $(E,\partial E)\to(M\times I,\partial)$ to pullback a local coefficient system on $M$.

*This should not be confused with the classifying space of $\widetilde{\mathrm{Diff}}(M\times I,\partial)$.

**Lemma 3.2.** *For a spherical 3-manifold $M$,*

$$\begin{aligned}&\mathrm{Map}((M\times I,\partial),(SO_4,1))\times \mathrm{Map}((M\times I,\partial),(M\times I,\partial))_{\mathrm{id}}\\ &\simeq \Omega\mathrm{Map}(M,SO_4)\times\Omega\mathrm{Map}(M,M)_{\mathrm{id}}.\end{aligned}$$

*Furthermore, there is a fibration sequence:*

$$\Omega^4 SO_4 \longrightarrow \Omega\mathrm{Map}(M,SO_4)\longrightarrow \Omega SO_4.$$

*Proof.* We have the following homotopy equivalences:

$$\begin{aligned}\mathrm{Map}((M\times I,\partial),(SO_4,1))&\simeq \Omega\mathrm{Map}(M,SO_4),\\ \mathrm{Map}(M\times I,\partial),(M\times I,\partial))_{\mathrm{id}}&\simeq\Omega\mathrm{Map}(M,M)_{\mathrm{id}},\end{aligned}$$

where the basepoints of $\mathrm{Map}(M,SO_4)$ and $\mathrm{Map}(M,M)_{\mathrm{id}}$ are the constant map to 1 and id, respectively. Furthermore, we have the following fibration sequence:

$$\mathrm{Map}_*(M,SO_4)\to\mathrm{Map}(M,SO_4)\overset{\mathrm{ev}}{\to} SO_4,$$

where ev is the evaluation at a fixed basepoint of $M$, and $\mathrm{Map}_*(-,-)$ is the subspace of $\mathrm{Map}(-,-)$ of pointed maps. Then the result follows by the homotopy equivalence: $\Omega\mathrm{Map}_*(M,SO_4)\simeq\mathrm{Map}_*(S^1\wedge M,SO_4)\simeq\mathrm{Map}_*(S^4,SO_4)$. □

**Lemma 3.3.** *Let $\Gamma_0(M\times I)$ denote the image of the natural map $\pi_1\widetilde{B\mathrm{Diff}}_{\mathrm{deg}}(M\times I,\partial)\to\pi_1 B\mathrm{Diff}_0(M\times I)$. The homomorphism $Z_\Theta^{\mathrm{even}}\colon\pi_1\widetilde{B\mathrm{Diff}}_{\mathrm{deg}}(M\times I,\partial)\to\mathscr{A}_\Theta^{\mathrm{even}}(W)$ descends to a map $\Gamma_0(M\times I)\to\mathscr{A}_\Theta^{\mathrm{even}}(W)$.*

*Proof.* Since $\pi_1\Omega SO_4=0$ and $\pi_1\Omega^4SO_4=\pi_5SO_4=\mathbb{Z}_2\oplus\mathbb{Z}_2$, $\pi_1\Omega\mathrm{Map}(M,SO_4)=\pi_2\mathrm{Map}(M,SO_4)$ is a quotient of $\mathbb{Z}_2\oplus\mathbb{Z}_2$. Moreover, a change of the choice of the lift of an element of $\Gamma_0(M\times I)$ to $\pi_1\widetilde{B\mathrm{Diff}}_{\mathrm{deg}}(M\times I,\partial)$ within the factor $\mathrm{Map}((M\times I,\partial),(M\times I,\partial))_{\mathrm{id}}$ in the homotopy fiber does not affect the value of $Z_\Theta^{\mathrm{even}}$ since the local coefficient system on the total space of the corresponding $(M\times I)$-bundle $p\colon E\to S^1$, which is needed to define $Z_\Theta^{\mathrm{even}}$, is determined by the homotopy class of the induced map $\pi_1E\to\pi_1M$, which is canonically fixed since we have the canonical decomposition $\pi_1E=\pi_1S^1\times\pi_1M$ by the van Kampen theorem and the homotopical triviality of elements of $\mathrm{Diff}_0(M\times I,\partial)$, and the map $\pi_1E\to\pi_1M$ is just the projection to the second factor. This completes the proof. □

Since $\rho_W\colon\mathbb{C}[\pi]\to W=\bigoplus_i\mathrm{End}(A_i)$ in Proposition 3.1 is surjective for $W=\mathrm{Ker}\,\varepsilon$ (see (2.1)), we have the following.

**Corollary 3.4.** *The homomorphism $Z_\Theta^{\mathrm{even}}\colon\Gamma_0(M\times I)\otimes\mathbb{C}\to\mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon)$ is surjective.*

## 4. Example 1: Poincaré homology sphere

4.1. **The group $\mathrm{SL}_2(\mathbb{F}_5)$.** Let $\hat{\pi}$ denote the set of conjugacy classes of $\pi = \mathrm{SL}_2(\mathbb{F}_5)$. It is known that $\hat{\pi}$ has 9 elements, represented respectively by the following elements:

$$\pm I = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \beta' = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad -\gamma = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad -\gamma' = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

We give a list of all elements of $\mathrm{SL}_2(\mathbb{F}_5)$ in Appendix A.

**Lemma 4.1.** *For $\pi = \mathrm{SL}_2(\mathbb{F}_5)$, $\hat{\pi}$ is invariant under taking the inverse. Namely, for each class $[x] \in \hat{\pi}$, we have $[x^{-1}] = [x]$.*

*Proof.* It suffices to see that the inverse of each element $x$ in the above list of 9 elements is conjugate to $x$, which can be checked by comparing the Jordan canonical forms, or from the list in Appendix A (obtained by using the Jordan canonical forms). Note that we only allow the conjugation $gxg^{-1}$ by $g \in \mathrm{SL}_2(\mathbb{F}_5)$. □

4.2. **Representation of $\mathrm{SL}_2(\mathbb{F}_5)$.** There are 9 distinct irreducible represenations $A_i$ $(i = 1, 2, \ldots, 9)$ of the group $\pi$ whose character is given as in Table 1, and any irreducible representation of $\pi$ over $\mathbb{C}$ is isomorphic to one of them. Irreducible representations of $\pi \times \pi$ are given by the external tensor product $A_i \boxtimes A_j$ (e.g., [FH, Exercise 2.36]). Since the values of the characters are real for $\pi = \mathrm{SL}_2(\mathbb{F}_5)$, we have

$$\mathbb{C}[\pi] \cong \bigoplus_{i=1}^{9} \mathrm{End}(A_i) \cong \bigoplus_{i=1}^{9} (A_i \boxtimes A_i), \quad \mathrm{Ker}\,\varepsilon \cong \bigoplus_{i=2}^{9} (A_i \boxtimes A_i) \tag{4.1}$$

as $\pi \times \pi$-modules†, where the $\pi \times \pi$-invariant $A_1 \boxtimes A_1$ in $\mathbb{C}[\pi]$ corresponds to the subspace spanned by the element $\sum_{g \in \pi} g \in \mathbb{C}[\pi]$, and $\mathrm{Ker}\,\varepsilon$ is a $\pi \times \pi$-submodule of $\mathbb{C}[\pi]$.

4.3. **Computation of the character.** To get the dimension of $\mathscr{A}_\Theta^{\mathrm{odd/even}}(\mathbb{C}[\pi])$, we compute the dimensions of the invariants $(\bigwedge^3 W)^{(\pi\times\pi)\rtimes\mathbb{Z}_2}$ and $(\mathrm{Sym}^3 W)^{(\pi\times\pi)\rtimes\mathbb{Z}_2}$ for $W = \bigoplus_{i=1}^{9}(A_i \boxtimes A_i)$. Here, recall that the semidirect product structure on $(\pi \times \pi) \rtimes \mathbb{Z}_2$ is given by the homomorphism $\psi \colon \mathbb{Z}_2 = \{1, \tau\} \to \mathrm{Aut}(\pi \times \pi)$; $\tau \mapsto ((x, y) \mapsto (y, x))$. This is suitable since we have

$$\begin{aligned} &\tau \cdot (x, y) \cdot \tau^{-1}(g \wedge h \wedge k) = \tau \cdot (xg^{-1}y^{-1} \wedge xh^{-1}y^{-1} \wedge xk^{-1}y^{-1}) \\ &= ygx^{-1} \wedge yhx^{-1} \wedge ykx^{-1} = (y, x)(g \wedge h \wedge k), \end{aligned}$$

which shows that the given actions of $\pi \times \pi$ and $\mathbb{Z}_2$ on $\bigwedge^3 \mathbb{C}[\pi]$ agrees with that of the semidirect product, and similarly for $\mathrm{Sym}^3 \mathbb{C}[\pi]$.

**Lemma 4.2.** *For $\pi = \mathrm{SL}_2(\mathbb{F}_5)$, we have the following.*

1. $\dim (\bigwedge^3 \mathbb{C}[\pi])^{(\pi\times\pi)\rtimes\mathbb{Z}_2} = 27$.
2. $\dim (\mathrm{Sym}^3 \mathbb{C}[\pi])^{(\pi\times\pi)\rtimes\mathbb{Z}_2} = 65$.

†If the characters are not real, we have $\mathrm{End}(A_i) \cong A_i \boxtimes A_i^*$.

| $\hat{\pi}$ | $I$ | $-I$ | $\alpha$ | $\beta$ | $\beta'$ | $\gamma$ | $\gamma'$ | $-\gamma$ | $-\gamma'$ |
|---|---|---|---|---|---|---|---|---|---|
| size | 1 | 1 | 30 | 20 | 20 | 12 | 12 | 12 | 12 |
| $A_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_2$ | 2 | $-2$ | 0 | $-1$ | 1 | $-\phi^*$ | $-\phi$ | $\phi^*$ | $\phi$ |
| $A_3$ | 2 | $-2$ | 0 | $-1$ | 1 | $-\phi$ | $-\phi^*$ | $\phi$ | $\phi^*$ |
| $A_4$ | 3 | 3 | $-1$ | 0 | 0 | $\phi$ | $\phi^*$ | $\phi$ | $\phi^*$ |
| $A_5$ | 3 | 3 | $-1$ | 0 | 0 | $\phi^*$ | $\phi$ | $\phi^*$ | $\phi$ |
| $A_6$ | 4 | 4 | 0 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $A_7$ | 4 | $-4$ | 0 | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 |
| $A_8$ | 5 | 5 | 1 | $-1$ | $-1$ | 0 | 0 | 0 | 0 |
| $A_9$ | 6 | $-6$ | 0 | 0 | 0 | 1 | 1 | $-1$ | $-1$ |

TABLE 1. The characters $\rho_{A_i}(g)$ for $\mathrm{SL}_2(\mathbb{F}_5)$. $\phi = \dfrac{1+\sqrt{5}}{2}$, $\phi^* = \dfrac{1-\sqrt{5}}{2}$. The values of the characters are taken from [FH, CP]. The order of the rows and columns followed the one in [Bo], although the original table in [Bo] includes few typos in the row of $A_7$ (signs of the last four entries).

*Proof.* Recall that the dimension $m$ of the invariant part in a representation $V$ of a finite group $G$ can be given by the following formula ([FH, (2.9)]):

$$m = \frac{1}{|G|}\sum_{g\in G}\mathrm{Trace}(g|_V) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g).$$

We apply this formula for $G = (\pi \times \pi) \rtimes \mathbb{Z}_2$ with the formulas: $\chi_{V\oplus V'}(g) = \chi_V(g) + \chi_{V'}(g)$, $\chi_{V\boxtimes V'}(g,h) = \chi_V(g)\chi_{V'}(h)$, and

$$\chi_{\bigwedge^3(V)}(g) = \frac{1}{6}(\chi_V(g)^3 - 3\chi_V(g^2)\chi_V(g) + 2\chi_V(g^3)),$$

$$\chi_{\mathrm{Sym}^3(V)}(g) = \frac{1}{6}\left(\chi_V(g)^3 + 3\chi_V(g^2)\chi_V(g) + 2\chi_V(g^3)\right),$$

with the character table (Table 1). First, for $W = \mathbb{C}[\pi]$ we compute

$$\dim(\textstyle\bigwedge^3 W)^{\pi\times\pi} = \frac{1}{6|\pi|^2}\sum_{g,h\in\pi}(\chi_W(g,h)^3 - 3\chi_W(g^2,h^2)\chi_W(g,h) + 2\chi_W(g^3,h^3)),$$

$$\dim(\mathrm{Sym}^3 W)^{\pi\times\pi} = \frac{1}{6|\pi|^2}\sum_{g,h\in\pi}(\chi_W(g,h)^3 + 3\chi_W(g^2,h^2)\chi_W(g,h) + 2\chi_W(g^3,h^3)),$$

where $\chi_W(g,h) = \sum_{i=1}^9 \chi_{A_i}(g)\chi_{A_i}(h)$, $\chi_W(g^2,h^2) = \sum_{i=1}^9 \chi_{A_i}(g^2)\chi_{A_i}(h^2)$, and so on. Substituting the values of the characters of Table 1 into these formulas, we get

$$\dim(\textstyle\bigwedge^3 \mathbb{C}[\pi])^{\pi\times\pi} = 33, \quad \dim(\mathrm{Sym}^3\,\mathbb{C}[\pi])^{\pi\times\pi} = 71.$$

The detail of these computations by Maxima can be found in [Oh], in which Table 2 is used to compute the characters of $g^2$ and $g^3$.

| $g$ | $I$ | $-I$ | $\alpha$ | $\beta$ | $\beta'$ | $\gamma$ | $\gamma'$ | $-\gamma$ | $-\gamma'$ |
|---|---|---|---|---|---|---|---|---|---|
| $g^2$ | $I$ | $I$ | $-I$ | $\beta$ | $\beta$ | $\gamma'$ | $\gamma$ | $\gamma'$ | $\gamma$ |
| $g^3$ | $I$ | $-I$ | $\alpha$ | $I$ | $-I$ | $\gamma'$ | $\gamma$ | $-\gamma'$ | $-\gamma$ |

TABLE 2. The conjugacy classes of $g^2$ and $g^3$ in $\mathrm{SL}_2(\mathbb{F}_5)$. See also Appendix A.

We need also to consider terms for the elements $\tau\cdot(g,h)$ given by the following formulas.

(4.2)

$$\frac{1}{6|\pi|^2}\sum_{g,h\in\pi}(\chi_W(\tau\cdot(g,h))^3-3\chi_W((\tau\cdot(g,h))^2)\chi_W(\tau\cdot(g,h))+2\chi_W((\tau\cdot(g,h))^3),$$

$$\frac{1}{6|\pi|^2}\sum_{g,h\in\pi}(\chi_W(\tau\cdot(g,h))^3+3\chi_W((\tau\cdot(g,h))^2)\chi_W(\tau\cdot(g,h))+2\chi_W((\tau\cdot(g,h))^3).$$

We simplify the computation as follows.

$$\begin{aligned}
&\sum_{g,h}\chi_W(\tau\cdot(g,h))^3=\sum_{g,h}\Big(\sum_i\chi_{A_i\boxtimes A_i}(\tau\cdot(g,h))\Big)^3=\sum_{g,h}\Big(\sum_i\chi_{A_i}(hg)\Big)^3\\
&=\sum_h\sum_g\Big(\sum_i\chi_{A_i}(hg)\Big)^3=\sum_h\sum_g\Big(\sum_i\chi_{A_i}(g)\Big)^3=|\pi|\sum_g\Big(\sum_i\chi_{A_i}(g)\Big)^3,\\
&\sum_{g,h}\chi_W((\tau\cdot(g,h))^2)\chi_W(\tau\cdot(g,h))=\sum_{g,h}\Big(\sum_i\chi_{A_i\boxtimes A_i}(hg,gh)\Big)\Big(\sum_j\chi_{A_j}(hg)\Big)\\
&=\sum_{g,h}\Big(\sum_i\chi_{A_i}(hg)\chi_{A_i}(gh)\Big)\Big(\sum_j\chi_{A_j}(hg)\Big)\\
&=\sum_{g,h}\Big(\sum_i\chi_{A_i}(hg)^2\Big)\Big(\sum_j\chi_{A_j}(hg)\Big)=\sum_h\sum_g\Big(\sum_i\chi_{A_i}(g)^2\Big)\Big(\sum_j\chi_{A_j}(g)\Big)\\
&=|\pi|\sum_g\Big(\sum_i\chi_{A_i}(g)^2\Big)\Big(\sum_j\chi_{A_j}(g)\Big),\\
&\sum_{g,h}\chi_W((\tau\cdot(g,h))^3)=\sum_{g,h}\sum_i\chi_{A_i\boxtimes A_i}((\tau\cdot(g,h))^3)=\sum_{g,h}\sum_i\chi_{A_i}(hghghg)\\
&=|\pi|\sum_g\sum_i\chi_{A_i}(g^3).
\end{aligned}$$

Here, we have the identity $\chi_{A_i\boxtimes A_i}(\tau\cdot(g,h))=\chi_{A_i}(hg)$ since $\tau$ acts on $A_i\boxtimes A_i$ by the flip $x\boxtimes y\mapsto y\boxtimes x$. Hence (4.2) can be computed respectively by the following formulas.

$$\frac{1}{6|\pi|}\sum_g\Big\{\Big(\sum_i\chi_{A_i}(g)\Big)^3-3\Big(\sum_i\chi_{A_i}(g)^2\Big)\Big(\sum_j\chi_{A_j}(g)\Big)+2\sum_i\chi_{A_i}(g^3)\Big\},$$

$$\frac{1}{6|\pi|}\sum_g\Big\{\Big(\sum_i\chi_{A_i}(g)\Big)^3+3\Big(\sum_i\chi_{A_i}(g)^2\Big)\Big(\sum_j\chi_{A_j}(g)\Big)+2\sum_i\chi_{A_i}(g^3)\Big\}.$$

Substituting the values of the characters in Table 1 into these formulas, we get the values 21 and 59, respectively. By taking the averages $(33 + 21)/2 = 27$, $(71 + 59)/2 = 65$, we get the result. $\square$

For $W = \operatorname{Ker}\varepsilon$, the following proposition holds.

**Proposition 4.3.** *For any finite group $G$, we have the following.*

(1) $\mathscr{A}_{\Theta}^{\mathrm{even}}(\mathbb{C}[G]) = \mathscr{A}_{\Theta}^{\mathrm{even}}(\operatorname{Ker}\varepsilon)$.

(2) $\mathscr{A}_{\Theta}^{\mathrm{odd}}(\mathbb{C}[G]) \cong \mathscr{A}_{\Theta}^{\mathrm{odd}}(\operatorname{Ker}\varepsilon) \oplus (\mathbb{C}\hat{G})_{\mathbb{Z}_2}$ *as vector spaces over $\mathbb{C}$, where $\hat{G}$ is the set of conjugacy classes in $G$, and the $\mathbb{Z}_2$-action on it is the one induced by the inversion.*

*Proof.* Instead of the $(G \times G) \rtimes \mathbb{Z}_2$-invariant part, we consider the submodule of coinvariants, and apply the formula $V_H \cong (V_K)_{H/K}$ (e.g., [Br, Ch.II-2 (Exercises 3)]) for a subgroup $K$ of $H$ and an $H$-module $V$ to $K = G \times G$, $H = (G \times G) \rtimes \mathbb{Z}_2$.

For (1), let $U$ be a trivial 1-dimensional $G \times G$-module and let $W$ be a $G \times G$-module. The formula $\bigwedge^n(U \oplus W) = \bigoplus_{p+q=n} \bigwedge^p U \otimes \bigwedge^q W$ and $\bigwedge^2 U = 0$ gives

$$\begin{aligned}\textstyle\bigwedge^3(U \oplus W) &= \textstyle\bigwedge^3 U \oplus \bigwedge^3 W \oplus ((\bigwedge^2 U) \otimes W) \oplus (U \otimes \bigwedge^2 W)\\ &\cong \textstyle\bigwedge^3 W \oplus \bigwedge^2 W.\end{aligned}$$

If $W = \operatorname{Ker}\varepsilon$ and if $\mathbb{Z}_2$ acts trivially on $U$, then we have $U \oplus W \cong \mathbb{C}[G]$ as both $G \times G$-modules and $\mathbb{Z}_2$-modules, and $((\bigwedge^2 W)_{G\times G})^{\mathbb{Z}_2} = 0$. Indeed, we have $((\bigwedge^2 \mathbb{C}[G])_{G\times G})^{\mathbb{Z}_2} = 0$, since in $(\bigwedge^2 \mathbb{C}[G])_{G\times G}$, we have $[g \wedge h] = [gh^{-1} \wedge 1] = -[1 \wedge gh^{-1}] = -[g^{-1} \wedge h^{-1}]$ $(g, h \in G)$ and the $\mathbb{Z}_2$-invariant is generated by $[g \wedge h] + [g^{-1} \wedge h^{-1}] = 0$. Hence we have

$$((\textstyle\bigwedge^3(U \oplus \operatorname{Ker}\varepsilon))_{G\times G})^{\mathbb{Z}_2} \cong ((\bigwedge^3 \operatorname{Ker}\varepsilon)_{G\times G})^{\mathbb{Z}_2}.$$

For (2), we use the formula $\mathrm{Sym}^n(U \oplus W) = \bigoplus_{p+q=n} \mathrm{Sym}^p U \otimes \mathrm{Sym}^q W$ to obtain

$$\begin{aligned}\mathrm{Sym}^3(U \oplus W) &= \mathrm{Sym}^3 U \oplus \mathrm{Sym}^3 W \oplus ((\mathrm{Sym}^2 U) \otimes W) \oplus (U \otimes \mathrm{Sym}^2 W)\\ &\cong U \oplus \mathrm{Sym}^3 W \oplus W \oplus \mathrm{Sym}^2 W \cong \mathrm{Sym}^3 W \oplus \mathrm{Sym}^2(U \oplus W).\end{aligned}$$

Considering the case when $W = \operatorname{Ker}\varepsilon$, it suffices to prove $((\mathrm{Sym}^2\mathbb{C}[G])_{G\times G})^{\mathbb{Z}_2} \cong (\mathbb{C}\hat{G})_{\mathbb{Z}_2}$. Let $\rho\colon \mathrm{Sym}^2\mathbb{C}[G] \to (\mathbb{C}\hat{G})_{\mathbb{Z}_2}$ be the $\mathbb{C}$-linear map defined by $\rho(g \cdot h) = [gh^{-1}]$ $(g, h \in G)$, which is well-defined since $\rho(h \cdot g) = [hg^{-1}] = [gh^{-1}] = \rho(g \cdot h)$. One can check that this map induces a well-defined $\mathbb{C}$-linear isomorphism

$$\bar{\rho}\colon ((\mathrm{Sym}^2\mathbb{C}[G])_{G\times G})^{\mathbb{Z}_2} \to (\mathbb{C}\hat{G})_{\mathbb{Z}_2}$$

with the inverse given by $\bar{\rho}^{-1}([x]) = \frac{1}{2}([1\cdot x^{-1}] + [1\cdot x])$, which is well-defined. Indeed, $\bar{\rho}(gx \cdot hx) = [gxx^{-1}h^{-1}] = \bar{\rho}(g \cdot h)$, $\bar{\rho}(xg \cdot xh) = [xgh^{-1}x^{-1}] = [gh^{-1}] = \bar{\rho}(g \cdot h)$, $\bar{\rho}(g \cdot h) = \bar{\rho}(gh^{-1} \cdot 1) = \bar{\rho}(1 \cdot gh^{-1}) = \bar{\rho}(g^{-1} \cdot h^{-1})$ etc. $\square$

**Proposition 4.4.** *When $\pi = \mathrm{SL}_2(\mathbb{F}_5)$, we have the following.*

(1) $\dim \mathscr{A}_{\Theta}^{\mathrm{even}}(\operatorname{Ker}\varepsilon) = \dim \mathscr{A}_{\Theta}^{\mathrm{even}}(\mathbb{C}[\pi]) = 27$.

(2) $\dim \mathscr{A}_{\Theta}^{\mathrm{odd}}(\operatorname{Ker}\varepsilon) = 56$, $\dim \mathscr{A}_{\Theta}^{\mathrm{odd}}(\mathbb{C}[\pi]) = 65$.

*Proof.* This follows from Lemma 4.2 and Proposition 4.3. Note that by Lemma 4.1 the action of $\mathbb{Z}_2$ on $\mathbb{C}\hat{\pi}$ is trivial, and hence $\dim(\mathbb{C}\hat{\pi})_{\mathbb{Z}_2} = \dim \mathbb{C}\hat{\pi} = 9$. $\square$

## 5. Example 2: Lens spaces

If $\pi$ is the cyclic group $\mathbb{Z}_n = \{1, t, t^2, \dots, t^{n-1}\}$, the exact values of the dimensions of the spaces $\mathscr{A}_\Theta^{\mathrm{even/odd}}(\mathbb{C}[\pi])$ and $\mathscr{A}_\Theta^{\mathrm{even/odd}}(\mathrm{Ker}\,\varepsilon)$ can be determined with the help of the $\mathbb{C}$-linear maps ("weight system")

$$W^{\mathrm{even}} \colon \mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi]) \to \textstyle\bigwedge^3 \mathbb{C}[\pi], \qquad W^{\mathrm{odd}} \colon \mathscr{A}_\Theta^{\mathrm{odd}}(\mathbb{C}[\pi]) \to \mathrm{Sym}^3\,\mathbb{C}[\pi]$$

defined respectively by

$$\begin{aligned} W^{\mathrm{even}}(t^a \wedge t^b \wedge t^c) &= t^{b-a} \wedge t^{c-b} \wedge t^{a-c} + t^{a-b} \wedge t^{b-c} \wedge t^{c-a}, \\ W^{\mathrm{odd}}(t^a \cdot t^b \cdot t^c) &= t^{b-a} \cdot t^{c-b} \cdot t^{a-c} + t^{a-b} \cdot t^{b-c} \cdot t^{c-a}, \end{aligned}$$

instead of calculating the characters.

**Lemma 5.1.** *For $\pi = \mathbb{Z}_n$, the maps $W^{\mathrm{even/odd}}$ are well-defined.*

*Proof.* We need only to check that $W^{\mathrm{even/odd}}$ is alternating/symmetric and is invariant under both the actions of $\pi \times \pi$ and the involution $(g, h, k) \mapsto (g^{-1}, h^{-1}, k^{-1})$. For $W^{\mathrm{even}}$, this can be checked as follows ($t^k \in \pi$):

$$\begin{aligned} W^{\mathrm{even}}(t^b \wedge t^a \wedge t^c) &= t^{a-b} \wedge t^{c-a} \wedge t^{b-c} + t^{b-a} \wedge t^{a-c} \wedge t^{c-b} \\ &= -t^{a-b} \wedge t^{b-c} \wedge t^{c-a} - t^{b-a} \wedge t^{c-b} \wedge t^{a-c} \\ &= -W^{\mathrm{even}}(t^a \wedge t^b \wedge t^c) \quad \text{etc.} \\ W^{\mathrm{even}}(t^k t^a \wedge t^k t^b \wedge t^k t^c) &= t^{b-a} \wedge t^{c-b} \wedge t^{a-c} + t^{a-b} \wedge t^{b-c} \wedge t^{c-a} \\ &= W^{\mathrm{even}}(t^a \wedge t^b \wedge t^c), \\ W^{\mathrm{even}}(t^{-a} \wedge t^{-b} \wedge t^{-c}) &= t^{-b+a} \wedge t^{-c+b} \wedge t^{-a+c} + t^{-a+b} \wedge t^{-b+c} \wedge t^{-c+a} \\ &= W^{\mathrm{even}}(t^a \wedge t^b \wedge t^c). \end{aligned}$$

The proof for $W^{\mathrm{odd}}$ is similar. □

Then $W^{\mathrm{even/odd}}$ is an embedding into a subspace isomorphic to the space spanned by $t^p \wedge t^q \wedge t^r$ or $t^p \cdot t^q \cdot t^r$ ($0 \le p, q, r < n$, $p + q + r = 0 \pmod n$) quotiented by the relation $t^p \wedge t^q \wedge t^r \sim t^{-p} \wedge t^{-q} \wedge t^{-r}$ or $t^p \cdot t^q \cdot t^r \sim t^{-p} \cdot t^{-q} \cdot t^{-r}$.

**Proposition 5.2.** *Let $\pi = \mathbb{Z}_n$ ($n \ge 1$), and for an integer $m \ge 0$, let $p_3(m)$ be the number of partitions of $m$ into at most three parts, namely, the number of integer solutions of the equation $x + y + z = m$ ($0 \le x \le y \le z$). We set $p_3(m) = 0$ for $m < 0$. Then we have the following.*

1. $\dim \mathscr{A}_\Theta^{\mathrm{odd}}(\mathbb{C}[\pi]) = p_3(n)$.
2. $\dim \mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi]) = p_3(n-6)$.
3. $\dim \mathscr{A}_\Theta^{\mathrm{odd}}(\mathrm{Ker}\,\varepsilon) = p_3(n-3)$.
4. $\dim \mathscr{A}_\Theta^{\mathrm{even}}(\mathrm{Ker}\,\varepsilon) = \dim \mathscr{A}_\Theta^{\mathrm{even}}(\mathbb{C}[\pi]) = p_3(n-6)$.

*Proof.* For (1), we consider the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the $n$ points $1, \omega, \omega^2, \dots, \omega^{n-1}$ on it, where $\omega = e^{2\pi\sqrt{-1}/n}$. We represent the element $t^a \cdot t^b \cdot t^c$ ($0 \le a \le b \le c \le n-1$) by the three points $\omega^a, \omega^b, \omega^c$ on $S^1$, which splits $S^1$ into three (possibly degenerate) arcs of lengths $\frac{2\pi}{n}(b-a)$, $\frac{2\pi}{n}(c-b)$, $\frac{2\pi}{n}(a-c)$ (mod $2\pi$), respectively. Similarly, $\omega^{-a}, \omega^{-b}, \omega^{-c}$ splits $S^1$ into three arcs of lengths $\frac{2\pi}{n}(a-b)$, $\frac{2\pi}{n}(b-c)$, $\frac{2\pi}{n}(c-a)$ (mod $2\pi$), which are reflections of the previous triple with respect to the real axis. In this way, the subspace spanned by the values of $W^{\mathrm{odd}}(t^a \cdot t^b \cdot t^c)$ bijectively correspond to the space spanned by partitions of $S^1$ by three roots of 1 up to $\frac{2\pi k}{n}$-rotation and reflection. The number of such classes of partitions is exactly the number $p_3(n)$.

Proof of (2) is similar. In this case partitions are slightly restricted. First, a partition should not have a zero part since such a partition comes from $t^a \wedge t^b \wedge t^c$ such that at least two of $a, b, c$ agree. Also, a partition should have different sizes since we consider the value of the weight system in the alternating product $\bigwedge^3 \mathbb{C}[\pi]$. It follows that $\dim \operatorname{Im} W^{\mathrm{even}}$ agrees with the number of partitions of $n$ into three nonzero parts with different sizes. Such partitions $n = x + y + z$ $(0 < x < y < z)$ correspond bijectively to the partitions $n - (1+2+3) = (x-1) + (y-2) + (z-3)$ $(0 \leq x-1 \leq y-2 \leq z-3)$. This completes the proof.

For (3), it follows from Proposition 4.3 that $\dim \mathscr{A}^{\mathrm{odd}}_{\Theta}(\operatorname{Ker}\varepsilon) = \dim \mathscr{A}^{\mathrm{odd}}_{\Theta}(\mathbb{C}[\pi]) - \dim(\mathbb{C}\hat{\pi})_{\mathbb{Z}_2}$, where $\dim(\mathbb{C}\hat{\pi})_{\mathbb{Z}_2}$ is the number of partitions of $n$ into at most two parts. Hence $\dim \mathscr{A}^{\mathrm{odd}}_{\Theta}(\operatorname{Ker}\varepsilon)$ is the number of partitions of $n$ into three parts with positive sizes. Such partitions $n = x + y + z$ $(0 < x \leq y \leq z)$ correspond bijectively to partitions $n - 3 = (x-1) + (y-1) + (z-1)$ $(0 \leq x-1 \leq y-1 \leq z-1)$.

(4) follows immediately from (2) and Proposition 4.3. □

## Appendix A. Elements of $\mathrm{SL}_2(\mathbb{F}_5)$

The following is a list of all the 120 elements in $\mathrm{SL}_2(\mathbb{F}_5)$.

$$\underline{c_1 = [I]}: \ g_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{c_2 = [-I]}: \ g_{96} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$\underline{c_3 = [\alpha]}$ :

$$g_1 = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \ g_6 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \ g_{11} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, \ g_{16} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}, \ g_{29} = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix},$$
$$g_{35} = \begin{pmatrix} 1 & 2 \\ 4 & 4 \end{pmatrix}, \ g_{37} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, \ g_{43} = \begin{pmatrix} 1 & 4 \\ 2 & 4 \end{pmatrix}, \ g_{46} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \ g_{47} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix},$$
$$g_{48} = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}, \ g_{49} = \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}, \ g_{50} = \begin{pmatrix} 2 & 0 \\ 4 & 3 \end{pmatrix}, \ g_{51} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \ g_{56} = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix},$$
$$g_{61} = \begin{pmatrix} 2 & 3 \\ 0 & 3 \end{pmatrix}, \ g_{66} = \begin{pmatrix} 2 & 4 \\ 0 & 3 \end{pmatrix}, \ g_{71} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}, \ g_{72} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \ g_{73} = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix},$$
$$g_{74} = \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}, \ g_{75} = \begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}, \ g_{76} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \ g_{81} = \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \ g_{86} = \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix},$$
$$g_{91} = \begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}, \ g_{104} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}, \ g_{110} = \begin{pmatrix} 4 & 2 \\ 4 & 1 \end{pmatrix}, \ g_{112} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \ g_{118} = \begin{pmatrix} 4 & 4 \\ 2 & 1 \end{pmatrix}$$

$\underline{c_4 = [\beta]}$ :

$$g_5 = \begin{pmatrix} 0 & 1 \\ 4 & 4 \end{pmatrix}, \ g_{10} = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}, \ g_{15} = \begin{pmatrix} 0 & 3 \\ 3 & 4 \end{pmatrix}, \ g_{20} = \begin{pmatrix} 0 & 4 \\ 1 & 4 \end{pmatrix}, \ g_{28} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix},$$
$$g_{32} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \ g_{40} = \begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}, \ g_{44} = \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix}, \ g_{54} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \ g_{60} = \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix},$$
$$g_{62} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \ g_{68} = \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, \ g_{78} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \ g_{82} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \ g_{90} = \begin{pmatrix} 3 & 3 \\ 4 & 1 \end{pmatrix},$$
$$g_{94} = \begin{pmatrix} 3 & 4 \\ 3 & 1 \end{pmatrix}, \ g_{105} = \begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}, \ g_{108} = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \ g_{114} = \begin{pmatrix} 4 & 3 \\ 3 & 0 \end{pmatrix}, \ g_{117} = \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}$$

$\underline{c_5 = [\beta']}$ :

$g_2 = \begin{pmatrix} 0 & 1 \\ 4 & 1 \end{pmatrix}$, $g_7 = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix}$, $g_{12} = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}$, $g_{17} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$, $g_{30} = \begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}$,
$g_{33} = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$, $g_{39} = \begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}$, $g_{42} = \begin{pmatrix} 1 & 4 \\ 1 & 0 \end{pmatrix}$, $g_{53} = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}$, $g_{57} = \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}$,
$g_{65} = \begin{pmatrix} 2 & 3 \\ 4 & 4 \end{pmatrix}$, $g_{69} = \begin{pmatrix} 2 & 4 \\ 3 & 4 \end{pmatrix}$, $g_{79} = \begin{pmatrix} 3 & 1 \\ 3 & 3 \end{pmatrix}$, $g_{85} = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$, $g_{87} = \begin{pmatrix} 3 & 3 \\ 1 & 3 \end{pmatrix}$,
$g_{93} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, $g_{103} = \begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix}$, $g_{107} = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$, $g_{115} = \begin{pmatrix} 4 & 3 \\ 4 & 2 \end{pmatrix}$, $g_{119} = \begin{pmatrix} 4 & 4 \\ 3 & 2 \end{pmatrix}$

$\underline{c_6 = [\gamma]}$ :

$g_3 = \begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$, $g_{18} = \begin{pmatrix} 0 & 4 \\ 1 & 2 \end{pmatrix}$, $g_{22} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $g_{25} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$, $g_{26} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,
$g_{41} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $g_{55} = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$, $g_{67} = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$, $g_{77} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$, $g_{95} = \begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix}$,
$g_{102} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$, $g_{120} = \begin{pmatrix} 4 & 4 \\ 4 & 3 \end{pmatrix}$

$\underline{c_7 = [\gamma']}$ :

$g_8 = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix}$, $g_{13} = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$, $g_{23} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $g_{24} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$, $g_{31} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,
$g_{36} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $g_{58} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$, $g_{64} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix}$, $g_{84} = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$, $g_{88} = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$,
$g_{109} = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$, $g_{113} = \begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$

$\underline{c_8 = [-\gamma]}$ :

$g_4 = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$, $g_{19} = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}$, $g_{27} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $g_{45} = \begin{pmatrix} 1 & 4 \\ 4 & 2 \end{pmatrix}$, $g_{52} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$,
$g_{70} = \begin{pmatrix} 2 & 4 \\ 4 & 1 \end{pmatrix}$, $g_{80} = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$, $g_{92} = \begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}$, $g_{97} = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix}$, $g_{100} = \begin{pmatrix} 4 & 0 \\ 4 & 4 \end{pmatrix}$,
$g_{101} = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}$, $g_{116} = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$

$\underline{c_9 = [-\gamma']}$ :

$g_9 = \begin{pmatrix} 0 & 2 \\ 2 & 3 \end{pmatrix}$, $g_{14} = \begin{pmatrix} 0 & 3 \\ 3 & 3 \end{pmatrix}$, $g_{34} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$, $g_{38} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$, $g_{59} = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$,
$g_{63} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$, $g_{83} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$, $g_{89} = \begin{pmatrix} 3 & 3 \\ 3 & 0 \end{pmatrix}$, $g_{98} = \begin{pmatrix} 4 & 0 \\ 2 & 4 \end{pmatrix}$, $g_{99} = \begin{pmatrix} 4 & 0 \\ 3 & 4 \end{pmatrix}$,
$g_{106} = \begin{pmatrix} 4 & 2 \\ 0 & 4 \end{pmatrix}$, $g_{111} = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}$

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