Frobenius splitting of Schubert varieties of semi-infinite flag manifolds*

Syu Kato†

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Abstract

We exhibit basic algebro-geometric results on the formal model of semi-infinite flag varieties and its Schubert varieties over an algebraically closed field $\mathbb{K}$ of characteristic $\neq 2$ from scratch. We show that the formal model of a semi-infinite flag variety admits a unique nice (ind)scheme structure, its projective coordinate ring has a $\mathbb{Z}$-model, and it admits a Frobenius splitting compatible with the boundaries and opposite cells in positive characteristic. This establishes the normality of the Schubert varieties of the quasi-map space with a fixed degree (instead of their limits proved in [K, Math. Ann. 371 no.2 (2018)]) when $\text{char } \mathbb{K} = 0$ or $\gg 0$, and the higher cohomology vanishing of their nef line bundles in arbitrary characteristic $\neq 2$. Some particular cases of these results play crucial roles in our proof [K, arXiv:1805.01718] of a conjecture by Lam-Li-Mihalcea-Shimozono [J. Algebra 513 (2018)] that describes an isomorphism between affine and quantum $K$-groups of a flag manifold.

Introduction

The semi-infinite flag varieties are variants of affine flag varieties that encode the modular representation theory of a semi-simple Lie algebra, representation theory of a quantum group at roots of unities, and representation theory of an affine Lie algebra at the critical level. They originate from the ideas of Lusztig [64] and Drinfeld, put forwarded by Feigin-Frenkel [26], and subsequently polished by the works of Braverman, Finkelberg, and their collaborators [31, 25, 2, 9, 10, 11, 12, 13]. They (mainly) employed the ind-model of semi-infinite flag varieties and achieved spectacular success on the geometric Langlands correspondence [2, 6], on the quantum $K$-groups of flag manifolds [11], and on their (conjectural) relation to the finite $W$-algebras [10].

In [52], we have initiated the study of the formal model of a semi-infinite flag variety (over $\mathbb{C}$) that follows the classical treatments of flag varieties [57, 58, 59, 58] more closely than the above. We refer this formal model of a semi-infinite flag variety as a “semi-infinite flag manifold” since we hope to justify that it is “smooth” in a sense. However, the analysis in [52] has two defects: the relation

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†Department of Mathematics, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502 JAPAN E-mail: syuchan@math.kyoto-u.ac.jp
with the ind-models of semi-infinite flag varieties is unclear, and the treatment there is rather ad hoc (it is just an indscheme whose set of \( C \)-valued points have the desired property, and lacks a characterization as a functor cf. \([1, 2]\)). The former defect produces difficulty in the discussion of deep properties on the identification between the equivariant \( K \)-group of a semi-infinite flag manifold and the equivariant quantum \( K \)-group of a flag manifold \([11]\), that is in turn inspired by the works of Givental and Lee \([34, 35]\). The goal of this paper is to study semi-infinite flag manifolds in characteristic \( \neq 2 \) from scratch, and resolve the above defects. In particular, we verify that the scheme in \([52]\) is the universal one among all the indschemes with similar set-theoretic properties, and provide new proofs of the normality of Zastava spaces \([11]\) and of the semi-infinite flag manifolds \([52]\).

It is possible to regard our works \((47, 48, 52, 49, 50)\) as a part of catch-up of Peterson’s original construction \([73]\) of his isomorphism \([61]\) between the quantum cohomology of a flag manifold and the equivariant cohomology of an affine Grassmannian in the \( K \)-theoretic setting. From this viewpoint, this paper gives \([73, \text{Lecture 11}]\) their appropriate compactifications. Hence, though there are still some missing pieces to complete the original program along the lines in \([73]\), this paper provides a step to fully examine his ideas.

To explain our results more precisely, we introduce more notation: Let \( g \) denote a simple Lie algebra (given in terms of root data and the Chevalley generators) over an algebraically closed field \( K \) of characteristic \( \neq 2 \). Let \( G \) denote the connected simply-connected algebraic group over \( K \) such that \( g = \text{Lie} \, G \). Let \( H \subset G \) be a Cartan subgroup and let \( N \) be an unipotent radical of \( G \) that is normalized by \( H \). We set \( B := HN \) and \( B \):= \( G=HN \) (the \( \mathbb{A} \)-manifold of \( G \)). Let \( I^+ \subset G(\mathbb{K}[z]) \) denote the Iwahori subgroup that contains \( B \), and let \( I^- \subset G(\mathbb{K}[z^{-1}]) \) be its opposite Iwahori subgroup. Let \( \tilde{g} \) denote the untwisted affine Kac-Moody algebra associated to \( g \). Let \( W \) and \( W_{\text{af}} \) be the finite Weyl group and the affine Weyl group of \( g \), respectively. The coroot lattice \( Q_{\text{af}} \) of \( g \) yields a natural subgroup \( \{ t_\beta \}_{\beta \in Q_{\text{af}}} \subset W_{\text{af}} \). Let \( w_0 \in W \) be the longest element.

Our first main result is as follows:

**Theorem A** \( (\doteq \text{Theorem 3.18 and Proposition 3.26}) \). There is an indscheme \( Q_{G}^{r} \) with the following properties:

1. The indscheme \( Q_{G}^{r} \) is expressed as the union of infinite type integral schemes flat over \( \mathbb{Z} \);
2. If we set \( (Q_{G}^{r})_K := Q_{G}^{r} \otimes \mathbb{Z} \mathbb{K} \), then we have
   \[
   (Q_{G}^{r})_K(\mathbb{K}) \cong G(\mathbb{K}(z))/H(\mathbb{K})N(\mathbb{K}(z))
   \]
   that intertwines the natural \( G(\mathbb{K}(z)) \times \mathbb{G}_m(\mathbb{K}) \)-actions on the both sides, where \( \mathbb{G}_m \) is the loop rotation;
3. The functor
   \[
   \text{Aff}^{\text{op}}_K \ni R \mapsto G(R(z))/H(R)N(R(z)) \in \text{Sets}
   \]
   is coarsely (ind-)representable by \( (Q_{G}^{r})_K \) (see \([4,3,1]\) for the convention).

One can equip \( (Q_{G}^{r})_K \) with an indscheme structure using the arc scheme of the basic affine space \( G/N \). Such an indscheme cannot coincide with ours.
Let $B$ instead, we prove $A$. If $B$, then the scheme $(Q^\text{rat}_G)_K$ is in bijection with $W_{af}$. Let $p_w \in (Q^\text{rat}_G)_K$ be the point corresponding to $w \in W_{af}$. We set $\mathcal{O}(w) := \mathfrak{I}^w p_w$ and $\mathcal{O}^-(w) := \mathfrak{I}^w p_w$ for each $w \in W_{af}$.

Theorem A has some applications to the theory of quasi-map spaces from $P^1$ to $\mathcal{B}$ \cite{11, 12, 13, 14, 15} as follows:

**Theorem B** \(\doteqdot \) Theorem 6.3.1. Proposition 1.2. Lemma 6.3, and Corollaries 6.3.8 and 1.7.26. In the above settings, it holds:

1. If $\text{char } K > 0$, then the scheme $(Q^\text{rat}_G)_K$ admits an $I^A$-canonical Frobenius splitting that is compatible with $\mathcal{O}(w)$’s and $\mathcal{O}^-(v)$’s $(w, v \in W_{af})$;

2. For each $w, v \in W_{af}$, the intersection $\mathcal{O}(v, w) := \mathcal{O}(w) \cap \mathcal{O}^-(v)$ is reduced.

3. For each $w, v \in W_{af}$, the scheme $\mathcal{O}(v, w)$ is weakly normal. It is (irreducible and) normal when $\text{char } K = 0$ or $\text{char } K \gg 0$;

4. For each $\beta \in Q^+_G$, the set of $K$-valued points of the scheme $\mathcal{O}(w_0 t_\beta, e)$ is in bijection with the set of $(K$-valued) Drinfeld-Plücker data. In particular, $\mathcal{O}(w_0 t_\beta, e)$ is isomorphic to the quasi-map space in $[\mathcal{B}]$ when $K = \mathbb{C}$.

Theorem B is a key result at the deepest part (correspondence between natural bases) in our proof (11.1.15) of a conjecture of Lam-Li-Mihalcea-Shimozono \cite{61} about the comparison between the equivariant $K$-group of the affine Grassmannian of $G$ and the equivariant small quantum $K$-group of $\mathcal{B}$. In \cite{54}, we also prove that $\mathcal{O}(w_0 t_\beta, w)$ admits only rational singularities (and hence it is Cohen-Macaulay) when $K = \mathbb{C}$ on the basis of Theorem 1.1. We remark that Theorem E 3) is proved in \cite{11, 12} when $v = w_0 t_\beta$, $w = e$, and $K = \mathbb{C}$.

Unlike the case of the usual flag varieties \(\doteqdot\) \cite{54}, our formulation of the De-mazure character formula for $Q^\text{rat}_G$ (11.1. Theorem A) does not produce something out of a point. In addition, the canonical bundle of $Q^\text{rat}_G$ (or $\mathcal{O}(w)$) does not make sense naively. These make us difficult to apply standard technique (as seen in [16]) to obtain a Frobenius splitting in Theorem B 1). Instead, we prove an auxiliary representation-theoretic result (Proposition 2.11) and combine it with the Frobenius splitting of an affine flag manifold \cite{54, Corollary B} to obtain Theorem B 1). This is the main technical idea of this paper.

In the rest introduction, we assume $K = \mathbb{C}$ for the sake of simplicity. Let $P$ be the weight lattice of $H$, and let $P_+ \subset P$ denote its subset corresponding to dominant weights. For each $\lambda \in P$, we have an equivariant line bundle $\mathcal{O}_{Q^\text{rat}_G}(\lambda)$ on $Q^\text{rat}_G$, whose restriction to $\mathcal{O}(v, w)$ is denoted by $\mathcal{O}_{Q(v, w)}(\lambda)$. Associated to $\lambda \in P_+$, we have a level zero extremal weight module $\mathcal{X}(\lambda)$ of $U(\mathfrak{g})$ in the sense of Kashiwara \cite{37}. We know that $\mathcal{X}(\lambda)$ is equipped with two kinds of Demazure modules, and a distinguished basis (the global basis).

**Corollary C** \(\doteqdot\) Theorem 5.3.5. Let $w, v \in W_{af}$. For each $\lambda \in P_+$, we have

$$H^{\geq 0}(\mathcal{O}(v, w), \mathcal{O}_{Q(v, w)}(\lambda)) = \{0\}.$$  

The space $H^0(\mathcal{O}(v, w), \mathcal{O}_{Q(v, w)}(\lambda))^\vee$ is the intersection of two Demazure modules of $\mathcal{X}(\lambda)$ spanned by a subset of the global basis of $\mathcal{X}(\lambda)$ if $\lambda$ is strictly dominant.
If we have \( w', v' \in W_{af} \) such that \( Q(v', w') \subset Q(v, w) \), then the restriction map

\[
H^0(Q(v, w), \mathcal{O}_{Q(v, w)}(\lambda)) \to H^0(Q(v', w'), \mathcal{O}_{Q(v', w')}(\lambda))
\]

is surjective.

Note that Corollary \( \mathfrak{B} \) adds new vanishing region to \([12, \text{Theorem 3.1 1}])]. We also provide parabolic versions of Theorem \( \mathfrak{A} \) and Corollary \( \mathfrak{A} \).

We also have a description of \( H^0(Q(v, w), \mathcal{O}_{Q(v, w)}(\lambda)) \) for general \( \lambda \in P_+ \) that is more complicated (Appendix B).

Let \( B_{2, \beta} \) be the space of genus zero stable maps with two marked points to \( B \) with the class of its image \( \beta \in Q^+_G \subset Q_G^+ \cong H_2(B, \mathbb{Z}) \). We have evaluation maps \( \mathfrak{e}_j : B_{2, \beta} \to B \) for \( j = 1, 2 \). The following purely geometric result is a byproduct of our proof, that maybe of independent interest.

**Corollary D** (\( \doteq \) Corollary \([12, 4.18]\)). Let \( \beta \in Q^+_G \), and let \( x, y \in B \). The space \( (\mathfrak{e}_1^{-1}(x) \cap \mathfrak{e}_2^{-1}(\overline{By})) \) is connected if it is nonempty.

Note that Corollary \( \mathfrak{D} \) is contained in \([17]\) whenever \( \{x\}, \overline{By} \subset B \) are in general position.

The plan of this paper is as follows: In section one, we collect basic material needed in the sequel. In section two, after recalling generalities on Frobenius splitting and representation theory of quantum loop algebras, we construct the indscheme \( Q_{rat}^G \) and equip it with a Frobenius splitting (Corollary \( \mathfrak{A} \)). In section three, we first interpret \( Q_{rat}^G \) as an indscheme (coarsely) representing the coset \( G((\mathbb{K}(z)))/H((\mathbb{K}))N((\mathbb{K}(z))) \) (Theorem \( \mathfrak{A} \)). Using this, we identify some Richardson varieties of \( Q_{rat}^G \) with quasi-map spaces (Theorem \( \mathfrak{B} \)) and present their cohomological properties (Theorem \( \mathfrak{C} \)), and hence prove (large parts of) Theorem \( \mathfrak{B} \) and Corollary \( \mathfrak{C} \). Since our construction equips quasi-map spaces with Frobenius splittings (Lemma \( \mathfrak{C} \)), they are automatically weakly normal. Moreover, we explain how to connect characteristic zero and positive characteristic (\( \mathfrak{X} \)). In section four, we analyze the fibers of the graph space resolutions of quasi-map spaces and deduce that Richardson varieties of semi-infinite flag manifolds (over \( \mathbb{C} \)) are normal based on the weak normality proved in the previous section. This proves the remaining part of Theorem \( \mathfrak{B} \). Our analysis here contains an inductive proof that the fibers of the evaluation maps of the space of genus zero stable maps to flag varieties are connected (Corollary \( \mathfrak{D} \)). In Appendix A, we prove the normality of (the ind-pieces of) our indscheme \( Q_{rat}^G \) and present an analogue of the Kempf vanishing theorem for them. Appendix B exhibits the structure of global sections of nef line bundles of Richardson varieties of \( Q_{rat}^G \).

Note that Theorem \( \mathfrak{B} \) equips a quasi-map space from \( \mathbb{P}^1 \) to \( B \) with a Frobenius splitting compatible with the boundaries. However, the notion of boundary in quasi-map spaces depend on a configuration of points in \( \mathbb{P}^1 \) (we implicitly set them to \( \{0, \infty\} \subset \mathbb{P}^1 \) throughout this paper). This makes our analogue of open Richardson variety not necessarily smooth contrary to the original case \([11, 8.4.1]\). The author hope to give more account of this, as well as the factorization structure \([11, 6.3]\) from the view point presented in this paper, in his future works.
1 Preliminaries

We work over an algebraically closed field \( \mathbb{K} \) unless stated otherwise at the beginning of each section. A vector space is a \( \mathbb{K} \)-vector space, and a graded vector space refers to a \( \mathbb{Z} \)-graded vector space whose graded pieces are finite-dimensional and its grading is bounded from the above or from the below. Tensor products are taken over \( \mathbb{K} \) unless specified otherwise.

Let \( A \) be a PID. For a graded free \( A \)-module \( M = \bigoplus_{m \in \mathbb{Z}} M_m \), we set \( M^\vee := \bigoplus_{m \in \mathbb{Z}} \text{Hom}_A(M_m, A) \), where \( \text{Hom}_A(M_m, A) \) is understood to be degree \(-m\).

As a rule, we suppress \( \emptyset \) and associated parenthesis from notation. This particularly applies to \( \emptyset = J \subset I \) frequently used to specify parabolic subgroups.

1.1 Groups, root systems, and Weyl groups

We refer to [22, 67] for precise expositions of general material presented in this subsection.

Let \( G \) be a connected, simply connected simple algebraic group of rank \( r \) over an algebraically closed field \( \mathbb{K} \), and let \( B \) and \( H \) be a Borel subgroup and a maximal torus of \( G \) such that \( H \subset B \). We set \( N := [B, B] \) to be the unipotent radical of \( B \) and let \( N^- \) be the opposite unipotent subgroup of \( N \) with respect to \( H \). We denote the Lie algebra of an algebraic group by the corresponding German small letter. We have a (finite) Weyl group \( W := N_G(H)/H \). For an algebraic group \( E \), we denote its set of \( \mathbb{K}[z] \)-valued points by \( E[z] \), its set of \( \mathbb{K}[z] \)-valued points by \( E[z] \), and its set of \( \mathbb{K}((z)) \)-valued points by \( E((z)) \) etc...

Let \( I \subset G[z] \) be the preimage of \( B \subset G \) via the evaluation at \( z = 0 \) (the Iwahori subgroup of \( G[z] \)). We set \( I^- \subset G[z^{-1}] \) be the opposite Iwahori subgroup of \( I \) in \( G((z)) \) with respect to \( H \). By abuse of notation, we might consider \( I \) and \( G[z] \) as group schemes over \( \mathbb{K} \) whose \( \mathbb{K} \)-valued points are given as these.

Let \( P := \text{Hom}_\mathbb{K}(H, G_m) \) be the weight lattice of \( H \), let \( \Delta \subset P \) be the set of roots, let \( \Delta_+ \subset \Delta \) be the set of roots that yield root subspaces in \( \mathfrak{h} \), and let \( \Pi \subset \Delta_+ \) be the set of simple roots. We set \( \Delta^- := -\Delta_+ \). Let \( Q^\vee \) be the dual lattice of \( P \) with a natural pairing \( \langle \bullet, \bullet \rangle : Q^\vee \times P \to \mathbb{Z} \). We define \( \Pi^\vee \subset Q^\vee \) to be the set of positive simple coroots, and let \( Q^\vee_+ \subset Q^\vee \) be the set of non-negative integer span of \( \Pi^\vee \). For \( \beta, \gamma \in Q^\vee \), we define \( \beta \geq \gamma \) if and only if \( \beta - \gamma \in Q^\vee_+ \). We set \( P_+ := \{ \lambda \in P \mid \langle \alpha_\gamma, \lambda \rangle \geq 0, \forall \alpha_\gamma \in \Pi^\vee \} \) and \( P_{++} := \{ \lambda \in P \mid \langle \alpha_\gamma, \lambda \rangle > 0, \forall \alpha_\gamma \in \Pi^\vee \} \). Let \( I := \{ 1, 2, \ldots, r \} \). We fix bijections \( \Pi \cong \Pi^\vee \) so that \( i \in I \) corresponds to \( \alpha_i \in \Pi \), its coroot \( \alpha_i^\vee \in \Pi^\vee \), and a simple reflection \( s_i \in W \) corresponding to \( \alpha_i \). Let \( \{ \varpi_i \}_{i \in I} \subset P_+ \) be the set of fundamental weights (i.e. \( \langle \alpha_i^\vee, \varpi_j \rangle = \delta_{ij} \) and \( \rho := \sum_{i \in I} \varpi_i = \frac{1}{2} \sum_{\alpha \in \Delta_+, \alpha} \alpha \in P_+ \).

For a subset \( J \subset I \), we define \( P(J) \) as the standard parabolic subgroup of \( G \) corresponding to \( J \). I.e. we have \( B \subset P(J) \subset G \) and \( P(J) \) contains the root subspace corresponding to \( -\alpha_i \) (i.e. \( i \in I \) if and only if \( i \in J \). Then the set of characters of \( P(J) \) is identified with \( P_J := \sum_{\iota \in \Pi(J)} \mathbb{Z} \iota \). We also set \( P_{J+} := \sum_{\iota \in \Pi(J)} \mathbb{Z}_{\geq 0} \iota = P_+ \cap P_J \) and \( P_{J++} := \sum_{\iota \in \Pi(J)} \mathbb{Z}_{> 0} \iota = P_{++} \cap P_J \). We define \( W_J \subset W \) to be the reflection subgroup generated by \( \{ s_i \}_{i \in J} \). It is the Weyl group of the semisimple quotient of \( P(J) \).

Let \( \Delta_{af} := \Delta \times \mathbb{Z}_{\delta} \cup \{ m\delta \}_{m \neq 0} \) be the untwisted affine root system of \( \Delta \) with its positive part \( \Delta_+ \subset \Delta_{af} \). We set \( \alpha_0 := -\delta + \delta, \Pi_{af} := \Pi \cup \{ \alpha_0 \}, \) and \( \Pi_{af} := I \cup \{ 0 \} \), where \( \delta \) is the highest root of \( \Delta_+ \). We set \( W_{af} := W \times Q^\vee_+ \) and
call it the affine Weyl group. It is a reflection group generated by \( \{ s_i \mid i \in \mathcal{I}_{af} \} \), where \( s_0 \) is the reflection with respect to \( \alpha_0 \). We also have a reflection \( s_{a_i} \in W_{af} \) corresponding to \( \alpha \in \Delta \times \mathbb{Z}_{\leq 0} \subseteq \Delta_{af} \). Let \( \ell : W_{af} \to \mathbb{Z}_{\geq 0} \) be the length function and let \( w_0 \in W \) be the longest element in \( W \subset W_{af} \). Together with the normalization \( t_{-\theta^\vee} := s_{\beta} s_0 \) (for the coroot \( \theta^\vee \) of \( \theta \)), we introduce the translation element \( t_{\beta} \in W_{af} \) for each \( \beta \in Q^\vee \).

For each \( i \in \mathcal{I}_{af} \), we have a connected algebraic group \( SL(2,i) \) that is isomorphic to \( SL(2) \) equipped with an inclusion \( SL(2,i)(\mathbb{K}) \subset G((\mathbb{Z})) \) as groups corresponding to \( \pm \alpha_i \in \mathcal{I}_{af} \). Let \( P_{i,\alpha_i} : G_m \to SL(2,i) \) denote the unipotent one-parameter subgroup corresponding to \( \pm \alpha_i \in \mathcal{I}_{af} \). We set \( B_i := SL(2,i) \cap I \), that is a Borel subgroup of \( SL(2,i) \). For each \( i \in I \), we set \( P_i := P((i)) \). For each \( i \in \mathcal{I}_{af} \), we set \( I(i) := SL(2,i)I \). Each \( I(i) \) is a pro-algebraic group.

As a variation of \([\text{[3]}],[19],[\text{Chap VI}]\), we say an indscheme \( X \) over \( \mathbb{K} \) admits a \( G((\mathbb{Z})) \)-action if it admits an action of \( I \) and \( SL(2,i) \) \((i \in \mathcal{I}_{af}) \) as (ind)schemes over \( \mathbb{K} \) that coincides on \( B_i = (I \cap SL(2,i)) \), and they generate a \( G((\mathbb{Z})) \)-action on the set of closed points of \( X \) (the latter is a group action on a set). We consider the notion of \( G((\mathbb{Z})) \)-equivariant morphisms accordingly.

Let \( W_{af}^- \) denote the set of minimal length representatives of \( W_{af}/W \) in \( W_{af} \). We set
\[
Q^\vee := \{ \beta \in Q^\vee \mid \langle \beta, \alpha_i \rangle < 0, \forall i \in I \}.
\]

Let \( \leq \) be the Bruhat order of \( W_{af} \). In other words, \( w \leq v \) holds if and only if a subexpression of a reduced decomposition of \( v \) yields a reduced decomposition of \( w \). We define the generic (semi-infinite) Bruhat order \( \leq^? \) as:
\[
w \leq^? v \Leftrightarrow wt_{\beta} \leq vt_{\beta} \quad \text{for every } \beta \in Q^\vee \text{ so that } \langle \beta, \alpha_i \rangle < 0 \text{ for } i \in I. \quad (1.1)
\]
By \([\text{[3]}],[19],[\text{Chap VI}]\), this defines a preorder on \( W_{af} \). Here we remark that \( w \leq v \) if and only if \( w \leq^? v \) for \( w, v \in W \). See also \([\text{[52]}, \S 2.2] \).

For each \( u \in W \) and \( \beta \in Q^\vee \), we set
\[
\ell_{\beta}^Z (ut_{\beta}) := \ell(u) + \sum_{\alpha \in \Delta_+} \langle \beta, \alpha \rangle = \ell(u) + 2 \langle \beta, \rho \rangle.
\]

**Theorem 1.1** (Lusztig \([\text{[3]}]) cf. \([\text{[1]}],[\text{[27]}],[\text{Proposition 4.4}]\). For each \( w, v \in W_{af} \) such that \( w \leq^? v \), then there exists \( \alpha \in \Delta_{af}^+ \) such that \( w \leq^? s_{\alpha} v \leq^? v \) and \( \ell^Z(s_{\alpha}v) = \ell^Z(v) + 1 \).

For each \( \lambda \in P_+ \), we denote the corresponding Weyl module by \( V(\lambda) \) (cf. \([\text{[11]}],[\text{Proposition 1.22}]\)). By convention, \( V(\lambda) \) is a finite-dimensional indecomposable \( G \)-module with a cyclic \( B \)-eigenvector \( v_\lambda^0 \) with its \( H \)-weight \( \lambda \) whose character obeys the Weyl character formula. For a semi-simple \( H \)-module \( V \), we set
\[
\text{ch } V := \sum_{\lambda \in P} e^\lambda \cdot \dim \text{Hom}_H(\mathbb{K}_\lambda, V).
\]
If \( V \) is a \( Z \)-graded \( H \)-module in addition, then we set
\[
\text{gch } V := \sum_{\lambda \in P, n \in \mathbb{Z}} q^n e^\lambda \cdot \dim \text{Hom}_H(\mathbb{K}_\lambda, V_n).
\]

Let \( \mathcal{B} := G/B \) and call it the flag manifold of \( G \). We have the Bruhat decomposition
\[
\mathcal{B} = \bigsqcup_{w \in W} \mathcal{O}_{\mathcal{B}}(w) \quad (1.2)
\]
into $B$-orbits such that $\dim \mathcal{O}_B(w) = \ell(w_0) - \ell(w)$ for each $w \in W \subset W_{af}$. We set $B(w) := \mathcal{O}_B(w) \subset B$.

For each $\lambda \in P$, we have a line bundle $O_B(\lambda)$ such that

$$H^0(B, O_B(\lambda)) \cong V(\lambda)^*, \quad O_B(\lambda) \otimes O_B(-\mu) \cong O_B(\lambda - \mu) \quad \lambda, \mu \in P_+.$$ 

For each $w \in W$, let $p_w \in \mathcal{O}_B(w)$ be the unique $H$-fixed point. We normalize $p_w$ (and hence $\mathcal{O}_B(w)$) so that the restriction of $O_B(\lambda)$ to $p_w$ is isomorphic to $K_{-w_0w}\lambda$ for every $\lambda \in P_+$. (Here we warn that the convention differs from [19].)

### 1.2 Representations of affine and current algebras

In the rest of this section, we work over $K = \mathbb{C}$, the field of complex numbers. Material in this subsection is transferred to every field in $\mathbb{F}$.

Let $\widehat{\mathfrak{g}}$ be the untwisted affine Kac-Moody algebra associated to $\mathfrak{g}$. I.e. we have

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] \oplus CK \oplus Cd,$$

where $K$ is central, $[d, X \otimes z^m] = mX \otimes z^m$ for each $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$, and for each $X, Y \in \mathfrak{g}$ and $f, g \in \mathbb{C}[z, z^{-1}]$ it holds:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)_0 \cdot K \cdot \text{Res}_{z=0} f \frac{\partial g}{\partial z},$$

where $\langle \alpha, \beta \rangle_0$ denotes the $G$-invariant bilinear form such that $\langle \alpha^\vee, \beta^\vee \rangle_0 = 2$ for a long simple root $\alpha$. Let $E_i, F_i$ ($i \in \mathbb{I}_{af}$) denote the Kac-Moody generators of $\widehat{\mathfrak{g}}$ corresponding to $\alpha_i$. We set $\mathfrak{h} := \mathfrak{h} \oplus CK \oplus Cd$. Let $\mathfrak{f}$ be the Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by $E_i$ ($i \in \mathbb{I}_{af}$) and $\mathfrak{h}$, and $\mathfrak{f}^-$ be the Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by $F_i$ ($i \in \mathbb{I}_{af}$) and $\mathfrak{h}$. For each $i \in \mathbb{I}_{af}$ and $n \geq 0$, we set $E_i^{(n)} := \frac{1}{n!} E_i^n$ and $F_i^{(n)} := \frac{1}{n!} F_i^n$.

We define

$$Q_{af, \vee} := \mathbb{Z} d \oplus \bigoplus_{i \in \mathbb{I}_{af}} \mathbb{Z} \alpha_i^\vee \quad \text{and} \quad P_{af} := \mathbb{Z} \delta \oplus \bigoplus_{i \in \mathbb{I}_{af}} \mathbb{Z} \Lambda_i$$

and a pairing $Q_{af, \vee} \times P_{af} \to \mathbb{Z}$ such that

$$\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij} \quad (i, j \in \mathbb{I}_{af}), \quad \langle \alpha_i^\vee, \delta \rangle = 0, \quad \langle d, \Lambda_i \rangle = \delta_{i0} \quad (i \in \mathbb{I}_{af}), \quad \langle d, \delta \rangle = 1.$$

This form extends the form $Q^\vee \times P \to \mathbb{Z}$ via the embedding

$$P \oplus \mathbb{Z}\delta \ni \varpi_i + m\delta \mapsto \Lambda_i - \langle \alpha_i^\vee + \vartheta^\vee, \Lambda_i \rangle \Lambda_0 + m\delta \in P_{af} \quad m \in \mathbb{Z}.$$ 

We refer the image of this embedding as the set of level zero weights. We have a projection

$$P_{af} \ni \Lambda \mapsto \hat{\Lambda} \in P \quad \Lambda_0, \delta \mapsto 0, \quad \Lambda_i \mapsto \varpi_i \quad (i \in \mathbb{I}).$$

We set $P_{af} := \sum_{i \in \mathbb{I}} \mathbb{Z}_{\geq 0} \Lambda_i$. Each $\Lambda \in P_{af}$ defines an irreducible integrable highest weight module $L(\Lambda)$ of $\widehat{\mathfrak{g}}$ with its highest weight vector $v_\Lambda$. In addition, each $\lambda \in P_+$ defines a level zero extremal weight module $X(\lambda)$ of $\widehat{\mathfrak{g}}$ by means of the specialization of the quantum parameter $q = 1$ in [18, Proposition 8.2.2].
and [13, §5.1], that is integrable and \( \mathfrak{K} \) acts by 0. The module \( X(\lambda) \) carries a cyclic \( \mathfrak{h} \)-weight vector \( \mathbf{v}_\lambda \) such that:

\[
H \mathbf{v}_\lambda = \lambda(H) \mathbf{v}_\lambda \quad (H \in \mathfrak{h}), \quad K \mathbf{v}_\lambda = 0 = d \mathbf{v}_\lambda, \quad E_i \mathbf{v}_\lambda = 0 \quad (i \in \mathfrak{I}), \quad \text{and} \quad F_0 \mathbf{v}_\lambda = 0.
\]

(We can deduce that \( X(\lambda) \) is the maximal integrable \( \mathfrak{g} \)-module that possesses a cyclic vector with the above properties [13, §8.1].) Moreover, each \( w = u t_\beta \in W_d \) \((u \in W, \beta \in Q^+)\) defines an element \( \mathbf{v}_{w\lambda} \in X(\lambda) \) such that

\[
H \mathbf{v}_{w\lambda} = (w\lambda)(H) \mathbf{v}_{w\lambda} \quad (H \in \mathfrak{h}), \quad K \mathbf{v}_{w\lambda} = 0, \quad d \mathbf{v}_{w\lambda} = -\langle \beta, \lambda \rangle \mathbf{v}_{w\lambda}
\]

up to sign (see [13, §8.1]). We call a vector in \( \{ \mathbf{v}_{w\lambda} \}_{w \in W_d} \) an extremal weight vector of \( X(\lambda) \).

We set \( \mathfrak{g}[z] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z] \) and regard it as a Lie subalgebra of \( \mathfrak{g} \). We have \( \mathfrak{J} \subset \mathfrak{g}[z] + \mathbb{C}K + \mathbb{C}d \). The Lie algebra \( \mathfrak{g}[z] \) is graded, and its grading is the internal grading of \( \mathfrak{g} \) given by \( d \).

For each \( \lambda \in P_+ \), we set

\[
W_w(\lambda) := U(\mathfrak{J}) \mathbf{v}_{w\lambda} \subset X(\lambda).
\]

These are the \( q = 1 \) cases of the Demazure modules of \( X(\lambda) \), as well as the generalized global Weyl modules in the sense of [13]. We set \( \mathbb{W}(\lambda) := W_w(\lambda) \).

By construction, the both of \( X(\lambda) \) and \( \mathbb{W}(\lambda) \) are semi-simple as \((H \times \mathbb{G}_m)\)-module, where \( \mathbb{G}_m \) acts on \( z \) by \( a : z^m \mapsto a^m z^m \) \((m \in \mathbb{Z})\).

**Theorem 1.2** [LNSSS 12, Chari-Ion 13, cf. [17] Theorem 1.6]. For each \( \lambda \in P_+ \), the \( \mathfrak{J} \)-action on \( \mathbb{W}(\lambda) \) prolongs to \( \mathfrak{g}[z] \), and it is isomorphic to the global Weyl module of \( \mathfrak{g}[z] \) in the sense of Chari-Pressley [27]. Moreover, \( \mathbb{W}(\lambda) \) is a projective module in the category of \( \mathfrak{g}[z] \)-modules whose restriction to \( \mathfrak{g} \) is a direct sum of modules in \( \{ V(\mu) \}_{\mu \leq \lambda} \).

**Theorem 1.3** ([17] Theorem 3.3 and Corollary 3.4). Let \( \lambda, \mu \in P_+ \) and \( w \in W \). We have a unique (up to scalar) injective degree zero \( \mathfrak{J} \)-module map

\[
\mathbb{W}_w(\lambda + \mu) \hookrightarrow \mathbb{W}_w(\lambda) \otimes \mathbb{W}_w(\mu).
\]

**Sketch of proof.** For each \( \lambda, \mu \in P_+ \), the projectivity of \( \mathbb{W}(\lambda + \mu) \) in the sense of Theorem 1.2 yields a unique graded \( \mathfrak{g}[z] \)-module map

\[
\mathbb{W}(\lambda + \mu) \longrightarrow \mathbb{W}(\lambda) \otimes \mathbb{W}(\mu)
\]

of degree 0. This map is injective by examining the specializations to local Weyl modules. By examining the \( \mathfrak{J} \)-cyclic vectors, it uniquely restricts to a map

\[
\mathbb{W}_w(\lambda + \mu) \longrightarrow \mathbb{W}_w(\lambda) \otimes \mathbb{W}_w(\mu)
\]

up to scalar. This map must be also injective as the ambient map is so. \( \square \)

### 1.3 Semi-infinite flag manifolds

We work over \( \mathbb{C} \) as in the previous subsection. Material in this section is reproved in the setting of characteristic \( \neq 2 \) in [24] and [13] (cf. [13]). We define the semi-infinite flag manifold as the reduced indscheme such that:
We have a closed embedding

\[ Q_G^{\text{rat}} \subset \prod_{i \in I} \mathbb{P}(V(\varpi_i) \otimes \mathbb{C}((z))); \quad \text{and} \]

We have an equality \( Q_G^{\text{rat}}(\mathbb{C}) = G((z))/(H(\mathbb{C}) \cdot N((z))). \)

This is a pure indscheme of ind-infinite type \([52]\). Note that the group \( Q_\mathbb{C}_H((z)) \) acts on \( Q_G^{\text{rat}} \) from the right. The indscheme \( Q_G^{\text{rat}} \) is equipped with a \( G((z)) \)-equivariant line bundle \( O_{Q_G^{\text{rat}}}(\lambda) \) for each \( \lambda \in P \). Here we normalized so that \( (Q_G^{\text{rat}}; O_{Q_G^{\text{rat}}}) \) is \( B((z)) \)-cocyclic to a \( H \)-weight vector with its \( H \)-weight \( -\lambda \). We warn that this convention is twisted by \( w_0 \) from that of \([49]\), and complies with \([52]\).

**Theorem 1.4** \([31, 25, 52, 64]\). We have an \( I \)-orbit decomposition

\[ Q_G^{\text{rat}} = \bigsqcup_{w \in W_{af}} O(w) \]

with the following properties:

1. Each \( O(w) \) is isomorphic to \( \mathbb{A}^\infty \) and have a unique \( (H \times \mathbb{G}_m) \)-fixed point;
2. The right action of \( \gamma \in Q' \) on \( Q_G^{\text{rat}} \) yields the translation \( O(w) \mapsto O(\gamma \cdot w) \);
3. The relative dimension of \( O(w \cdot \beta) \) (\( w \in W, \beta \in Q' \)) and \( O(e) \) is \( \ell(\beta) \);
4. We have \( O(w) \subset O(v) \) if and only if \( w \leq v \).

For each \( w \in W_{af} \), let \( Q_G(w) \) denote the closure of \( O(w) \). We refer \( Q_G(w) \) as a Schubert variety of \( Q_G^{\text{rat}} \) (corresponding to \( w \in W_{af} \)).

Let \( S = \bigoplus_{\lambda \in P_{+,+}} S(\lambda) \) be a \( P_{+,+} \)-graded commutative ring such that \( S(0) = A \) is a PID, \( S \) is torsion-free over \( A \), and \( S \) is generated by \( \bigoplus_{i \in I \setminus J} S(\varpi_i) \). We define

\[ \text{Proj} S = (\text{Spec} S \setminus E)/H \subset \prod_{i \in I \setminus J} \mathbb{P}_A(S(\varpi_i)^\vee) \]  

as the \( P_{+,+} \)-graded proj over \( \text{Spec} A \), where \( E \) is the locus that whole of \( S(\varpi_i) \) vanishes for some \( i \in I \setminus J \) (the irrelevant locus).

**Theorem 1.5** \([52\] Theorem 4.26 and Corollary 4.27). For each \( w \in W_{af} \), it holds:

\[ Q_G(w) \cong \text{Proj} \bigoplus_{\lambda \in P_+} W_{w_0}(\lambda)^\vee, \]

where the multiplication is given by Theorem 1.4.

### 1.4 Quasi-map spaces and Zastava spaces

We work over \( \mathbb{C} \) as in the previous subsection unless stated otherwise. Here we recall basics of quasi-map spaces from \([31, 68]\).

We have \( W \)-equivariant isomorphisms \( H^2(B, \mathbb{Z}) \cong P \) and \( H_2(B, \mathbb{Z}) \cong Q' \). This identifies the (integral points of the) nef cone of \( B \) with \( P_+ \subset P \) and the
effective cone of \( B \) with \( Q^Y \). A quasi-map \((f, D)\) is a map \( f : \mathbb{P}^1 \to B \) together with a \( \Pi^Y \)-colored effective divisor
\[
D = \sum_{\alpha \in \Pi^Y, x \in \mathbb{P}^1(C)} m_x(\alpha^Y) \alpha^Y \otimes [x] \in Q^Y \otimes \mathbb{Z} \text{ Div } \mathbb{P}^1 \text{ with } m_x(\alpha^Y) \in \mathbb{Z}_{\geq 0}.
\]
For \( i \in \mathbb{I} \), we set \( D_i := \langle D, \varpi_i \rangle \in \text{ Div } \mathbb{P}^1 \). We call \( D \) the defect of the quasi-map \((f, D)\). Here we define the degree of the defect by
\[
|D| := \sum_{\alpha \in \Pi^Y, x \in \mathbb{P}^1(C)} m_x(\alpha^Y) \alpha^Y \in Q^Y_+.
\]
For each \( \beta \in Q^Y \), we set
\[
\Omega(\beta) := \{ f : \mathbb{P}^1 \to X \mid \text{ quasi-map s.t. } f_*[\mathbb{P}^1] + |D| = \beta \},
\]
where \( f_*[\mathbb{P}^1] \) is the class of the image of \( \mathbb{P}^1 \) multiplied by the degree of \( \mathbb{P}^1 \to \text{ Im } f \).
We denote \( \Omega(\beta) \) by \( \Omega(\beta) \) in case there is no danger of confusion.

**Definition 1.6** (Drinfeld-Plücker data). Consider a collection \( \mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in \mathcal{P}_+} \) of inclusions \( \psi_\lambda : \mathcal{L}^\lambda \supseteq V(\lambda) \otimes \mathcal{O}_{\mathbb{P}^1} \) of line bundles \( \mathcal{L}^\lambda \) over \( \mathbb{P}^1 \). The data \( \mathcal{L} \) is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of \( G \)-modules
\[
\eta_{\lambda, \mu} : V(\lambda + \mu) \leftrightarrow V(\lambda) \otimes V(\mu)
\]
induces an isomorphism
\[
\eta_{\lambda, \mu} \otimes \text{id} : \psi_{\lambda+\mu}(\mathcal{L}^{\lambda+\mu}) \stackrel{\sim}{\longrightarrow} \psi_\lambda(\mathcal{L}^\lambda) \otimes \mathcal{O}_{\mathbb{P}^1} \psi_\mu(\mathcal{L}^\mu)
\]
for every \( \lambda, \mu \in \mathcal{P}_+ \).

**Theorem 1.7** (Drinfeld, see Finkelberg-Mirković [31]). The variety \( \Omega(\beta) \) is isomorphic to the variety formed by isomorphism classes of the DP-data \( \mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in \mathcal{P}_+} \) such that \( \text{ deg } \mathcal{L}^\lambda = \langle w_0 \beta, \lambda \rangle \). In addition, \( \Omega(\beta) \) is irreducible.

For each \( w \in W \), let \( \mathcal{Z}(\beta, w) \subseteq \Omega(\beta) \) be the locally closed subset consisting of quasi-maps that are defined at \( z = 0 \), and their values at \( z = 0 \) are contained in \( B(w) \subseteq B \). We set \( \Omega(\beta, w) := \mathcal{Z}(\beta, w) \). (Hence, we have \( \Omega(\beta) = \Omega(\beta, e) \).)

**Theorem 1.8** (Finkelberg-Mirković [31]). Let \( \mathbb{C} \) be an algebraically closed field, and let \( \Omega(\beta)_\mathbb{C} \) and \( \mathcal{Z}(\beta, w_0)_\mathbb{C} \) be the spaces obtained by replacing the base field \( C \) with \( \mathbb{C} \) in Definition 1.5. For each \( \beta \in Q^Y_+ \), the space \( \mathcal{Z}(\beta, w_0) \) is an irreducible affine scheme equipped with a \((B \times \mathbb{G}_m)\)-action with a unique fixed point.

**Remarks on proof.** Theorem 1.8 is proved in [31] for \( \mathbb{K} = \mathbb{C} \) using [32], and is proved in the current setting in [4] using [8]. One can also replace the usage of [6] with [32], Corollary 5.3.8 along the lines of [31].

For each \( \lambda \in \mathcal{P} \) and \( w \in W \), we have a \( G \)-equivariant line bundle \( \mathcal{O}_\mathcal{Q}(\beta, w)(\lambda) \) (and its pro-object \( \mathcal{O}_\mathcal{Q}(\lambda) \)) obtained by the (tensor products of the) pull-backs \( \mathcal{O}_\mathcal{Q}(\beta, w)(\varpi_i) \) of the \( i \)-th \( \mathcal{O}(1) \) via the embedding
\[
\Omega(\beta, w) \hookrightarrow \prod_{i \in \mathbb{I}} \mathbb{P}(\mathcal{V}(\varpi_i) \otimes \mathbb{C}[z]_{\leq -(w_0 \beta, \varpi_i)}), \tag{1.4}
\]
for each \( \beta \in Q^Y_+ \).

We have embeddings \( B \subseteq \Omega(\beta) \subseteq \mathcal{Q}_G(e) \) such that the line bundles \( \mathcal{O}(\lambda) \) \((\lambda \in \mathcal{P})\) correspond to each other by restrictions ([12] [13] [32]).
2 Semi-infinite flag manifolds over $\mathbb{Z}[\frac{1}{2}]$

We keep the settings of the previous section. In this section, we sometimes work over a (commutative) ring or a non-algebraically closed field. For a commutative ring $S$ or a scheme $X$, we may write $S_A$ and $X_A$ if it is defined over $A$. In addition, we may consider their scalar extensions $S_B := S_A \otimes_A B$ and $X_B$ for a ring map $A \to B$.

2.1 Frobenius splittings

Let $k$ be a field, and let $p$ be a prime. We assume $\text{char } k = p > 0$ throughout this subsection.

We follow the generality on Frobenius splittings in [16], that considers separated schemes of finite type. We sometimes use the assertions from [16] without finite type assumption when the assertion is independent of that, whose typical disguises are properness, finite generation, and the Serre vanishing theorem.

**Definition 2.1** (Frobenius splitting of a ring). Let $R$ be a commutative ring over $k$, and let $R^{(1)}$ denote the set $\mathbb{Z}(\mathbb{N})$ equipped with the map $R \ni (r, m) \mapsto r^p m \in R^{(1)}$.

This equips $R^{(1)}$ an $R$-module structure over $k$ (the $k$-vector space structure on $R^{(1)}$ is also twisted by the $p$-th power operation), together with an inclusion $\iota : R \to R^{(1)}$. An $R$-module map $\phi : R^{(1)} \to R$ is said to be a Frobenius splitting if $\phi \circ \iota$ is an identity.

**Definition 2.2** (Frobenius splitting of a scheme). Let $X$ be a separated scheme defined over $k$. Let $\text{Fr}$ be the (relative) Frobenius endomorphism of $X$ (that induces a $k$-linear endomorphism). We have a natural inclusion $\iota : \mathcal{O}_X \to \text{Fr}_* \mathcal{O}_X$. A Frobenius splitting of $X$ is a $\mathcal{O}_X$-linear morphism $\phi : \text{Fr}_* \mathcal{O}_X \to \mathcal{O}_X$ such that the composition $\phi \circ \iota$ is the identity.

**Definition 2.3** (Compatible splitting). Let $Y \subset X$ be a closed immersion of separated schemes defined over $k$. A Frobenius splitting $\phi$ of $X$ is said to be compatible with $Y$ if $\phi(\text{Fr}_* I_Y) \subset I_Y$, where $I_Y := \ker(\mathcal{O}_X \to \mathcal{O}_Y)$.

**Remark 2.4.** A Frobenius splitting of $X$ compatible with $Y$ induces a Frobenius splitting of $Y$ (see e.g. [16, Remark 1.1.4 (ii)]).

**Theorem 2.5** ([16] Lemma 1.1.11 and Exercise 1.1.E). Let $X$ be a separated scheme of finite type over $k$ with semiample line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$. If $X$ admits a Frobenius splitting, then the multi-section ring

$$\bigoplus_{n_1, \ldots, n_r \geq 0} \Gamma(X, \mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_r^{\otimes n_r})$$

admits a Frobenius splitting $\phi$. Moreover, a closed subscheme $Y \subset X = \text{Proj } S$ admits a compatible Frobenius splitting if and only if the homogeneous ideal $I_Y \subset S$ that defines $Y$ satisfies $\phi(I_Y) \subset I_Y$. \qed
**Definition 2.6** (Canonical splitting). Let $X$ be a separated scheme equipped with a $B$-action. A Frobenius splitting $\phi$ is said to be $B$-canonical if it is $H$-fixed, and each $i \in I$ yields

\[
\rho_{\alpha_i}(z^p)\phi(\rho_{\alpha_i}(-z)f) = \sum_{j=0}^{p-1} z^j \phi_{i,j}(f),
\]

where $\phi_{i,j} \in \text{Hom}_{\mathcal{O}_X}(\text{Fr}_{*}\mathcal{O}_X, \mathcal{O}_X)$. We similarly define the notion of $B^-$-canonical splitting (resp. $B^+$-canonical splitting and $I^-$-canonical splitting) by using $\{\rho_{-\alpha_i}\}_{i \in I}$ (resp. $\{\rho_{\alpha_i}\}_{i \in I}$ and $\{\rho_{-\alpha_i}\}_{i \in I}$) instead. Canonical splittings of a commutative ring $S$ over $k$ is defined through its spectrum.

**Proposition 2.7** ([14] Proposition 4.1.8). Let $S = \bigoplus_{m \geq 0} S_m$ be a graded ring with $S_0 = k$ such that

- $S$ is equipped with a degree preserving $I$-action;
- Each $S_m$ is a graded $k$-vector space compatible with the multiplication;
- We have an $I$-canonical Frobenius splitting $\phi : S^{(1)} \to S$.

Then, the induced map

\[
\phi^\vee : S_m^\vee \longrightarrow S_{pm}^\vee \quad m \in \mathbb{Z}_{\geq 0}
\]

satisfies

\[
\phi^\vee(E_i^{(n)}v) = E_i^{(pn)}\phi^\vee(v) \quad \forall i \in I, n \in \mathbb{Z}_{\geq 0}, v \in S_m^\vee.
\]

Similar results hold for the $I^-$- and $B^\pm$-actions.

**Remark 2.8.** In the opinion of the author, a merit of Proposition [14] over [14] Proposition 4.1.8 is that it becomes obvious that a projective variety $X$ with a $B$-action has at most one $B$-canonical splitting whenever the space of global sections of all ample line bundles are (or can be made) $B$-cyclic compatible with multiplications (cf. [14] Theorem 4.1.15) and Corollary [4.5.7].

**Proof of Proposition 2.7.** The condition that $S_m$ is a graded vector space implies $S_m^\vee \cong (S_m^{(1)})^\vee$ for each $m \in \mathbb{Z}_{\geq 0}$. By [14] Proposition 4.1.8, each $w \in S_{pm} \subset S^{(1)}$ satisfies $\phi(E_i^{(pn)}w) = E_i^{(n)}\phi(w)$ for $i \in I$ and $n \geq 0$. Using the natural non-degenerate invariant pairing $\langle \cdot, \cdot \rangle$ between $S_m^\vee$ and $S_m$, we compute the most LHS of

\[
\langle v, \phi(E_i^{(p)}w) \rangle = \langle v, E_i\phi(w) \rangle = -\langle \phi^\vee(E_i^\vee v), w \rangle
\]

as

\[
\langle v, \phi(E_i^{(p)}w) \rangle = - \sum_{k_1=1}^{p} \langle E_i^{(k_1)}\phi^\vee(v), E_i^{(p-k_1)}w \rangle
\]

\[
\ldots = \sum_{m=1}^{p} \sum_{k_1>0, k_2>0, \ldots, k_m=p} (-1)^m \langle E_i^{(k_1)}E_i^{(k_2)} \cdots E_i^{(k_m)} \phi^\vee(v), w \rangle
\]

\[
= - \langle E_i^{(p)}\phi^\vee(v), w \rangle
\]
since we have \( E_i^{(k_1)} E_i^{(k_2)} \cdots E_i^{(k_m)} \in pZ E_i^{(p)} \) except for \( k_1 = p, 0 = k_2 = \cdots \). This implies the case \( n = 1 \).

Similarly, we have

\[
\left\langle \mathbf{v}, \phi(E_i^{(m)}) w \right\rangle = \sum_{m=1}^{n} \sum_{k > 0, k_1 + k_2 + \cdots + k_m = n} (-1)^m \left\langle E_i^{(pk_1)} E_i^{(pk_2)} \cdots \phi' v, w \right\rangle.
\]

Compared with

\[
\left\langle \mathbf{v}, E_i^{(n)} \phi(w) \right\rangle = \sum_{m=1}^{n} \sum_{k > 0, k_1 + k_2 + \cdots + k_m = n} (-1)^m \left\langle \phi(E_i^{(k_1)} E_i^{(k_2)} \cdots \mathbf{v}), w \right\rangle
\]

using induction on \( n \), we conclude the result.

\[ \square \]

2.2 Representations of affine Lie algebras over \( \mathbb{Z} \)

In this section, we systematically use the global basis theory [13, 13, 13, 13, 13, 13] by specializing the quantum parameter \( q \) to 1. Therefore, we might refer these references without an explicit declaration that we specialize \( q \).

We consider the Kostant-Lusztig \( \mathbb{Z} \)-form \( U^+_Z \) (resp. \( U^-_Z \)) of \( U(\mathfrak{g}, \mathfrak{g}) \) (resp. \( U(\mathfrak{g}^{-}, \mathfrak{g}^{-}) \)) obtained as the specialization \( q = 1 \) of the \( \mathbb{Z}[q, q^{-1}] \)-integral form of the quantized enveloping algebras [63, §23.2].

Remark 2.9. We remark that \( U^+_Z \) and \( U^-_Z \) are the same integral forms dealt in [63], and also coincide with the integral forms obtained through the Drinfeld presentation (13, §2) and [62, Lemma 2.5]).

Note that \( U^\pm_Z \) are equipped with the \( \mathbb{Z} \)-bases \( B(\mp \infty) \) obtained by the specialization \( q = 1 \) of the lower global basis [13] (see also [63, §25]). In view of [63, 13], we have an idempotent \( \mathbb{Z} \)-integral form

\[
\hat{U}_Z = \bigoplus_{\Lambda \in P^{af}} U^- Z U^+_Z a_{\Lambda}
\]

such that

\[
a_{\Lambda} a_{\Gamma} = \delta_{\Lambda, \Gamma} a_{\Lambda} \quad A, \Gamma \in P^{af} \quad \text{and,}
\]

\[
E_i^{(m)} a_{\Lambda} = a_{\Lambda + m \alpha_i} E_i^{(m)} , \quad F_i^{(m)} a_{\Lambda} = a_{\Lambda - m \alpha_i} F_i^{(m)} \quad i \in I_{af}, m \in \mathbb{Z}_{\geq 0}.
\]

We set \( \hat{U}^+_Z \subset \hat{U}_Z \) to be the subalgebra (topologically) generated by \( \{ F_i^{(m)} \}_{i \in I, m \in \mathbb{Z}_{\geq 0}}, \{ a_{\Lambda} \}_{\Lambda \in P^{af}}, \) and \( U^- Z \).

If a \( \hat{U}_Z \)-module \( M \) over a field \( k \) admits a decomposition

\[
M = \bigoplus_{\Lambda \in P^{af}} a_{\Lambda} M,
\]

then we call this the \( P^{af} \)-weight decomposition. If \( \Lambda \in P^{af} \) satisfies \( a_{\Lambda} M \neq 0 \), then we call \( \Lambda \) a \( P^{af} \)-weight of \( M \). We define the \( P^{af} \)-character of \( M \) as

\[
gch M := \sum_{\Lambda \in P^{af}} e^{\Lambda} \dim_k a_{\Lambda} M
\]

whenever the RHS makes sense. Note that this is consistent with §13 through the identification \( q = e^\delta \) etc... As in §13, we refer a linear combination of vectors in \( a_{\Lambda} M \) for various \( \Lambda \in P^{af} \) with \( \langle d, \Lambda \rangle = d_0 \) to have \( d \)-degree \( d_0 \).
For each $\lambda \in P$, we set
\[a^0_\lambda M := \sum_{\Lambda \in P^{af}, \Lambda = \lambda} a_\Lambda M.\]
We call the decomposition
\[M = \bigoplus_{\lambda \in P} a^0_\lambda M,\]
the $P$-weight decomposition. We call a non-zero element of $a_\Lambda M$ (resp. $a^0_\lambda M$) a $P^{af}$-weight vector of $M$ (resp. a $P$-weight vector of $M$). We also call $\lambda \in P$ with $a^0_\lambda M \neq \{0\}$ a $P$-weight of $M$.

We set $U_2^0 \subset \hat{U}_Z$ to be the subalgebra of $\hat{U}_Z$ (topologically) generated by \{(\(E^{(m)}_i, F^{(m)}_i\))\}_{i \in \mathbb{I}, m \in \mathbb{Z}_{\geq 0}}$, \{a_\Lambda\}_{\Lambda \in P^{af}}. For a field $k$, a $\hat{U}_k^{\geq 0}$-module $M$ with a $P^{af}$-weight decomposition is said to be $\hat{U}_k^{\geq 0}$-integrable if its \{(E^{(m)}_i, F^{(m)}_i)\}_{m \geq 0}\text{-action}
induces a $SL(2, i)$-action whose $(SL(2, i) \cap H)$-eigenvalues are given by the $P^{af}$-weights for each $i \in \mathbb{I}$.

Note that if a $U(\hat{g})$-module $V$ over $\mathbb{C}$ carries a cyclic $\hat{h}_C$-weight vector whose weight belongs to $P^{af}$ and each of its $\hat{h}_C$-weight space is finite-dimensional, then we have its $U_2$-lattice $V_Z$ inside $V$. In such a case, the module $V_Z \otimes \mathbb{C}$ admits $P^{af}$- or $P$-weight decompositions.

We have the Chevalley involution of $\hat{U}_Z$ defined as:
\[\theta(E^{(m)}_i) = F^{(m)}_i, \quad \theta(F^{(m)}_i) = E^{(m)}_i, \quad \text{and} \quad \theta(a_\Lambda) = a_{-\Lambda} \quad i \in \mathbb{I}_{af}, m \in \mathbb{Z}_{\geq 0}, \Lambda \in P^{af}.\]

**Definition 2.10** ([**HK**] Definition 2.4 and §2.8). A $U(\hat{g})$-module $V$ over $\mathbb{C}$ with a cyclic $\hat{h}$-weight vector $v$ is said to be compatible with the negative global basis if we have
\[U_2^- v = \bigoplus_{b \in \text{B}(\infty)} \mathbb{Z} bv \subset V.\]

If $(V, v)$ is compatible with the negative global basis, then we set
\[\text{B}^-(V) := \text{B}^-(V, v) := \{bv \mid b \in \text{B}(\infty) \text{ s.t. } bv \neq 0\} \subset V\]
and refer them as the negative global basis of $V$.

Compatibility with the positive global basis of $V$ and the positive global basis $\text{B}^+(V) = \text{B}^+(V, v)$ of $V$ is defined similarly.

**Theorem 2.11** (Kashiwara [**K**] Theorem 5). We have:

1. For each $\Lambda \in P^+_af$, the $\hat{g}$-module $L(\Lambda)_{\mathbb{C}}$ is compatible with the negative global basis;
2. For each $\lambda \in P^+_af$, we have
\[V(\lambda)_{\mathbb{C}} = \bigoplus_{b \in \text{B}(\infty) \cap U_2^0} \mathbb{C} bv_\lambda^0.\]

We set $\text{B}(L(\Lambda)) := \text{B}^-(L(\Lambda), v_\Lambda)$ for each $\Lambda \in P^+_af$.

For each $\Lambda \in P^+_af$ and $\lambda \in P^+_af$, we set
\[L(\Lambda)_{\mathbb{Z}} := U_2^- v_\Lambda \subset L(\Lambda)_{\mathbb{C}} \quad \text{and} \quad V(\lambda)_{\mathbb{Z}} := (U_2^- \cap U_2^0) v_\lambda^0 \subset V(\lambda)_{\mathbb{C}}.\]
We set

\[ M(\lambda)_Z := U_Z^{2\mathbb{Z}} \otimes U_Z^{2\mathbb{Z}} V(\lambda)_Z. \]

Since \( \theta^* M(\lambda)_Z \) can be seen as a limit of \( L(\Lambda)_Z \otimes \mathbb{Z}_{\langle K, \Lambda \rangle} \Lambda_0 \) (with \( -\Lambda = X \) and \( \langle d, \Lambda \rangle = 0 \)), where \( \mathbb{Z}_{\langle K, \Lambda \rangle} \Lambda_0 \) shifts the \( P^{af} \)-weight by \( -\langle K, \Lambda \rangle \Lambda_0 \), we conclude that \( \theta^* M(\lambda)_C \) is compatible with the negative global basis. (Hence \( M(\lambda)_C \) is compatible with the positive global basis.)

**Corollary 2.12.** We have:

1. For each \( \Lambda, \Gamma \in P_+^f \), we have a natural inclusion \( L(\Lambda + \Gamma)_Z \hookrightarrow L(\Lambda)_Z \otimes L(\Gamma)_Z \) of \( U_Z \)-modules, that is a direct summand as \( \mathbb{Z} \)-modules;

2. For each \( \lambda, \mu \in P_+ \), we have a natural inclusion \( V(\lambda + \mu)_Z \hookrightarrow V(\lambda)_Z \otimes V(\mu)_Z \) of \( U_Z^{2\mathbb{Z}} \)-modules, that is a direct summand as \( \mathbb{Z} \)-modules.

**Proof.** Since the two cases are completely parallel, we only prove the first case. The \( \mathfrak{g} \)-module \( L(\Lambda)_C \otimes L(\Gamma)_C \) decomposes into the direct sum of integrable highest weight modules ([13, Proposition 9.10]), with a direct summand \( L(\Lambda + \Gamma)_C \). In view of [13, Theorem 3], it gives rise to the \( \mathbb{Z}[q] \)-lattice of the quantized version \( L(\Lambda) \otimes L(\Gamma) \) compatible with those of \( L(\Lambda + \Gamma) \) via the natural embedding. By setting \( q = 1 \), we obtain a direct sum decomposition of \( L(\Lambda)_Z \otimes L(\Gamma)_Z \) as \( \mathbb{Z} \)-modules with its direct summand \( L(\Lambda + \Gamma)_Z \).

**Theorem 2.13** (Kashiwara [13] Proposition 8.2.2). For each \( \lambda \in P_+ \), the \( \mathfrak{g} \)-module \( \mathcal{X}(\lambda)_C \) is compatible with the negative/positive global basis (for every extremal weight vector).

**Theorem 2.14** (Kashiwara [10]). Let \( \lambda \in P_+ \), there exists a \( \mathbb{Z} \)-basis \( \mathcal{B}(\mathcal{X}(\lambda)) \) of \( \mathcal{X}(\lambda)_Z \) that contains the negative/positive global basis of \( \mathcal{X}(\lambda)_Z \) constructed from every extremal weight vector of \( \mathcal{X}(\lambda) \).

**Proof.** We set \( \mathcal{B}(\mathcal{X}(\lambda)) \) to be the specialization of the global basis of a quantum loop algebra module [31, Proposition 8.2.2]. Then, it is compatible with the global basis generated from an extremal weight vectors by [10, Theorem 3.3].

For each \( \lambda \in P_+ \), we set

\[ \mathcal{X}(\lambda)_Z := U_Z \mathcal{V}_\lambda \subset \mathcal{X}(\lambda)_C. \]

For each \( w \in W_{af} \), we define

\[ \mathcal{W}_w(\lambda)_Z := U_Z^{2\mathbb{Z}} \mathcal{V}_{w\lambda} \subset \mathcal{X}(\lambda)_C \quad \text{and} \quad \mathcal{W}^{-}_w(\lambda)_Z := U_Z^{2\mathbb{Z}} \mathcal{V}_{w\lambda} \subset \mathcal{X}(\lambda)_C. \]

We set \( \mathcal{W}(\lambda)_Z := \mathcal{W}_{w_0}(\lambda)_Z \) and \( \mathcal{W}^{-}(\lambda)_Z := \mathcal{W}^{-}_{w_0}(\lambda)_Z \).

**Lemma 2.15** (Naito-Sagaki). For \( \lambda \in P_+ \) and \( w, v \in W_{af} \), we have \( \mathcal{W}_{w_0 w}(\lambda) \subset \mathcal{W}_{w w_0}(\lambda) \) if \( w \leq w_0 v \). If we have \( \lambda \in P_{++} \) in addition, then we have \( \mathcal{W}_{w_0 w}(\lambda) \subset \mathcal{W}_{w w_0}(\lambda) \) if and only if \( w \leq w_0 v \).

**Proof.** Apply the inclusion relation of the (labels of the) global basis in [31, Corollary 3.2.5] (see also [10, §2.8]).
Proposition 2.16. Let $k$ be a field. The module $\mathcal{W}(\lambda)_k$ is the projective cover of $V(\lambda)_k$ in the category of $U^+_k$-modules that are $U^+_k$-integrable and whose $P$-weights are contained in $\text{Conv}W \lambda \subset P \otimes \mathbb{R}$, where $\text{Conv}$ denote the $\mathbb{R}$-convex hull.

Proof. In view of [10, Theorem 4.16], the module $\mathcal{W}(\lambda)_k$ is free over a polynomial ring and we have a finite-dimensional quotient $W(\lambda)_k$ with $\dim_k a^0_0 W(\lambda)_k = 1$.

We equip $\hat{U}^+_k := \bigoplus_{\lambda \in P} U^+_k a^0_0$ with the structure of an algebra, where $a^0_0 a^0_0 = \delta_{\lambda, \mu} a^0_0 (\lambda, \mu \in P)$,

$$E^{(m)}_0 a^0_\lambda = a^0_{\lambda-ma} E^{(m)}_0, \quad E^{(m)}_i a^0_\lambda = a^0_{\lambda+ma}, E^{(m)}_i (i \in I, m \in \mathbb{Z}_{\geq 0}, \lambda \in P),$$

and the assignment $\hat{U}^+_k \ni \xi a^0_\lambda \mapsto \xi \in U^+_k \ (\forall \lambda \in P, \xi \in U^+_k)$ gives rise to an algebra homomorphism. We sometimes regard $\mathcal{W}(\lambda)_k$ as $\mathbb{Z}$-graded $\hat{U}^+_k$-module whose gradings is given by the $d$-degree. We consider the Demazure functor $\mathcal{D}_w$ for $w \in W_{ad}$ with respect to $\hat{U}^+_k$ (cf. [11, 17, 21]). In view of [10, §2.8], the character part of the calculation in [27, Theorem 4.13] carries over to our setting, and hence we have

$$L^* \mathcal{D}_{t_\beta}(\mathcal{W}(\lambda)_k) = \mathcal{D}_{t_\beta}(\mathcal{W}(\lambda)_k) \cong \mathcal{W}(\lambda)_k \otimes_k k_{-(\beta, w_0)\delta} \quad \beta \in Q^\vee.$$ 

From this, we also derive that

$$L^* \mathcal{D}_{t_\beta}(W(\lambda)_k) = \mathcal{D}_{t_\beta}(W(\lambda)_k) \cong W(\lambda)_k \otimes_k k_{-(\beta, w_0)\delta} \quad \beta \in Q^\vee$$

using the Koszul resolution (as in [21, §5.1.4]).

The $d$-gradings of $\hat{U}^+_k$, $\mathcal{W}(\lambda)$, and $W(\mu)_k$ are concentrated in non-negative degrees. It follows that the $d$-grading of $\text{Ext}^*_\hat{U}^+_k(\mathcal{W}(\lambda), W(\mu)_k^*)$ is bounded from the above. Moreover, we have

$$\text{Ext}^*_\hat{U}^+_k(\mathcal{D}_{t_\beta w_0}(\mathcal{W}(\lambda)_k), W(\mu)_k^*) \cong \text{Ext}^*_\hat{U}^+_k(\mathcal{W}(\lambda)_k, \mathcal{D}_{t_{-(\beta, w_0)\mu}}(W(\mu)_k)^*)$$

for every $\lambda, \mu \in P_+$ and $\beta \in Q^\vee$ (by a repeated application of [27, Proposition 5.7]) (whose argument carries over to our setting). By varying $\beta$, we conclude that

$$\text{Ext}^*_\hat{U}^+_k(\mathcal{W}(\lambda)_k, W(\mu)_k^*) \equiv \{0\} \quad \lambda \neq -w_0 \mu$$

since $-(\beta, w_0) \lambda \neq -(\beta, w_0) \mu$ for some choice of $\beta$.

Consider the simple integrable $\hat{U}^+_k$-module quotient $L(\lambda)_k$ of $V(\lambda)_k$ for each $\lambda \in P_+$. We have

$$\text{gch} W(\lambda)_k \equiv \text{gch} V(\lambda)_k \equiv \text{gch} L(\lambda)_k \mod \sum_{\lambda > \mu \in P_+} \mathbb{Z}[q] \text{gch} L(\mu)_k$$

by [22] (cf. [19]). It follows that

$$\text{Ext}^*_\hat{U}^+_k(\mathcal{W}(\lambda)_k, W(\mu)_k^*) = \text{Ext}^*_\hat{U}^+_k(\mathcal{W}(\lambda)_k, L(\mu)_k) \equiv \{0\} \quad \lambda > \mu \in P_+.$$ 

Since the both of $\mathcal{W}(\lambda)_k$ and $L(\lambda)_k$ are $\hat{U}^+_k$-integrable, we find

$$\text{Ext}^1_{\hat{U}^+_k}(\mathcal{W}(\lambda)_k, L(\mu)_k) \equiv \{0\} \quad \lambda > \mu \in P_+.$$
From these, it suffices to prove
\[ \text{Ext}^1_{\mathcal{U}_k^{\geq 0}}(\mathcal{W}(\lambda)_k, L(\lambda)_k) \equiv \{0\} \quad \lambda \in P_+ \] (2.2)
in order to deduce the assertion. By the description of the \( \mathcal{U}_k^{\geq 0} \)-module endomorphism of \( \mathcal{W}(\lambda)_k \) in terms of the imaginary (PBW-type) weight vectors ([3, Proposition 3.17] or [8, Lemma 4.5]), we derive that the \( P \)-weight \( \lambda \)-part of \( \mathcal{W}(\lambda)_k \) is maximal possible as a cyclic module with a cyclic vector of \( P \)-weight \( \lambda \) in the category of \( \mathcal{U}_k^{\geq 0} \)-modules that is \( \mathcal{U}_k^{\leq 0} \)-integrable and whose \( P \)-weights are contained in \( \text{Conv} \ W \lambda \subset P \otimes \mathbb{Z} \mathbb{R} \). Therefore, (2.2) vanishes and we conclude the assertion. \( \square \)

2.3 Frobenius splitting of \( \mathcal{Q}_{G,J} \)

**Lemma 2.17.** For each \( \lambda \in P_+ \) and \( w \in W_{\text{af}} \), it holds:

1. Each \( \beta \in Q' \) defines an \( \mathcal{U}_\lambda \)-module automorphism \( \tau_\beta \) on \( \mathcal{X}(\lambda)_\mathbb{C} \) determined by \( \tau_\beta(\mathfrak{v}_\lambda) := \mathfrak{v}_{t_{\beta} \lambda} \). Moreover, we have \( \tau_\beta \mathcal{B}(\mathcal{X}(\lambda)) = \mathcal{B}(\mathcal{X}(\lambda)) \);
2. We have \( \theta^*(\mathcal{X}(\lambda)_\mathbb{C}) \cong \mathcal{X}(w_0 \lambda)_\mathbb{C} \) as \( \mathcal{U}_\lambda \)-modules. Moreover, we have \( \theta^* \mathcal{B}(\mathcal{X}(\lambda)) = \mathcal{B}(\mathcal{X}(w_0 \lambda)) \);
3. We have \( \mathcal{W}_w(\lambda)_\mathbb{C} = \mathcal{W}_w(\lambda)_\mathbb{C} \cap \mathcal{X}(\lambda)_\mathbb{C} \);
4. We have a \( \mathcal{U}_\lambda \)-cyclic vector of \( \theta^* (\mathcal{W}_w(\lambda)_\mathbb{C}) \) with weight \( -w \lambda = w w_0 (w_0 \lambda) \).

**Proof.** We borrow the setting of [13, §8.1 and §8.2].

We prove the first assertion. Since \( \mathfrak{v}_\lambda \) and \( \mathfrak{v}_{t_{\beta} \lambda} \) obey the same relation, \( \tau_\beta \) defines an automorphism as \( \mathfrak{g} \)-modules. The latter assertion follows from Theorem 2.3.

We prove the second assertion. The defining equation of \( \theta^* (\mathfrak{v}_\lambda) \) is the same as the cyclic vector \( \mathfrak{v}_{- \lambda} \in \mathcal{X}(w_0 \lambda)_\mathbb{C} \) as \( \mathfrak{g} \)-modules. This yields a \( \mathfrak{g} \)-module isomorphism \( \eta : \theta^*(\mathcal{X}(\lambda)_\mathbb{C}) \rightarrow \mathcal{X}(w_0 \lambda)_\mathbb{C} \). Since \( \theta \) exchanges \( \mathcal{U}_\lambda \) and \( \mathfrak{v}_\lambda \) is cyclic, we deduce that \( \eta (\theta^*(\mathcal{X}(\lambda)_\mathbb{C})) = \mathcal{U}_\lambda \mathfrak{v}_{- \lambda} \subset \mathcal{X}(w_0 \lambda)_\mathbb{C} \). By Theorem 2.3, we conclude \( \theta^* \mathcal{B}(\mathcal{X}(\lambda)) = \mathcal{B}(\mathcal{X}(w_0 \lambda)) \) as required.

We prove the third assertion. By Theorem 2.3, the \( \mathbb{Z} \)-basis of \( \mathcal{X}(\lambda)_\mathbb{Z} \) is formed by the non-zero elements of \( \mathcal{B}(- \infty) \mathfrak{v}_{w_0 \lambda} \) and forms a direct summand of \( \mathcal{X}(\lambda)_\mathbb{C} \) as \( \mathcal{X}(\lambda)_\mathbb{C} \)-modules. Hence, the case \( w = w_0 \) follows. For \( w \in W \), we apply [13, Lemma 8.2.1] repeatedly to deduce the assertion from the \( w = w_0 \) case by using \( \mathcal{B}(- \infty) \mathfrak{v}_{w \lambda} \subset \mathcal{B}(- \infty) \mathfrak{v}_{w_0 \lambda} \). For \( w = u \beta \in W_{\text{af}} \) with \( u \in W, \beta \in Q' \), we additionally apply \( \tau_{u \beta} \) to conclude the assertion.

We prove the fourth assertion. The vector \( \theta^* (\mathfrak{v}_{w \lambda}) \) is a \( \mathcal{U}_\lambda \)-cyclic vector of \( \theta^* (\mathcal{W}_w(\lambda)_\mathbb{C}) \), and its weight is
\[ -w \lambda = w w_0 (w_0 \lambda). \]

Hence, we conclude the assertion (using the fact that \( \theta \) is an involution). \( \square \)
Theorem 2.18. For each \( \Lambda \in P^+ \), we have a surjective map \( L(\Lambda) \rightarrow \mathcal{W}^- (\Lambda) \) of \( \mathfrak{g}[z^{-1}] \)-modules. In addition, this map yields a surjection \( L(\Lambda)_Z \rightarrow \mathcal{W}^- (\Lambda)_Z \) of \( U^*_Z \)-modules.

Proof. By [1], Theorem A], the graded \( \mathfrak{g}[z] \)-module \( \theta^*(L(\Lambda)_C) \) admits a filtration by the grading shifts of \( \{ \mathcal{W}(\mu)_C \}_{\mu \in P_+} \). Since the \((d-)\)degrees of \( L(\Lambda)_C \) is concentrated in \( \mathbb{Z}_{\leq 0} \) and the degree 0-part of \( L(\Lambda)_C \) is \( V(\Lambda)_C \), the first quotient of \( \theta^*(L(\Lambda)_C) \) in our filtration must be \( \theta^*(\mathcal{W}^- (\Lambda)_C) \). Hence, we obtain the surjection \( \eta : L(\Lambda)_C \rightarrow \mathcal{W}^- (\Lambda)_C \) of \( \mathfrak{g}[z^{-1}] \)-modules.

Since the both modules share the \( U^-_Z \)-cyclic vector and compatible with the negative global basis, we conclude that the \( Z \)-basis of \( \mathcal{W}^- (\Lambda)_Z \) is obtained as a \( Z \)-basis of \( L(\Lambda)_Z \) that is not annihilated by \( \eta \). Hence, we conclude that \( \eta \) induces a surjection \( L(\Lambda)_Z \rightarrow \mathcal{W}^- (\Lambda)_Z \) of \( U^-_Z \)-modules.

Corollary 2.19. Let \( k \) be a field. For \( \Lambda \in P^+_k \), we have a \( U^-_k \)-module generator set \( \{ u_m \}_{m \in \mathbb{Z}_{\geq 0}} \) of \( \text{ker}(L(\Lambda)_k \rightarrow \mathcal{W}^- (\Lambda)_k) \) that satisfies:

- Each element \( u_m \) satisfies \( a_{\Lambda_m} u_m = u_m \) for some \( \Lambda_m \in P^+ \);
- For each \( m \in \mathbb{Z}_{\geq 0} \), we have \( \overline{\Lambda}_m \notin \text{Conv} W \Lambda \subset P \otimes \mathbb{R}.

Proof. Note that \( L(\Lambda)_k \) has at most countable rank over \( k \), which implies that the generator set is at most countable. As the both of \( L(\Lambda)_k \) and \( \mathcal{W}^- (\Lambda)_k \) admit \( P^+_k \)-weight decompositions, we deduce the first assertion. Since the both modules are \( U^-_k \)-integrable and \( U^-_k \)-cyclic, the second assertion follows by Proposition 2.16.

Theorem 2.20 ([15] Proposition 8.6 and [2] Corollary 4.15). Let \( \lambda \in P^+_1 \). The unique degree zero \( U^*_Z \)-module map

\[
\Psi_{\lambda} : \mathcal{W}(\lambda)_Z \hookrightarrow \bigotimes_{i \in I} \mathcal{W}(\varpi_i)_Z^{(\alpha^*_i, \lambda)}
\]  

(2.3)

is injective and defines a direct summand as \( Z \)-modules.

Proof. We set \( X^\otimes := \bigotimes_{i \in I} X(\varpi_i)^{\otimes (\alpha^*_i, \lambda)} \). The map \( \Psi_{\lambda} \) exists as \( \bigotimes_{i \in I} \mathbb{V}(\varpi_i)^{\otimes (\alpha^*_i, \lambda)} \) obeys the same defining condition as the extremal weight vector \( v_{\lambda} \in X(\lambda) \), and the comultiplication of \( \mathfrak{g} \) induces an algebra map \( \hat{U}_Z \subset \hat{U}_Z \otimes \hat{U}_Z \) ([15] [23.1.5]).

The map \( \Psi_{\lambda} \) is injective by [1, Corollary 4.15 and Remark 4.17]. By [15] Theorem 2.5], the \( q = 1 \) specializations of the global bases yield the \( \hat{U}_Z \)-spans of the extremal weight vectors in \( X(\lambda) \) and \( X^\otimes \) up to the action of some rings of (partially symmetric) polynomials with integer coefficients, respectively. This yields a splitting of \( \Psi_{\lambda} \) as \( Z \)-modules, where the \( Z \)-module structure of the RHS is coming from the crystal lattice of (the \( q \)-version of) \( X^\otimes \). We call this crystal lattice \( L_1 \).

We have another \( \mathbb{Z}[q] \)-lattice inside (the \( q \)-version of) \( X^\otimes \) obtained by the tensor product of the crystal lattices of (the \( q \)-versions of) \( X(\varpi_i) \)'s. We call this \( \mathbb{Z}[q] \)-lattice \( L_2 \).

By [15], Proposition 8.6], the global basis of \( L_1 \) is written in terms of the (tensor product) global bases of \( L_2 \) by an upper-unitriangular matrix \( C \) (valued in \( \mathbb{Q}[q] \), with finitely many non-zero entry in each row). Hence, the corresponding bases in \( X^\otimes \) are related by an upper-unitriangular matrix \( C \) obtained as
the $q = 1$ specialization of $C$ (valued in $\mathbb{Q}$). In view of the fact that $\Delta$ is an algebra morphism (and [15, Theorem 8.5]), we deduce $L_1 \subset L_2$. Hence, we have $\mathbb{Z} \otimes_{\mathbb{Z}[q]} L_1 \subset \mathbb{Z} \otimes_{\mathbb{Z}[q]} L_2 \subset \mathbb{C}^\otimes$. In particular, the matrix $C$ must be valued in $\mathbb{Z}$. This implies $\mathbb{Z} \otimes_{\mathbb{Z}[q]} L_1 = \mathbb{Z} \otimes_{\mathbb{Z}[q]} L_2$.

Hence, the $\mathbb{Z}$-module splitting in the second paragraph is indeed what we wanted.

**Proposition 2.21.** For each $\Lambda, \Gamma \in P_+^\text{af}$, we have the following commutative diagram of $U_+^\text{af}$-modules:

$$
\begin{array}{ccc}
L(\Lambda + \Gamma)_Z & \longrightarrow & L(\Lambda)_Z \otimes Z \ Z(\Gamma)_Z \\
\downarrow & & \downarrow \\
\mathcal{W}^- (\Lambda + \Gamma)_Z & \longrightarrow & \mathcal{W}^- (\Lambda)_Z \otimes Z \mathcal{W}^- (\Gamma)_Z.
\end{array}
$$

Moreover, all the maps define direct summands as $\mathbb{Z}$-modules.

**Proof.** The injectivity of the top horizontal arrow and the fact that it defines a direct summand as $\mathbb{Z}$-modules is Corollary 2.12. The surjectivity of the vertical arrows are Theorem 2.18. Since they are obtained by annihilating parts of $\mathbb{Z}$-bases, these maps define direct summands as $\mathbb{Z}$-modules.

Since all the modules are generated by the cyclic vectors $\mathcal{V}_\Lambda \otimes \mathcal{V}_\Gamma$ as $\mathfrak{g}[z^{-1}]$-modules or $\mathfrak{g}[z^{-1}]^{\otimes 2}$-modules, Theorem 1.3 (twisted by $\theta^*$) implies the injectivity of $m$ after extending the scalar to $\mathbb{C}$. Hence, we deduce

$$
m(\mathcal{W}^- (\Lambda + \Gamma)_Z) \subset \mathcal{W}^- (\Lambda)_Z \otimes \mathcal{W}^- (\Gamma)_Z. \quad (2.4)
$$

Therefore, it suffices to prove that (2.4) has torsion-free cokernel to complete the proof. By a repeated use of (2.4), we arrive the setting of Theorem 2.20 in view of Theorem 1.3. Thus, the map $m$ defines a direct summand of $\mathcal{W}(\Lambda)_Z \otimes \mathcal{W}(\Gamma)_Z$ as $\mathbb{Z}$-modules.

**Corollary 2.22.** For each $\lambda, \mu \in P_+$ and $w \in W$, we have the following commutative diagram of $U_+^\text{af}$-modules:

$$
\begin{array}{ccc}
\mathcal{X}(\lambda + \mu)_Z & \longrightarrow & \mathcal{X}(\lambda)_Z \otimes \mathcal{X}(\mu)_Z \\
\downarrow & & \downarrow \\
\mathcal{W}_w (\lambda + \mu)_Z & \longrightarrow & \mathcal{W}_w (\lambda)_Z \otimes \mathcal{W}_w (\mu)_Z
\end{array}
$$

Moreover, this inclusion is compatible with positive global basis and commutes with the automorphism $\tau_\beta$ ($\beta \in Q^+$) of $\mathcal{X}(\lambda)_Z, \mathcal{X}(\mu)_Z$, and $\mathcal{X}(\lambda + \mu)_Z$.

**Proof.** In view of Proposition 2.21, the $w = w_0$ case follows from Lemma 2.14 (1). Thanks to Theorem 1.3 and Lemma 2.17 (3), we deduce the general case from the $w = w_0$ case.

Let $w \in W_\text{af}$ and $J \subset I$. We define $P_{\ast}$- and $P_{\ast, \text{af}}$-graded $\mathbb{Z}$-modules:

$$
\begin{align*}
R^\text{af} & := \bigoplus_{\lambda \in P_+^\text{af}} L(\lambda)_Z^{\text{af}} \quad \text{and} \quad R_w (J) := \bigoplus_{\lambda \in P_{\ast, \text{af}}} \mathcal{W}_w (\lambda)_Z^{\text{af}}.
\end{align*}
$$
Lemma 2.23. The $\mathbb{Z}$-duals of the horizontal maps in Proposition 2.22 equip $R^\text{af}$ and $R_w(J)$ structures of $(P_+\text{- and } P_{1+})$ graded commutative rings.

Proof. The maps in Proposition 2.22 are characterized as the $d$-degree zero maps of cyclic $U_Z^+$-modules, that are unique up to a scalar. Therefore, the composition

$$W(\lambda + \mu + \gamma)Z \hookrightarrow W(\lambda + \mu)Z \otimes W(\gamma)Z$$

is the same map as

$$W(\lambda + \mu + \gamma)Z \hookrightarrow W(\lambda)Z \otimes W(\mu)Z \otimes W(\gamma)Z$$

for every $\lambda, \mu, \gamma \in P_+$ as the images of the cyclic vectors are the same. Taking their restricted duals implies the associativity of the multiplication of $R_w(J)$. In view of Lemma 2.17 and Corollary 2.22, we deduce the associativity of the multiplication of $R_w(J)$ for each $w \in W_{\text{af}}$. The associativity of $R^\text{af}$ is proved similarly (cf. [18]). The commutativity of $R^\text{af}$ and $R_w(J)$ follow as the $q = 1$ coproduct of $U_Z$ is symmetric ([18], Lemma 3.1.4, §23.1.5]).

Corollary 2.24 (of the proof of Lemma 2.23). For each $w \in W_{\text{af}}$ and $\beta \in Q^+ \backslash Q^\vee$, we have an isomorphism of $(R_w)_{\mathbb{F}_p}$ and $(R_{w\beta})_{\mathbb{F}_p}$ as graded rings equipped with $U_Z^+$-actions up to grading twists (given by Lemma 2.17).

We set $R := R_w$. Note that $R_w(J) \subset R$ is a subring. We also define

$$R^+(J) := \bigoplus_{\lambda, \mu \in P_+} \text{Span}_{\mathbb{Z}} \prod_{i \in I} (\mathbb{Z}(\varpi_i)^\vee_{\mathbb{Z}}(\lambda^i, \lambda)) \subset \bigoplus_{\lambda \in P_+} \mathbb{Z}(\lambda)^\vee_{\mathbb{Z}} := \hat{R},$$

where the multiplication is defined through the projective limit formed by the duals of Corollary 2.22. Here we warn that $\hat{R}$ is not a ring as the rank of the $(H \times \mathbb{G}_m)$-weight space of $\mathbb{X}(\lambda)^\vee_{\mathbb{Z}}$ is not bounded, while the rank of the $(H \times \mathbb{G}_m)$-weight space of $\mathbb{X}(\varpi_i)^\vee_{\mathbb{Z}}$ is bounded for each $i \in I$ by [18], Proposition 5.16]. By construction, all of these (three kinds of) rings are free over $\mathbb{Z}$.

For each $\lambda \in P_+$ and $w \in W_{\text{af}}$, we have a unique $P^\text{af}$-weight vector

$$v_w^\vee_{\lambda, \mu} \in W_w(\lambda)^\vee_{\mathbb{Z}}$$

with paring $1$ with $v_w^\vee_{\lambda, \mu} \in W_w(\lambda)^\vee_{\mathbb{Z}}$. This vector $v_w^\vee_{\lambda, \mu}$ yields an $U^+_k$-cocyclic vector of $W_w(\lambda)^\vee_k$ for a field $k$. By the construction of the ring structure on $R_w$, we have $v_w^\vee_{\lambda, \mu} \cdot v_w^\vee_{\lambda', \mu'} = v_w^\vee_{\lambda + \lambda', \mu + \mu'}$ in $\hat{R}$ for every $\lambda, \mu \in P_+$.

Lemma 2.25. For each $w, v \in W_{\text{af}}$ and $J \subset I$, the ring $R_w(J)$ is a quotient of $R_v(J)$ if $w \leq v$ and $J$ is irreducible. In addition, the ring $R_w$ is a quotient of $R_v$ if and only if $w \leq v$.

Proof. We have $W_v^\vee \subseteq W_w^\vee$ if and only if $v_w \lambda^\vee \subseteq v_v \lambda^\vee$. Now we apply Lemma 2.21 to deduce the result.

Lemma 2.26. We have the following morphisms of rings with $U^+_Z$-actions

$$R^+ \longrightarrow R \hookrightarrow R^\text{af},$$

that admit $\mathbb{Z}$-module splittings, where the $U^+_Z$-action on $R^\text{af}$ is twisted by $\theta$. 20
Proof. Apply Proposition 2.14 and Corollary 2.18.

For each \( w \in W \) and \( J \subseteq \mathcal{I} \), we set
\[
(Q_{G,2}(w))_w := \text{Proj} R_w(J) \quad \text{and} \quad (Q_{G,2}^{\text{af}}(w))_w := \bigcup_{w \in W} (Q_{G,2}(w))_w.
\]
These schemes and ind-schemes are flat over \( \mathbb{Z} \).

**Theorem 2.27** (HS Corollary B). Let \( p \) be a prime. Then, the ring \( R_{\mathbb{F}_p}^{\text{af}} \) admits a Frobenius splitting, that is \( I^- \) - and \( I^+ \) -canonically split.

**Theorem 2.28.** Let \( p \) be a prime. The ring \( R_{\mathbb{F}_p} \) admits a Frobenius splitting, that is \( I^- \) -canonically split.

**Proof.** The \( I^- \)-canonical Frobenius splitting \( \phi \) of \( R_{\mathbb{F}_p}^{\text{af}} \) gives rise to the following maps, whose composition is the identity:
\[
L(\Lambda)_{\mathbb{F}_p} \xrightarrow{\phi^\vee} L(p\Lambda)_{\mathbb{F}_p} \longrightarrow L(\Lambda)_{\mathbb{F}_p} \quad \Lambda \in P_{+-}.
\]
In view of Proposition 2.21, it prolongs to
\[
L(\Lambda)_{\mathbb{F}_p} \xrightarrow{\phi^\vee} L(p\Lambda)_{\mathbb{F}_p} \longrightarrow L(\Lambda)_{\mathbb{F}_p} \quad \Lambda \in P_{+-}.
\]
The right square is automatic (and is canonically defined) from the adjunction of the Frobenius push-forward (by taking the restricted dual). In order to show that \( \phi \) descends to a Frobenius splitting of \( R_{\mathbb{F}_p}^{\text{af}} \), it suffices to show that the dotted map \( \phi^\vee \) is a well-defined linear map (induced from \( \phi^\vee \) and so that the left square is commutative).

By Corollary 2.18, \( \ker \pi_{\Lambda} \) is generated by the \( P \)-weight \( (P \setminus \text{Conv} \mathcal{W} \mathcal{A}) \)-part of \( L(\Lambda)_{\mathbb{F}_p} \). By the cyclicity of \( L(\Lambda)_{\mathbb{F}_p} \) as \( U_{\mathbb{F}_p} \)-modules and Proposition 2.21, we deduce that \( \phi^\vee(\ker \pi_{\Lambda}) \) is contained in the \( U_{\mathbb{F}_p} \)-submodule of \( L(p\Lambda)_{\mathbb{F}_p} \) generated by the \( P \)-weight \( p(P \setminus \text{Conv} \mathcal{W} \mathcal{A}) \)-part of \( L(p\Lambda)_{\mathbb{F}_p} \). The latter is contained in \( \ker \pi_{p\Lambda} \) by
\[
\ker \pi_{p\Lambda} \subseteq \ker \pi_{\Lambda}.
\]
Therefore, we conclude that \( \phi^\vee \) is a well-defined linear map, and hence \( \theta^*(R_{\mathbb{F}_p}) \) admits a Frobenius splitting induced from \( \phi \). The unipotent part of the \( I^- \)-canonical splitting condition is in common with subrings. It remains to twist the grading given by \( \alpha_0^\vee \) with that given by \( -\theta \) and twist the \( I^- \)-action into the \( I^- \)-action by \( \theta \) to conclude that our Frobenius splitting on \( R_{\mathbb{F}_p} \) is \( I^- \)-canonical.

**Corollary 2.29.** Let \( p \) be a prime, and let \( w \in W \). The \( I^- \)-canonical splitting of \( R_{\mathbb{F}_p} \) obtained in Theorem 2.28 induces an \( I^- \)-canonical splitting of \( (R_w)_{\mathbb{F}_p} \).

**Proof.** We set \( L^w(\Lambda)_{\mathbb{F}_p} := U_{\mathbb{F}_p} w_{W} \Lambda, \) where \( Z_{W} w_{W} \Lambda \) is the \( P_{af} \)-weight \( w_{W} \Lambda \)-part of \( L(\Lambda)_{\mathbb{F}_p} \), that is rank one over \( \mathbb{Z} \).

The subspace \( \mathcal{W}^-(\Lambda)_{\mathbb{F}_p} \subseteq \mathcal{W}^-(\Lambda)_{\mathbb{F}_p} \) is the image of \( L^w(\Lambda)_{\mathbb{F}_p} \subseteq L(\Lambda)_{\mathbb{F}_p} \) (with \( w = \Lambda \)) under Theorem 2.28 as \( L^w(\Lambda)_{\mathbb{F}_p} \) is spanned by a subset of \( B(L(\Lambda)) \).
Our Frobenius splitting $\phi$ is obtained from that of $R^e_p$, which is compatible with $\bigoplus_{\lambda \in P} L^w(\lambda)_{\overline{F}_p}$ by [15, Corollary B]. Therefore, $\phi$ must descend to a Frobenius splitting of $(R_{w_{af}})_{\overline{F}_p}$. 

**Corollary 2.30.** An I-canonical splitting of $R^e_p$ is unique.

**Proof.** The behavior of the vectors in (23) (with $w = e$) under an I-canonical Frobenius splitting is uniquely determined as they form a polynomial ring isomorphic to $\mathbb{F}_p P_n$ such that each of its $(P_+)$-graded component is a multiplicity-free $P^0$-weight space in $W(\lambda)_{\overline{F}_p}$'s. By Proposition 2.24, this completely determines the behavior of our splitting (through its dual map).

**Corollary 2.31.** An I-canonical splitting of $R^e_p$ is compatible with $(R_w)_{\overline{F}_p}$ for every $w \in W_{af}$ such that $w \leq_{\triangleright} e$.

**Proof.** By [15, Proposition 4.1.17 and Remark 4.1.18 (ii)] and [17, Theorem 4.12], we derive that the I-canonical Frobenius splitting of $(R_w)_{\overline{F}_p}$ (where $w \in W$) obtained from $R^e_p$ recovers the original splitting uniquely by the $G$-action (arguing by restricting to $SL(2, i)$-actions for each $i \in I$).

Let $w \in W$ such that $w_{af} = s_0 w_{af} \leq_{\triangleright} e$. Then, $R^e_p$ is a quotient of $R_w$. Again by [15, Proposition 4.1.17 and Remark 4.1.18 (ii)], the sets of I-canonical splittings of $(R_w)_{\overline{F}_p}$ is in bijection with that of $(R_w)_{\overline{F}_p}$ compatible with $(R_w)_{\overline{F}_p}$. Similarly, the set of I-canonical splittings of $(R_w)_{\overline{F}_p}$ is in bijection with that of $(R^e_p)_{\overline{F}_p}$ compatible with $(R^e_p)_{\overline{F}_p}$ for every $w \in W$.

By applying [15, Proposition 4.1.17] for $SL(2, i)$-actions ($i \in I$), we derive that the set of I-canonical splittings of $(R^e_p)_{\overline{F}_p}$ compatible with $(R^e_p)_{\overline{F}_p}$ for every $w \in W$ is in bijection with the set of I-canonical splittings of $R^e_p$ compatible with $(R^e_p)_{\overline{F}_p}$ and $(R^e_p)$ for each $s_0^e = e \leq_{\triangleright} e \in W$.

By Corollary 2.24 and Corollary 2.30, we deduce that our canonical splitting must be compatible with $(R^e_p)_{\overline{F}_p}$ for every $w \in W$ and $m \in \mathbb{Z}_{\geq 0}$.

Every $w \leq_{\triangleright} e$ satisfies $t^\beta_\rho \leq_{\triangleright} w \leq_{\triangleright} e$ for some $\beta \in Q^+_+$. (cf. [17, Theorem 4.6]). As we can find $m$ such that $m\gamma^\beta \geq \beta$, we deduce that $t^\beta_\rho \leq_{\triangleright} e \leq_{\triangleright} e$. Hence, we apply [15, Proposition 4.1.17] for $SL(2, i)$-actions ($i \in I_{af}$) repeatedly to deduce that our canonical splitting is compatible with $(R^e_p)_{\overline{F}_p}$ for every $w \in W_{af}$ such that $w \leq_{\triangleright} e$ (cf. Lemma 2.24).

**Theorem 2.32.** Let $p$ be a prime. The ring $R^+(J)_{\overline{F}_p}$ admits a Frobenius splitting that is I- and $\Gamma^+$-canonically split. This splitting is compatible with $R^+(J)_{\overline{F}_p}$ and the image of $R^+(J)_{\overline{F}_p} \subset R^e_p$ under the quotient map $R^e_p \twoheadrightarrow \theta^*(R_{w_{af}})_{\overline{F}_p}$ for each $w \in W_{af}$.

**Proof.** Since the case of $J \neq \emptyset$ follows by the restriction to a part of the $P_+$-grading, we concentrate into the case $J = \emptyset$.

The ring structure of $R^e_p$ is determined by $R^e_p$ through the application of $U^*_-$ before taking duals. By Corollary 2.31 (and its proof), it defines an I-canonical splitting $\phi$ of $R^e_p$ compatible with $(R^e_p)_{\overline{F}_p}$ for each $w \in W_{af}$. By [15, Proposition 2.10], this splitting $\phi$ is also $\Gamma^+$-canonical.

The I-cocyclic $P^0$-weight vector $v^e_{w_{af}} \in W_{w_{af}}(\lambda)_{\overline{F}_p}$ is uniquely characterized by its $P^0$-weight. Hence, we obtain a map

$$X(\lambda)_{\overline{F}_p} \supset \text{Span}_{\mathbb{Z}} \prod_{i \in I} (X(\pi_i \gamma^\beta_\rho))^{(\sigma_i, \lambda)} \rightarrow W_{w_{af}}(\lambda)_{\overline{F}_p} \rightarrow F^e_p v^e_{w_{af}}.$$
It gives rise to the ring surjections
\[ R_p^+ \longrightarrow (R_w)_p \longrightarrow \bigoplus_{\lambda \in P^+} \mathbb{F}_p \nu w_{w_0 \lambda} \]
that is compatible with \( \phi \) by construction in the first surjection and by examining the \( P_{\text{rat}} \)-weights in the second surjection (we denote this composition surjective ring map by \( \xi \)). Consider the ideal
\[ I(w) := R_p^+ \cap \bigcap_{g \in \Gamma(p)} g(\mathbb{F}_p \otimes \mathbb{F}_p \ker \xi) \subset R_p^+. \]
(Here the action of \( \Gamma^{-}(\mathbb{F}_p) \) is obtained by the unipotent one-parameter subgroups \( \{p_{-\alpha_i}\}_{i \in I_{\alpha}} \) defined through the exponentials, that are well-defined as we have all the divided powers.) This ideal is the maximal \( U_{\mathbb{F}_p} \)-invariant ideal of \( R_p^+ \) that is contained in \( \ker \xi \). Let us denote the quotient ring by
\[ Q = \bigoplus_{\lambda \in P^+} Q(\lambda) := (R^+/I(w))_p. \]
By the construction of \( I(w) \), we deduce that
\[ U_{\mathbb{F}_p} \nu w_{w_0 \lambda} \subset Q(\lambda)^{\circ} \subset X(\lambda)^{\circ} \]
for each \( \lambda \in P^+ \) (otherwise we can derive a vector in \( I(w) \) to obtain a non-zero element of \( \bigoplus_{\lambda \in P^+} \mathbb{F}_p \nu w_{w_0 \lambda} \)). Since \( \theta^*(\mathbb{W}_{w_0}(-w_0 \lambda)_{\mathbb{F}_p}) \) is a cyclic \( U_{\mathbb{F}_p} \)-submodule of \( X(\lambda)_{\mathbb{F}_p} \), we have \( U_{\mathbb{F}_p} \nu w_{w_0 \lambda} = \theta^*(\mathbb{W}_{w_0}(-w_0 \lambda)_{\mathbb{F}_p}) \). In particular, we deduce a vector space surjection
\[ (R^+/I(w))_p \longrightarrow \theta^*((R_{w_0})_{\mathbb{F}_p}) \]
(cf. Corollary 2.22). Since the RHS is naturally a ring, we conclude
\[ (R^+/I(w))_p \cong \theta^*((R_{w_0})_{\mathbb{F}_p}) \]
by the maximality of \( I(w) \).

The ideal \( I(w) \subset R_p^+ \) also splits compatibly by \( \phi \) (since \( \phi \) is \( \Gamma^- \)-canonical and ker \( \xi \) splits compatibly). In particular, each \( \theta^*((R_{w_0})_{\mathbb{F}_p}) \) compatibly split under \( \phi \) as required.

**Corollary 2.33.** For each \( J \subset I \), the indscheme \( (Q_G^{\text{rat}})_{\mathbb{F}_p} \) admits an \( I^- \) and \( \Gamma^- \)-canonical Frobenius splitting that is compatible with \( Q_{G,J}(w)_{\mathbb{F}_p} \) for each \( w \in W_{af} \).

**Corollary 2.34.** For each \( J \subset I \), the indscheme \( (Q_G^{\text{rat}})_{\mathbb{F}_p} \) and the schemes \( Q_{G,J}(w)_{\mathbb{F}_p} \) (\( w \in W_{af} \)) are reduced. In addition, every finite, bijective, birational morphism of schemes from \( X \) to an open subscheme of \( Q_{G,J}(w)_{\mathbb{F}_p} \) that shares an isomorphic Zariski open dense subset is an isomorphism.

**Proof.** For the first assertion, apply [11, Proposition 1.2.1] to Corollary 2.33. For the second assertion, we walk-around the part of the proof of [11, Proposition 1.2.5], where the Noetherian hypothesis is used (cf. [11, Example 01RQ]).
3 Frobenius splitting of quasi-map spaces

We retain the settings of the previous section. In particular, we sometimes work over a ring or a non-algebraically closed field. Moreover, the notational convention explained in the beginning of §2 continue to apply.

3.1 The scheme $Q'_J(v, w)$ and its Frobenius splitting

Let $v, w \in W_{af}$ and $J \subseteq I$. We set

$$R^+_w(J) := R^+(J)/\left(\ker(R^+(J) \rightarrow R_w(J)) + \ker(R^+(J) \rightarrow \theta^*(R_{vw0})\right).$$

By construction, $R^+_w(J)$ is a $P_{1,+}$-graded ring. We set

$$R^w_J(v, w) := \bigoplus_{\lambda \in P_{1,+}} R^w_J(J, \lambda).$$

**Lemma 3.1.** For each $w, v \in W_{af}$, the multiplication map

$$R^w_J(v, \lambda) \otimes_Z R^w_J(J, \mu) \rightarrow R^w_J(J, \lambda + \mu) \quad (\lambda, \mu \in P_{1,+})$$

is surjective.

**Proof.** We have a quotient

$$R_w(J) = \bigoplus_{\lambda \in P_{1,+}} W_{w0}(\lambda)_Z \twoheadrightarrow \bigoplus_{\lambda \in P_{1,+}} R^w_J(J, \lambda) = R^w_J(J)$$

of homogeneous rings. Corollary 2.22 implies that the multiplication map of $R_w(J)$ is surjective. Hence, so is the quotient ring. □

We set

$$Q'_J(v, w) := \text{Proj} R^w_J(J),$$

where our definition of Proj is (1.3). In case $v = w_0 \beta$ for $\beta \in Q'$, we write $R^w_J(J)$ and $Q'_J(v, w)$ by $R^w_{w0}(J)$ and $Q'_J(\beta, w)$, respectively.

**Lemma 3.2.** For each $w, v \in W$, the Chevalley involution induces an isomorphism

$$Q'(v, w) \xrightarrow{\sim} Q'(wv_0, vw_0).$$

**Proof.** Apply Lemma 2.17 to the construction of $R^w_J$ to deduce an isomorphism

$$\theta^* : R^w_{w0} \xrightarrow{\sim} R^w_{wv_0}$$

of graded rings, that yields the assertion. □

**Lemma 3.3.** The scheme $Q'_J(v, w)$ is flat over $\mathbb{Z}$.

**Proof.** The ring $R^+(J)$ has a $\mathbb{Z}$-basis that is dual to $\bigsqcup_{\lambda \in P_{1,+}} B(\lambda(\lambda))$. The rings $R_w(J)$ and $\text{Im} (R^+(J) \rightarrow \theta^*(R_{vw0}))$ are quotients by subsets of such basis elements by Lemma 2.17. Hence, we have a free $\mathbb{Z}$-basis of $R^+_w(J)$ as required. □

**Lemma 3.4.** For each $w, v \in W_{af}$, the scheme $Q'_J(v, w)$ is of finite type.
Proof. By Proposition 5.16, we have rank $R_c^\nu_i(v_i) < \infty$ for each $i \in I$. By Lemma 3.5, this forces $Q'(v, w)$ to be of finite type as required.

Lemma 3.5. We have $Q'_1(v, w) \neq \emptyset$ if $v \leq w$.  

Proof. We have $R_c^\nu(J, \lambda) \neq \emptyset$ if $v(\lambda) \in W_{w_0(\lambda)}(w_0\lambda)$ by Lemma 4.4. Here $v(\lambda) \in W_{w_0}(w_0\lambda)$ is equivalent to $W_{w_0}(w_0\lambda) \subset W_{w_0}(w_0\lambda)$. Now apply Lemma 4.5 to obtain the assertion.

Remark 3.6. In view of Theorem 3.5, we have $Q'(v, w) \neq \emptyset$ only if $v \leq w$ as otherwise we have no $(H \times G_m)$-fixed point in the (flat) specializations.

Lemma 3.7. Let $v, w \in W_{af}$, $\beta \in Q'$, and $J \subset I$, we have $Q_1'(v, w) \cong Q_1'(v, w)$.  

Proof. We borrow notation from Lemma 4.4. By the definition of our ring $R_c^\nu(J)$, the assertion follows from $W_{w_0(\lambda)} = \tau_{w_0}(W_{w_0}(\lambda))$ and $\tau_{w_0}(\theta^*(W_{w_0}(\lambda))) = \theta^*(W_{w_0}(\lambda))$ for each $w \in W_{af}$, $\beta \in Q'$, and $\lambda \in P$. This assertion itself follows by chasing the weights of the cyclic vectors.

Lemma 3.8. Let $p$ be a prime. For each $v, w \in W_{af}$, the ring $R_c^\nu(J)_{F_p}$ admits a Frobenius splitting that is compatible with the quotient $R_c^\nu(J)_{F_p} / \theta^*(R_c^\nu(J)_{F_p})$ such that $v \leq w$. In particular, the scheme $Q'_1(v, w)_{F_p}$ is reduced and weakly normal.

Proof. By construction, $\ker(R_c^\nu(J)_{F_p} \rightarrow (R_c^\nu(J)_{F_p})$ and $\ker(R_c^\nu(J)_{F_p} \rightarrow \theta^*(R_c^\nu(J)_{F_p}))$ are ideals of a ring $R_c^\nu(J)_{F_p}$ that are compatible with the canonical Frobenius splitting of $R_c^\nu(J)_{F_p}$. Hence, so is their sum. It must be compatible with every quotient of the form $R_c^\nu(J)_{F_p}$ with the above condition by Lemma 4.6, Lemma 4.5, and Lemma 4.3. This proves the first assertion. We apply Proposition 1.2.1 and 1.2.5 to deduce the second assertion.

Corollary 3.9. Let $p$ be a prime. For each $v, w \in W$, the scheme $Q'_1(v, w)_{F_p}$ admits a Frobenius splitting compatible with $Q'_1(v', w')_{F_p}$ for every $v', w' \in W_{af}$ such that $v \leq w$. In particular, the scheme $Q'_1(v, w)_{F_p}$ is reduced.

Proof. Apply Theorem 4.3 to Lemma 4.3.

Corollary 3.10. The scheme $Q'_1(v, w)$ is reduced.

Proof. By Lemma 4.4, every non-zero element of $R_c^\nu(J)$ is annihilated by reduction mod $p$ for finitely many primes. Now it remains to apply Lemma 4.3.

3.2 Modular interpretation of $Q_{G, J}^{rat}$

We have an identification

$$W_{af} \cong \mathcal{N}_{G(\zeta)}(H(\mathbb{K}))/H(\mathbb{K})$$

regardless of the (algebraically closed) base field $\mathbb{K}$. We consider a lift of $w \in W_{af}$ in $\mathcal{N}_{G(\zeta)}(H(\mathbb{K}))/H(\mathbb{K})$ by $\bar{w}$. 

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Lemma 3.11. For each \( w \in W \), the scheme \( Q_{G,2}(w)_{\mathbb{K}} \) contains an affine Zariski open \( 1 \)-orbit \( \mathcal{O}(J, w)_{\mathbb{K}} \) that is isomorphic to

\[
I / (H \cdot (\text{Ad}(\hat{w}_0))([P(J), P(J)]([z])) \cap I)
\]

as a scheme over \( \mathbb{K} \). (By abuse of notation, here we identify the set of \( \mathbb{K} \)-valued points \( (\text{Ad}(\hat{w}_0))([P(J), P(J)]([z])) \cap I \) with its Zariski closure in \( I \).) It is an open neighbourhood of the unique \((H \times G_m)_{\mathbb{K}}\)-fixed point of \( \mathcal{O}(J, w)_{\mathbb{K}} \).

Proof. Recall that \( v_{w_0, \lambda} = v_{w_0, (\lambda + \mu)} \) for each \( \lambda, \mu \in \mathbb{P} \). Since \( \mathbb{W}(\lambda) \) is compatible with the positive global basis, the ring

\[
\mathbb{Z}[w] := \sum_{\lambda \in \mathbb{P}^+} (v_{w_0, \lambda})^{-1} \mathbb{W}(\lambda) \subseteq (U^+_{\mathbb{C}})^{v}
\]

admits its dual basis. By construction, \( \mathbb{Z}[w] \) is the coordinate ring of an affine Zariski open set of \( Q_{G,1}(w)_{\mathbb{C}} \). In addition, it inherits a natural \( P_{\mathbb{C}} \)-grading from \( R_{G,1} \). Therefore, \( \mathbb{Z}[w] \) defines an open neighbourhood of a \((H \times G_m)_{\mathbb{K}}\)-fixed point of \( Q_{G,1}(w)_{\mathbb{K}} \) obtained by the linear forms \( \{v_{w_0, \lambda}\}_\lambda \). We set \( \mathbb{C}[w] := \mathbb{C} \otimes \mathbb{Z}[w] \) and \( \mathbb{K}[w] := \mathbb{K} \otimes \mathbb{Z}[w] \).

The defining relation of \( (\mathbb{C}[e])^{v} \) in terms of the \( U^{+\mathbb{C}} \)-action is

\[
U^{+\mathbb{C}}(\text{Ad}(\hat{w}_0))([P(J), P(J)]) \otimes \mathbb{C}[z] \cap \mathfrak{I}.
\]

by (the limit of) Chari-Fourier-Khandai [13, Proposition 3.3]. Since \( \mathbb{W}(\lambda) \subseteq \mathbb{W}(\lambda) \) for each \( u \in W \), the defining relation of \( (\mathbb{C}[w])^{v} \) in terms of the \( U^{+\mathbb{C}} \)-action is

\[
U^{+\mathbb{C}}(\text{Ad}(\hat{w}_0))([P(J), P(J)]) \otimes \mathbb{C}[z] \cap \mathfrak{I}
\]

by applying the action of \( u \in N_{G}(H)_{\mathbb{C}} \) that lifts \( u \in W \). In particular, we have

\[
\text{Spec} \mathbb{C}[w] \cong \mathfrak{I} / (H \cdot \text{Ad}(\hat{w}_0))([P(J), P(J)]([z])) \cap \mathfrak{I} \quad (3.2)
\]

as schemes over \( \mathbb{C} \). We put \( \mathfrak{I} = [\mathbb{I}_2, \mathbb{I}_3] \). Let \( \mathfrak{I}_{\mathbb{C}} \) and \( \mathfrak{R}_{\mathbb{C}} \) be pro-unipotent subgroups of \( \mathfrak{I}_{\mathbb{C}} \) whose closed points are

\[
\text{Spec} \mathbb{C}[w] \quad \text{and} \quad \text{Ad}(\hat{w}_0)([P(J), P(J)]([z])) \cap \mathfrak{I},
\]

respectively and they are stable by the natural \((H \times G_m)_{\mathbb{C}}\)-action on \( \mathfrak{I}_{\mathbb{C}} \). The isomorphism \( \mathfrak{I}_{\mathbb{C}} \rightarrow \mathfrak{I}_{\mathbb{C}} \) gives rise to an isomorphism

\[
m : \mathfrak{I}_{\mathbb{C}} \times \mathfrak{R}_{\mathbb{C}} \rightarrow \mathfrak{I}_{\mathbb{C}}
\]

of schemes over \( \mathbb{C} \), where \( m \) is the multiplication map. The Hopf algebra \( \mathbb{Z}[\mathfrak{I}_{\mathbb{C}}] := (U^{+\mathbb{C}})^{v} \) is the coordinate ring of \( \mathfrak{I}_{\mathbb{C}} \) (cf. [13, Theorem 1.3]). By sending \( \mathbb{Z}[\mathfrak{I}_{\mathbb{C}}] \) by the restriction morphisms \( \mathbb{C}[\mathfrak{I}_{\mathbb{C}}] \rightarrow \mathbb{C}[\mathfrak{L}_{\mathbb{C}}] \) and \( \mathbb{C}[\mathfrak{R}_{\mathbb{C}}] \rightarrow \mathbb{C}[\mathfrak{R}_{\mathbb{C}}] \), we have the corresponding group schemes \( \mathfrak{L}_{\mathbb{Z}} \) and \( \mathfrak{R}_{\mathbb{Z}} \) over \( \mathbb{Z} \) (we denote their coordinate rings as \( \mathbb{Z}[\mathfrak{L}] \) and \( \mathbb{Z}[\mathfrak{R}] \), respectively). For each real root \( \alpha \in \Delta_{af, +} \), one of the the restricted dual rings \( \mathbb{C}[\mathfrak{L}_{\mathbb{C}}]^{v} \) or \( \mathbb{C}[\mathfrak{R}_{\mathbb{C}}]^{v} \) contains the image of a primitive element of \( U^{+\mathbb{Z}} \) with \( \theta \)-weight \( \alpha \) (obtained by conjugations of \( \{E_i\}_{i \in I_{af}} \)’s cf. [15, Proposition 40.1.3] or [13, Lemma 6.6]), and hence the corresponding one-parameter subgroup lands in either of \( \mathfrak{L}_{\mathbb{Z}} \) or \( \mathfrak{R}_{\mathbb{Z}} \). In view of [3, Theorem 3.13] (or [13, Theorem 5.8]), they generates a closed normal subgroup scheme \( \mathfrak{N}_{\mathbb{Z}} \) of
The indscheme $\mathbb{L}_Z$ that is a projective limit of extensions of $G_\mathbf{a}$ over $\mathbb{Z}$. (The same is true and exhaust the whole $\mathbb{R}_Z$ if $J = \emptyset$.)

We examine the action of the imaginary PBW generators $\{\hat{P}_{i,m\delta}\}_{i\in I, m > 0}$ (weight of $\hat{P}_{i,m\delta}$ is $m\delta$) in [3, (3.7)]. By applying them on the direct sum of $P$-weight $\mu\mu_0\lambda$-parts of $X(\lambda)_C$ for all $\lambda \in P_{J,+}$, we obtain a quotient group scheme $\mathbb{L}_Z \rightarrow \mathbb{T}_Z$ whose kernel is $N_Z$. By [2, Proposition 3.22], the group scheme $\text{Spec}\mathbb{Z}[\mathbb{P}_{i,m\delta} | m > 1]^c$ for each $i \in I$ is isomorphic to a projective limit of extensions of $G_\mathbf{a}$ (given by truncations with respect to the duals of $\mathbb{P}_{i,m\delta}$ for $N \in \mathbb{Z}_{>0}$), which is flat over $\mathbb{Z}$. Thus, so is $\mathbb{T}_Z$. By induction on the rank of $G$ (to deal with the Levi part of $P(J)_C$ that effects on $\mathbb{R}_C$), we conclude that the both of $\mathbb{L}_Z$ and $\mathbb{R}_Z$ are group subschemes of $\mathbb{I}_Z$ flat over $\mathbb{Z}$, and the multiplication map yields an isomorphism $\mathbb{Z}[I^1] \cong \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathbb{Z}[R]$ (cf. [3, Theorem 5.8]).

In view of the flatness of the one-parameter subgroups over $\mathbb{Z}$ afforded by the above, we conclude that

$$\mathbb{R}_K = \text{Ad}(\hat{w}_0)([P(J)_K, P(J)_K][z]) \cap I_K$$

as being a projective limit of bijective morphisms between smooth varieties (as our construction factors through an algebraic closure of a prime field, the both are extensions of $G_\mathbf{a}$ [2, Exposé XVII Corollarie 4.1.3]). By construction, $\mathbb{Z}[w]$ is precisely the subring of $\mathbb{Z}[I^1]$ that is invariant under the $\mathbb{R}_Z$-action. It follows that the image of the composition map

$$\mathbb{Z}[w] \hookrightarrow \mathbb{Z}[I^1] \xrightarrow{\mathfrak{m}} \mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathbb{Z}[R]$$

is equal to $\mathbb{Z}[L] \otimes_{\mathbb{Z}} \mathbb{Z}$. Hence, we have

$$\text{Spec} \mathbb{K}[w] = \mathbb{L}_K \cong I_K / (H_K \cdot \text{Ad}(\hat{w}_0)([P(J)_K, P(J)_K][z]) \cap I_K).$$

Therefore, each $\mathbb{Q}_G(w)_K$ contains $\text{Spec} \mathbb{K}[w]$ as a Zariski open $I$-orbit, and it admits a unique $(H \times G_m)_K$-fixed point as required.

**Corollary 3.12.** The ring $R(J)_K$ is integral for each $J \subset I$.

**Remark 3.13.** We refer Proposition 3.11 for general case of Corollary 3.12.

**Proof of Corollary 3.12.** It suffices to prove the case $J = \emptyset$ since a subring of an integral ring is integral. We borrow notation from the proof of Lemma 3.11. We have an inclusion

$$R_e = \bigoplus_{\lambda \in P_+} \mathbb{W}(\lambda)_Z \hookrightarrow \mathbb{Z}[e]$$

as an inclusion of subrings of $\mathbb{L}_Z$. Taking tensor product $\mathbb{K} \otimes_{\mathbb{Z}} \bullet$ preserves the inclusion in view of the definition of $\mathbb{Z}[e]$. Hence, $(R_e)_K$ is integral as a subring of an integral ring $\mathbb{K}[e]$.

**Proposition 3.14.** The indscheme $(\mathbb{Q}_{G,J}^{\text{rat}})_K$ admits a $G((z))$-action. We have a subset $\mathbb{Q}'_K$ of the set of $\mathbb{K}$-valued points of $(\mathbb{Q}_{G,J}^{\text{rat}})_K$ that is in bijection with

$$G((z)) / (H(\mathbb{K}) \cdot [P(J), P(J)]((z))).$$
Proof. In view of Lemma 5.7.3.11, the proof for general $J$ is completely parallel to the case of $J = \emptyset$. Hence, we concentrate into the case $J = \emptyset$ during this proof.

Let $w \in W_{af}$ and $\beta \in Q^\vee$. We have an isomorphism $R_w \cong R_{wt, \beta}$ as rings with $U^+_{Z}$-action by Corollary 4.40.

In particular, we have an isomorphism $Q_G(w) \cong Q_G(wt, \beta)$ of schemes with $I$-actions for each $\beta \in Q^\vee$. This implies that $Q_G(wt, \beta)$ has a Zariski open subset of the shape

$$I/ (H \cdot (\text{Ad}(\bar{w}I\bar{u}
(\gamma)(z)) \cap I)) \cong I/ (H \cdot (\text{Ad}(\bar{w}I\bar{u}
(\gamma)(z)) \cap I)),$$

where we used that $\text{Ad}(\bar{w}I\bar{u}
(\gamma))$ is invariant under the $\text{Ad}(\bar{w})$-action.

The ring $(R_w)_K$ admits an action of $SL(2,i)$ whenever $s,w \leq w$ ($i \in I_{af}$) since each $\mathcal{W}_{awu}(\lambda)_K$ ($\lambda \in P^+_i$) is integrable with respect to the action of $SL(2,i)$. Hence, $\text{Proj} (R_w)_K$ admits an action of $SL(2,i)$ if $s,w \leq w$. In particular, the pro-algebraic group $I(i)$ acts on $\text{Proj} (R_w)_K$ if $s,w \leq w$. Hence, the ind-limit

$$Q^G_{\text{rat}} := \lim_{\to \mathcal{W}} \text{Proj} R_w$$

admits an action of $I(i)$ for each $i \in I_{af}$ (that coincide on $I$). By rank two calculations, they induce an action of $N_{G(Q)}(H(K))$ on $(Q^G_{\text{rat}})_K$. The intersections of $N_{G(Q)}(H(K))$ and $I(K)$ or $I(i)(K)$ ($i \in I_{af}$) inside $G(Q)$ define common actions on $(Q^G_{\text{rat}})_K$. The system of groups $(I(K), N_{G(Q)}(H(K)), I(i)(K); i \in I_{af})$ (in the sense of [57, Definition 5.1.6]) admits a map from with the system of groups in $[57, \S 6.1.16]$ for $\mathfrak{g}$ (with exponential maps replaced by one-parameter subgroups). Hence, Theorem 5.1.17 asserts that $G(Q)$ acts on $(Q^G_{\text{rat}})_K$.

The Bruhat decomposition of $SL(2,i)$ asserts that $Q(w)_K \cup Q(s,w)_K$ admits an $SL(2,i)$-action. This induces an action of $SL(2,i)(K)$ ($i \in I$) on the union

$$Q^G_K := \bigcup_{w \in W_{af}} Q(w)_K(K) \subset Q^G_{\text{rat}}(K).$$

Taking into account the fact that each $Q(w)_K$ admits an $I$-action (Lemma 5.7.11), we conclude that $Q^G_K$ admits an action of $G(Q)$ thanks to the Iwasawa decomposition (cf. [40, Theorem 2.5])

$$G(Q) = \bigcup_{w \in W_{af}} I(K)\bar{w}\bar{u}H(K) \cdot N(Q)$$

as required.

For $\lambda, \mu \in P_+$, we have a unique injective $U^i_Z$-module map

$$V(\lambda + \mu)_Z \rightarrow V(\lambda)_Z \otimes_Z V(\mu)_Z,$$

that is in fact a direct summand (as $Z$-modules). By extending the scalar, we obtain a unique injective $(U^i_Z, K[z])$-bimodule map

$$\eta_{\lambda, \mu} : V(\lambda + \mu)_K \otimes K[z] \rightarrow (V(\lambda)_K \otimes K[z]) \otimes_{K[z]} (V(\mu)_K \otimes K[z]).$$

**Lemma 3.15.** For each $\lambda, \mu, \gamma \in P_+$, we have

$$\eta_{\lambda + \mu, \gamma} \circ (\text{id} \boxtimes \eta_{\lambda, \mu}) = \eta_{\lambda, \mu + \gamma} \circ (\text{id} \boxtimes \eta_{\mu, \gamma}).$$
Proof. Straight-forward from the construction (cf. Lemme 2.23). \qed

**Proposition 3.16.** Assume that \( \text{char} \, K \neq 2 \). For each \( w \in W_{af} \) and \( J \subset I \), we have an \( I \)-equivariant rational map

\[
\psi_w : Q_G(J)(w)_K \longrightarrow \bigcup_{m \in \mathbb{Z}, \, i \in I \setminus J} \mathbb{P}(V(w_i)_K \otimes z^m K[[z]]) = \prod_{i \in I \setminus J} \mathbb{P}(V(w_i)_K \otimes K((z))),
\]

that gives rise to a \( G((z)) \)-equivariant rational map

\[
\psi : (Q_{G(J)}^I)_K \longrightarrow \prod_{i \in I \setminus J} \mathbb{P}(V(w_i)_K \otimes K((z))).
\]

In addition, the set \( (\text{Im} \, \psi)(K) \) is in bijection with \( Q'_K \), and it defines a closed \((\text{ind}-)\)subscheme of \( \prod_{i \in I \setminus J} \mathbb{P}(V(w_i)_K \otimes K((z))) \).

**Proof.** We have a surjective map \( \mathbb{W}(w_i)_K \rightarrow V(w_i)_K \otimes K[z] \) as \( \hat{U}^0_K \)-modules since we have the corresponding map over \( \hat{U}^0_K \) such that the \( P \)-weight \( w_i \)-part is the same (and \( V(w_i)_K \) is cyclic as a \( \hat{U}^0_K \)-module). The identification of the \( P \)-weight \( w_i \)-part also implies that this map commutes with the action of \( \tau_\beta \) (\( \beta \in Q' \)), and extends to a surjective map

\[
X(w_i)_K \rightarrow V(w_i)_K \otimes [z, z^{-1}]
\]

of \( \hat{U}_K \)-modules.

This gives a rational map

\[
\mathbb{P}(\mathbb{W}(w_i)_K) \longrightarrow \mathbb{P}(V(w_i)_K \otimes K[z])
\]

and its graded completion

\[
\mathbb{P}(\mathbb{W}(w_i)_K^\wedge) \longrightarrow \mathbb{P}(V(w_i)_K \otimes K[[z]]).
\]

Taking Corollary 2.22 into account, we have an embedding:

\[
Q_G(J)(e)_K \hookrightarrow \prod_{i \in I \setminus J} \mathbb{P}(\mathbb{W}(w_i)_K^\wedge). \tag{3.4}
\]

This yields a rational map

\[
\psi_e : Q_G(J)(e)_K \longrightarrow \prod_{i \in I \setminus J} \mathbb{P}(V(w_i)_K \otimes K[[z]])
\]

as a composition. This map is \( G[[z]] \)-equivariant by construction.

For each \( w \in W \), we can choose \( \beta \in Q' \) such that \( Q_{G,J}(w)_K \cong Q_{G,J}(wt_\beta)_K \subset Q_{G,J}(e)_K \) by Corollary 2.21. Hence we obtain the map \( \psi_w \) for every \( w \in W_{af} \) as the composition of the above maps. Since the \( \tau_\beta \)-action transfers \( \psi_w \) to \( \psi_{wt_\beta} \) for each \( \beta \in Q' \), we obtain the map \( \psi \) of ind-schemes. This map is \( G((z)) \)-equivariant in our sense. This proves the first assertion.

From now on, we concentrate into the second assertion.
Each \((H \times G_m)\)-fixed point of \(\mathcal{O}(w)_K\) \((w \in W_{\text{rat}})\) is contained in the domain of \(\psi\), and their images are distinct by inspection. It follows that \(\mathcal{Q}_K^\text{rat}\) is contained in the domain of \(\psi\), and the restriction of \(\psi\) to \(\mathcal{Q}_K^\text{rat}\) is injective by examining the stabilizer of \(I(K)\)-actions at each \((H \times G_m)\)-fixed points.

In view of Theorem 2.2.4, the maps \(\{\eta_{\lambda, \mu}\}_{\lambda, \mu} (\lambda, \mu \in P_{\text{rat}})\) induce a commutative diagram of \(U_{\text{rat}}^{>0}\)-modules:

\[
\begin{array}{ccc}
\mathbb{W}(\lambda)_K^\text{rat} & \rightarrow & \bigotimes_{i \in I} \mathbb{W}(\varpi_i)_K^{(\alpha_i^\vee, \lambda)} \\
\kappa_\lambda & \downarrow & \downarrow \\
V(\lambda)_K \otimes K[z]^\text{rat} & \rightarrow & \left( \bigotimes_{i \in I} (V(\varpi_i)_K^{(\alpha_i^\vee, \lambda)}) \otimes K[z] \right) \rightarrow \left( \bigotimes_{i \in I} (V(\varpi_i)_K^{(\alpha_i^\vee, \lambda)}) \otimes K[z] \right).
\end{array}
\]

(3.5)

Here the map \(\kappa_\lambda\) is well-defined by examining the degree 0-part and the action of \(E_0 = F_0 \otimes z\) (where \(F_0\) is a non-zero vector in the \(-v\)-weight space of \(\mathfrak{n}^+\)). The above commutative diagram also commutes with the translation by \(\tau_\beta (\beta \in Q^+)\) by construction. Moreover, we have \(\kappa_\lambda(\nu_{a, \lambda}) \neq 0\) for each \(w \in W_{\text{rat}}\). Therefore, the map \(\kappa_\lambda\) must be surjective whenever its \(d\)-degree belongs to \(\sum_{i \in I} \mathbb{Z} [\alpha_i^\vee, \lambda]\).

For each \(i \in I\), the \(\mathbb{Z}[\frac{1}{2}]\)-integral structure of \(V(2\varpi_i) \otimes \mathbb{C}[z, z^{-1}]\) at the odd degree and even degree must be the same as \(U_{\text{rat}}\)-modules (as we can connect the extremal weight vectors of even degree part and the odd degree part using the \(\mathfrak{sl}(2)\)-strings of length 3). Therefore, for \(\lambda = \varpi_i, \varpi_i + \varpi_j, 2\varpi_i (i,j \in I)\), the map \(\kappa_\lambda\) is surjective.

Consider a \((\text{representative of the})\) image

\[
\psi(x) = (x_i) \in \prod_{i \in I \setminus J} V(\varpi_i)_K \otimes K(z)
\]

of a \(K\)-valued point \(x \in \mathcal{Q}_{G,J}(\epsilon)\) under \(\psi\). We consider its lifts \(\tilde{x}_i \in \mathbb{W}(\varpi_i)_K^\text{rat}\) and \(\tilde{x}_j \in \mathbb{W}(\varpi_j)_K^\text{rat}\). They must belong to

\[
\text{Im} (\mathbb{W}(\varpi_i + \varpi_j)_K^\text{rat} \rightarrow (\mathbb{W}(\varpi_i)_K \otimes \mathbb{W}(\varpi_j)_K)^\text{flag})
\]

in order to satisfy the defining relations of \(R_K\). In view of the commutative diagram (3.5) for \(\lambda = \varpi_i + \varpi_j\), we deduce an equation

\[
x_i \otimes K[z] x_j \in \text{Im} \eta_{\varpi_i, \varpi_j} \quad i,j \in I \setminus J.
\]

Since the relation of the ring \(\bigoplus_{\lambda \in P_{\text{rat}}} V(\lambda)_K^\text{rat}\) is generated by \(P\)-degrees \(2\varpi_i\) and \(\varpi_i + \varpi_j\) for \(i,j \in I\) (Theorem 3.5.3]), this defines an element of

\[
G((z))/[P(J),P(J)]((z)) \subset \overline{G}/[P(J),P(J)](K((z)))
\]

(6.3)

through quadratic relations (see [311, §4]), where

\[
\overline{G}/[P(J),P(J)] = \text{Spec } K[G/[P(J),P(J)]]
\]

is the basic affine space. Therefore, applying some \(\tau_\beta (\beta \in Q^+)\) if necessary, we conclude that if a \(K\)-valued point \(x \in \mathcal{Q}_{G,J}^\text{rat}\) belongs to the domain of \(\psi\), then \(\psi(x)\) belongs to the set of \(K\)-valued points of the image given by (3.6). Taking Proposition 4.1.3 (and its \(G((z))\)-action into account, we conclude the second assertion.\]
Corollary 3.17. Keep the setting of Proposition \ref{prop:embedding}. The map $\psi$ induces a bijection of $K$-valued points between the domain and the range.

Proof. Let $x'$ be a preimage of a $(H \times G_m)(K)$-fixed point $x \in Q^t_{G,J}(K)$ through $\psi$. Consider the embedding

$$ (Q^t_{G,J})_K \hookrightarrow \prod_{i \in 1 \setminus J} \mathbb{P}(\mathbb{H}(\varpi_i)^2_K) $$

that prolongs \((\ref{eq:embedding})\). Let $x' = (x'_i)_{i \in 1 \setminus J}$ (resp. $x = (x_i)_{i \in 1 \setminus J}$) be the coordinate of $x'$ (resp. $x$) through the above embedding. We can regard $x'_i \in \mathbb{H}(\varpi_i)^2_K$, and it admits a decomposition

$$ x'_i = \prod_{\mu} x'_i[\mu], $$

where $\mu \in P$ runs over the $P$-weights of $\mathbb{H}(\varpi_i)^2_K$ (or $\mathbb{W}(\varpi_i)^2_K$). Let $d'_i[\mu]$ be the degree of the lowest $d$-degree non-zero contribution of $x'_i[\mu]$ (or $\infty$ if $x'_i[\mu] = 0$) for each $i \in 1 \setminus J$ and $\mu \in P$. Let $d_i$ be the $d$-degree of $x_i$ for each $i \in 1 \setminus J$ (remember that $x$ is $(H \times G_m)(K)$-fixed).

For each $\gamma \in Q^v$, we have a collection of the automorphisms of the vector spaces $\{\mathbb{H}(\lambda)\}_{\lambda \in P_+}$ by shifting (only) the $(d)$-gradings of the weight $\mu$-parts by $(\gamma, \mu)$ ($\mu \in P$). This defines an automorphism of $R^\gamma_{G,J}$, and hence defines an automorphism of $(Q^t_{G,J})_K(K)$. By using this twist for an appropriate $\gamma \in Q^v$, we can assume that

$$ d'_i[\mu] \geq d'_i[u \varpi_i] = d_i, \quad \mu \neq u \varpi_i, \quad (\text{3.7}) $$

for each $i \in 1 \setminus J$ and fixed $u \in W$ from the fact that every $P$-weight of $W(\varpi_i)_K$ is $\leq \varpi_i$. We have $x_i \in \mathbb{W}(u \varpi_i)_K$ for some $w \in W_{af}$ and all $i \in 1 \setminus J$. The inequality \((\text{3.7})\) implies $x'_i \in \mathbb{W}(u \varpi_i)_K$ for all $i \in 1 \setminus J$ \((\text{\text{[3]}}, \text{Theorem 5.17})\).

Hence, we have $x' \in Q^t_{G,J}(w)_K$ for $w \in W_{af}$ such that $x$ is the unique $(H \times G_m)(K)$-fixed point of $\mathfrak{O}(J, w)_K$. By twisting the whole situation by $N_{G(z)}(H(K))$, we can further assume $w = e$.

By the fact that $\mathbb{W}(\varpi_i)_K$ and $V(\varpi_i)_K \otimes K[z]$ shares the same $P$-weight $u_0 \varpi_i$-parts for each $i \in I$, the $H(K)$-action shrinks $x'$ to $x$. Hence, we can further modify $x'$ using the $H(K)$-action if necessary to assume $x' \in \mathfrak{O}(J, e)_K(K)$ by Lemma \ref{lem:modification}. As $\mathfrak{O}(J, e)_K(K) \subset Q^t_{G,J}$ by Proposition \ref{prop:embedding} and its proof, we deduce $x = x'$. By the description of $Q^t_{G,J}$, every $K$-valued point of the range of $\psi$ is $I(K)$-equivariant. Thus, the assertion follows. \qed

Theorem 3.18. Assume that $\text{char } K \neq 2$. For each $J \subset I$, we have a closed immersion of ind-schemes

$$ (Q^t_{G,J})_K \longrightarrow \prod_{i \in 1 \setminus J} \mathbb{P}(V(\varpi_i)_K \otimes K((z))). $$

In particular, the set of $K$-valued points of the indscheme $(Q^t_{G,J})_K$ is in bijection with

$$ \mathbb{G}(z)/ (H(K) \cdot [P(J), P(J)]((z))). $$

Proof. By construction, the loci $E \subset Q^t_{G,J}$ on which $\psi$ (borrowed from Proposition \ref{prop:embedding}) is not defined is an ind-subscheme. The map $\psi$ is $G(z)$-equivariant.

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It follows that $E$ admits a $G((z))$-action as ind-schemes. The map (borrowed from the proof of Proposition 3.16) 

$$Q_{G,J}(e_K) \mapsto \mathbb{P}(\mathbb{W}(\pi_i)_{K_i}^2)$$

sends $(E \cap Q_{G,J}(e_K))$ onto a closed subscheme of $\mathbb{P}(\ker(\mathbb{W}(\pi_i)_{K_i}^2 \to V(\pi_i)_{K_i} \otimes K[[z]]))$ for some $i \in I$. In particular, $(E \cap Q_{G,J}(e_K)) \subset Q_{G,J}(e_K)$ is a closed subscheme. Taking into account the $\{\tau_s\}_{s}$-actions (Lemma 3.17), we have $E \neq \emptyset$ if and only if $(E \cap Q_{G,J}(e_K)) \neq \emptyset$, and $\psi$ is a closed immersion if and only if $\psi_e$ is a closed immersion (cf. Lemma 3.17).

The one-parameter subgroup $a = (\xi, 1) : G_m \to (H \times G_m)$ for $\xi \in Q^e_K$ attracts every point of $\mathbb{P}(\mathbb{W}(\pi_i)_{K_i}^2)$ into its $(H \times G_m)$-fixed points (by setting $t \to 0$). In particular, $E$ has a $(H \times G_m)$-fixed point that is not realized by $Q_{K}^e$. Since the $(H \times G_m)$-fixed point of bounded $d$-degree is captured by the corresponding degree terms of $R_{K}$, it follows that the indscheme $E$ intersects with $\mathbb{Q}_{J}(v,e)$ for some $v \in \mathbb{W}_{st}$. The intersection of $\mathbb{Q}_{J}(v,e)_{K}$ and $Q_{K}^e$ (as the set of $K$-valued points) defines a closed subset $Y$ of (a product of) finite-dimensional projective space by Proposition 5.14 and Corollary 6.4. In particular, $Y$ acquires the structure of a proper scheme through $\psi$. From this viewpoint, Corollary 6.4 provides a $\psi$-section of $Y$ that defines a bijection of $K$-valued points with a Zariski open subset $\mathbb{Q}_{J}(v,e) \setminus E$ of $\mathbb{Q}_{J}(v,e)_{K}$. Since $Q_{K}^e$ admits a homogeneous $G((z))$-action, we can think of $\psi$ as an everywhere defined section of a vector bundle over the image (whose fiber is a product of $\ker(\mathbb{W}(\pi_i)_{K_i}^2 \to V(\pi_i)_{K_i} \otimes K[[z]])$’s). It implies that $\mathbb{Q}_{J}(v,e)_{K} \setminus E$ can be seen as an everywhere defined section of a vector bundle over the image (whose fiber is a product of finite-dimensional subspaces of $\ker(\mathbb{W}(\pi_i)_{K_i}^2 \to V(\pi_i)_{K_i} \otimes K[[z]])$’s). Since $\mathbb{Q}_{J}(v,e)_{K}$ is a finite type scheme defined over an algebraically closed field $K$, it has a dense subset formed by its $K$-valued points (see Corollaire 10.4.8]). Thus, the section $\psi$ can be seen as such of schemes. Therefore, $\mathbb{Q}_{J}(v,e)_{K} \setminus E$ is proper by itself. In conclusion, $\mathbb{Q}_{J}(v,e)_{K} \setminus E$ can be seen as a connected component of $\mathbb{Q}_{J}(v,e)_{K}$ (as being open and closed subset) which does not intersect with $E$. It follows that if $E \neq \emptyset$ as a scheme, then we find that the scheme $\mathbb{Q}_{J}(v,e)$ has at least two connected components. Therefore, the projective coordinate ring $R^e_{K}(J)_{K}$ of $\mathbb{Q}_{J}(v,e)_{K}$ must be non-integral. The same is true if we replace $v$ with a smaller element with respect to $<_{\mathfrak{m}}$.

Hence, we can find $f,g \in R^e_{K}(J)_{K} \setminus \{0\}$ such that $fg = 0$ for $v \leq_{\mathfrak{m}} e$. Since $E$ and its complement are $G_m$-stable, we can assume that $f$ and $g$ are $G_m$-eigenfunctions. Since $E$ defines a connected component of $\mathbb{Q}_{J}(v,e)_{K}$ for every $v \leq_{\mathfrak{m}} e$, we can fix the degrees of $f$ and $g$ for every $v \leq_{\mathfrak{m}} e$. For a fixed degree, the $d$-graded component of $R^e_{K}(J)_{K}$ and $R_{d}(J)_{K}$ are in common for $v \leq_{\mathfrak{m}} e$. Since $R^e_{K}(J)_{K}$ is a quotient ring of $R_{d}(J)_{K}$, we can find $G_m$-eigenfunctions $f,g \in R_{d}(J)_{K} \setminus \{0\}$ such that $fg = 0$. This implies that the ring $R_{d}$ is also non-integral. This contradicts with Corollary 6.4, and hence we deduce $E = \emptyset$ as an ind-scheme. Therefore, we conclude that $\psi$ is in fact a genuine morphism (instead of a rational map) of ind-schemes.

Next, we prove that $\psi$ (or rather $\psi_e$) defines a closed immersion. By Lemma 3.17 and the fact that $\psi$ is a morphism, we deduce that $\psi$ induces an isomorphism of the function field of $Q_{G,J}(e)$ and its image under $\psi$. This isomorphism between function fields descends to irreducible components of subschemes $\mathbb{Q}_{J}(v,e)_{K} \subset Q_{G,J}(e)$. Once we fix an integer $d_0$, two rings $R^e_{d}(J)_{K}$ and
$R_e(J)_K$ share the same $(d \cdot)$-grading $d_0$-part for $v \ll 2e$. Hence, it suffices to prove that $\psi$ restricts to a closed immersion of $Q_j(v, e)_K$ for $v \ll 2e$. Here $R_e(J)_K$ is weakly normal by Lemma 3.8. Let $R'$ be the multi-homogeneous coordinate ring of the (reduced induced structure of the) closed subscheme of $\prod_{i \in I} P(V(x_i) \otimes K((x)))$ defined by $\psi(Q_j(v, e)_K)$. Then, if we consider via each maximal integral quotient, we deduce that the weak normalization of $R'$ is precisely $R_e(J)_K$. This implies that $\psi(Q_j(v, e)_K)$ defines a closed subscheme for each $v \in W_{af}$ as required.

Since $\psi$ is a closed immersion, the rest of assertion follows from Proposition 3.13. These complete the proof. 

**Corollary 3.19** (of the proof of Theorem 3.18). Let $w \in W_{af}$ and $J \subset I$. Every two rational functions on $Q(J, \omega)_K$ are distinguished by a pair of $K$-valued points of $Q_j(v, w)_K$ for some $v \in W_{af}$. In particular, the union $\bigcup_{v \in W_{af}} Q_j(v, w)_K$ is Zariski dense in $Q(J, \omega)_K$.

**Proof.** In view of Theorem 3.18, all the results in Theorem 3.14 and Theorem 3.16 except for Theorem 3.16 3) is a consequence of the corresponding set-theoretic consideration. Theorem 3.16 3) is obtained by the fact that the $SL(2, i)$-action makes $O(w)_K$ into a $A^1$-fibration over $O(s_i w)_K$ for $i \in I_{af}$ when $w \geq 2s_i w$. 

### 3.3 Coarse representability of the scheme $Q_{\mathbb{G}}^{\text{rat}}$

Material in this subsection is rather special throughout this paper, and is irrelevant to the arguments in the later part, such as the normality of quasi-map spaces. In this subsection, we assume that \text{char} $K \neq 2$, and we also drop subscripts $K$ from $(Q_{\mathbb{G}}^{\text{rat}})_K$ and its subschemes in order to simplify notation.

Let $\text{Aff}^\text{op}$ be the category of affine schemes over $K$. We identify $\text{Aff}^\text{op}$ with the category of commutative rings over $K$. Let $\text{Zar}_K$ denote a big Zariski site over $K$ [GR, Section 00WN]. For $X \in \text{Zar}_K$, the assignment

$$Zar_K^{\text{op}} \ni U \mapsto \text{Hom}_{\text{Zar}_K}(U, X) \in \text{Sets}$$

defines a sheaf $h_X$ on $\text{Zar}_K$ [GR, Definition 00WR).

For the definition on the coarse moduli functors, we refer to [GR, Definition 1.10]. However, we employ some modified definition given in the below:

**Definition 3.21** (Strict indscheme). Let $\mathfrak{X} = \bigcup_{n \geq 0} X_n$ be an increasing union of schemes in $\text{Zar}_K$. We call $(\mathfrak{X}, \{X_n\}_n)$ (or simply refer as $\mathfrak{X}$) a strict indscheme if each inclusion $X_k \subset X_{k+1}$ $(k \geq 0)$ is a closed immersion.

**Definition 3.22** (Filtered sheaf on $\text{Zar}_K$). A filtered sheaf $(\mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0})$ on $\text{Zar}_K$ is a family of sheaves such that $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for each $k \in \mathbb{Z}_{\geq 0}$ and $\mathcal{F} = \bigcup_{n} \mathcal{F}_n$. Let $(\mathcal{F}, \{\mathcal{F}_n\}_{n \geq 0})$ and $(\mathcal{G}, \{\mathcal{G}_n\}_{n \geq 0})$ be filtered sheaves on $\text{Zar}_K$. A morphism $f : \mathcal{F} \to \mathcal{G}$ of sheaves is said to be continuous if for each $n \in \mathbb{Z}_{\geq 0}$, there is some $m \in \mathbb{Z}_{\geq 0}$ such that

$$f(\mathcal{F}_n) \subset \mathcal{G}_m \quad \text{and} \quad f^{-1}(\mathcal{G}_m) \subset \mathcal{F}_m.$$
Let \((\mathfrak{X}, \{F_n\})\) be a strict indscheme. Then, we call \(h_{\mathfrak{X}} := (\bigcup_n h_{F_n}, \{h_{F_n}\}_{n \geq 0})\) the filtered sheaf associated to \(\mathfrak{X}\).

**Definition 3.23** (Coarse indrepresentability). Let \(\mathcal{X}\) be a filtered sheaf on \(\text{Zar}_K\). Let \(\mathfrak{X}\) be a strict indscheme over \(K\). We say that \(\mathcal{X}\) is coarsely indrepresentable by \(\mathfrak{X}\) if the following conditions hold:

- We have a continuous morphism \(u : \mathcal{X} \to h_{\mathfrak{X}}\) of filtered sheaves;
- We have \(\mathcal{X}(k) = h_{\mathfrak{X}}(k)\) for an overfield \(k \supset K\);
- Let \(Q\) be a strict indscheme and we have a continuous morphism \(f : \mathcal{X} \to h_Q\), then it factors as:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{u} & h_{\mathfrak{X}} \\
\downarrow f & & \downarrow g \\
h_Q
\end{array}
\]

where \(g\) is a morphism of sheaves. It is automatic that \(g\) is continuous, and hence is induced by a morphism of indschemes.

We consider the assignment \(Q\) on \(\text{Aff}^{\text{op}}\) defined as:

\[
\text{Aff}^{\text{op}}_K \ni R \mapsto Q(R) := G(R(\|z\|))/H(R)N(R(\|z\|)) \in \text{Sets}.
\]

For each \(n \in \mathbb{Z}_{\geq 0}\), we consider an assignment

\[
\text{Aff}^{\text{op}}_K \ni R \mapsto Q_n(R) := \{g \mod H(R)N(R(\|z\|)) \in Q(R) \mid (\ast)\} \in \text{Sets},
\]

where

\[(\ast) \ g v_{\omega_i} \text{ has at worst pole of order } n \text{ on } V(\omega_i) \otimes \mathbb{Z} R(\|z\|) \text{ for each } i \in I.\]

**Lemma 3.24.** The assignments \((Q, \{Q_n\})\) define a filtered sheaf on \(\text{Zar}_K\) that we denote by \(Q\).

**Proof.** We first prove that \(Q\) is a sheaf. Since the sheaf condition on \(\text{Zar}_K\) can be checked on \(\text{Aff}^{\text{op}}_K\) ([* Lemma 020W*]), it suffices to check that

\[
Q(R) \to Q(R_1) \times Q(R_2) \Rightarrow Q(R_{12})
\]

is exact for \(R, R_1, R_2, R_{12} \in \text{Aff}^{\text{op}}_K\) such that:

- \(R_1\) and \(R_2\) are localizations of \(R\);
- \(\text{Spec } R_{12} = \text{Spec } R_1 \cap \text{Spec } R_2\) and \(R = R_1 \cap R_2 \subset R_{12}\).

For each \(h \in H(R_{12})N(R_{12}(\|z\|))\), we have some \(h_1 \in H(R_i)N(R_i(\|z\|))\) \((i = 1, 2)\) such that \(h = h_2 h_1^{-1}\) by inspection (as \(N\) is unipotent and \(HN\) is solvable).

Let \(g_1 \in G(R_i(\|z\|))\) and \(g_2 \in G(R_2(\|z\|))\) be elements such that \(g_1 = g_2 h\) for \(h \in H(R_{12})N(R_{12}(\|z\|))\). It follows that \(g_1 h_1 = g_2 h_2 \in G(R(\|z\|))\) for some \(h_1 \in H(R_i)N(R_i(\|z\|))\) and \(h_2 \in H(R_2)N(R_2(\|z\|))\). This yields the desired lift.

By the condition \((\ast)\), twisting by \(H(R)N(R(\|z\|))\) does not change our counting of the pole order. Therefore, we deduce that each \(Q_n\) defines a sheaf such that \(Q = \bigcup_n Q_n\). \(\square\)
Lemma 3.25. The indscheme $Q^G_{\text{rat}}$ defines a filtered sheaf on $Zar_X$ given by a strict indscheme structure.

Proof. A scheme over $k$ defines a sheaf over $Zar_K$, and so is its increasing union. In view of Proposition 3.16, the pole order $n$ condition amounts to choose the $I$-orbits $O(ut_\beta)$ ($u \in W, \beta \in Q^R$) such that $\langle \beta, \varpi_i \rangle \geq -n$ (for every $i \in I$), that makes the smaller one to be a closed subscheme of the larger one (cf. Lemma 2.25).

Proposition 3.26. The scheme $Q^G_{\text{rat}}$ coarsely (ind-)represents the sheaf $Q_g$.

Proof. We first construct an injective continuous morphism $Q \to hQ^G_{\text{rat}}$ as filtered sheaves on $Zar_K$ such that $Q(k) = hQ^G_{\text{rat}}(k)$ for an overfield $k \supset k$. For $R \in \text{Aff}_{\text{rat}}^+$, the set $Q(R)$ is represented by a class of $g \in G(R((z)))$ modulo the right action of $H(R)N(R((z)))$. It defines a point of $Q^G_{\text{rat}}(R)$ by applying $g$ on $\{v_{\varpi_i}\}_{i \in I} \in \prod_{i \in I} F_R(V(\varpi_i)z \otimes Z R((z)))$. Since the $G(R((z)))$-stabilizer of $v_{\varpi_i}$ is precisely $H(R)N(R((z)))$, we conclude an inclusion $Q(R) \subset hQ^G_{\text{rat}}(R)$. By examining the construction, we deduce that this defines an injective continuous morphism of filtered sheaves.

By the Bruhat decomposition, we have

$$G(k((z))/N(k((z)))) = (G/N)(k((z)))$$

for an overfield $k \supset k$ (and hence $k((z))$ is a field). In view of Theorem 3.5.8, we conclude that $Q \subset hQ^G_{\text{rat}}$ is a subsheaf with $Q(k) = hQ^G_{\text{rat}}(k)$ when $k \supset k$ is an overfield.

We verify the versality property. Suppose that we have a strict indscheme $(\mathcal{X}, \{X_n\}_{n})$ and we have a continuous morphism $Q \to hX$. By Lemma 3.11, we deduce that each $I$-orbit of $Q^G_{\text{rat}}$ defines a subsheaf of $Q$. The zero-th ind-piece $(Q^G_{\text{rat}})_0$ in Lemma 3.23 is $Q_G(e)$.

For each $t_\beta \leq e$ ($\beta \in Q^Z_+$), we find a reduced expression $t_\beta^{-1} = s_{i_1} \cdots s_{i_t}$ (that we record as $i := (i_1, \ldots, i_t)$) and form a scheme

$$Z(i) := I(i_1) \times I(i_2) \times \cdots \times I(t_\beta)$$

and the map

$$Z(i) = I(i_1) \times I(i_2) \times \cdots \times I(t_\beta) \to Q_G(e)$$

(see e.g. Kumar [24, Chapter VIII]; cf. [37, §6]). In view of Lemma 3.11, the image of this map contains an open neighbourhood of $O(t_\beta)$, and the domain is formally smooth. In view of the $G((z))$-action (or the various $SL(2, i)$-actions for $i \in I_{ad}$) on $Q^G_{\text{rat}}$ and $Q$, we have a morphism

$$f_i : hZ(i) \to Q_0.$$ 

By varying $i$ (and consequently varying $t_\beta \leq e$), we deduce that the union of the image of the morphisms $\{f_i\}_{i}$ exhausts $Q_0(k)$ for an overfield $k \supset k$. From the Yoneda embedding, we derive a map

$$Z(i) \to X_n$$

of schemes for some fixed $n \in \mathbb{Z}$. This map factors through a scheme $Z$ that glues (among $i$'s) all the closed points that maps to the same points in $Q_0$.  

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Such a scheme is integral as $Z(\bar{i})$’s are so and the gluing identifies the Zariski open dense subset $O(e)$ for distinct $\bar{i}$’s. In addition, we have a birational map $\pi: Z \to Q_G(e)$, and hence we have

$$Z(k) = Q_G(e)(k) = Q_0(k) \quad \text{for an overfield} \quad k \ni k.$$ 

We prove that $Z = Q_G(e)$ by induction. For each $m \in \mathbb{Z}_{\geq 0}$, let $Q_G(e)_{<m}$ (resp. $Q_G(e)_{\leq m}$) be the union of $I$-orbits in $Q_G(e)$ of the shape $\ell(v)$ for $\ell \geq 1$ (resp. $\ell \leq m$).

Assume that the map $\pi$ is an isomorphism when restricted to $Q_G(e)_{<m}$, and we prove the same is true when restricted to $Q_G(e)_{\leq m}$. The $m = 1$ case is afforded by $O(e) \subset Z$ already used in the construction of the above.

We have a partial compactification $Z(\bar{i})$ of $Z(\bar{i})^\circ$ with a map $f_1^+$ given as:

$$Z(\bar{i}) := I(i_1) \times I(i_2) \times I \times I Q_G(t_\beta) \xrightarrow{f_1^+} Q_G(e).$$

Note that we have a surjective morphism induced by $O(t_\beta) \to \text{Spec } \mathbb{K}$

$$\eta: Z(\bar{i}) \longrightarrow I(i_1) \times I(i_2) \times I \times I \text{Spec } \mathbb{K},$$

where we denote the image (the RHS term) by $Z'(\bar{i})$. Since $Z'(\bar{i})$ is a finite successive $\mathbb{P}^1$-fibration, it is proper. The map $f_1^+$ is proper as the product map

$$(\eta \times f_1^+): Z(\bar{i}) \to Z'(\bar{i}) \times I(i_1) \times I(i_2) Q_G(t_\beta) = Z'(\bar{i}) \times Q_G(e)$$

is a closed immersion. In view of the isomorphism $Q_G(t_\beta) \cong Q_G(e)$, we transplant $Q_G(e)_{\leq m}$ to $Q_G(t_\beta)_{\leq m}$.

\textbf{Claim A.} \textit{For each closed point $x \in Q_G(e)_{\leq m}$, the scheme}

$$(f_1^+)^{-1}(x) \setminus ((f_1^+)^{-1}(x) \cap (I(i_1) \times I(i_2) \times I \times I Q_G(t_\beta)_{\leq m})) \subset (f_1^+)^{-1}(x)$$

\textit{is a closed subscheme that is zero-dimensional. In other words, it is a finite union of points (that is potentially an empty set).}

\textbf{Proof.} For each sequence $(j_1, \ldots, j_s) \in I_{\mathbb{K}}(s \in \mathbb{Z}_{\geq 0})$ and $v \leq \ell$, the image of the map

$$f : I(j_1) \times I(j_2) \times I \times I I(j_s) \times I Q_G(v) \longrightarrow Q_G^{\mathbb{K}},$$

induced by the multiplication is a union of $I$-orbits $O(v')$ with $\ell \geq \ell - s$ (as we have $\ell \geq 1$ for each $i \in I_{\mathbb{K}}$ and $\ell \in W$ by [21, Lecture 13, Proposition 13]). In addition, if the image of the map $f$ contains $O(v')$ for $\ell \geq \ell - s$, then the map $f$ is an isomorphism along $O(v')$ (as the isomorphism between open subsets). By collecting these for $I$-orbits in the closed subset $Q_G(t_\beta) \setminus Q_G(t_\beta)_{\leq m}$ of $Q_G(t_\beta)$ in the construction of the (proper) map $f_1^+$, we conclude the result. \hfill \square

We return to the proof of Proposition 25.24. By Claim A, we deduce that

$$\overline{(f_1^+)^{-1}(x) \cap I(i_1) \times I(i_2) \times I \times I Q_G(t_\beta)_{\leq m}} \subset (f_1^+)^{-1}(x)$$

is a union of connected components of $(f_1^+)^{-1}(x)$ for each closed point $x \in Q_G(e)_{\leq m}$.
We have

\[ (Z \ni) \pi^{-1}(Q_G(e) \leq m) \to Q_G(e) \leq m \]

defines an isomorphism as schemes (as \( \pi^{-1}(Q_G(e) \leq m) \to Q_G(e) \leq m \) is finite bijective, birational, and shares the same Zariski open subset, cf. \[ (\text{Section } 02LQ) \]). Therefore, induction on \( m \) proceeds and we conclude \( Z \cong Q_G(e) \) as schemes. Thus, we obtain a morphism \( Q_G(e) \to \mathcal{X} \) of schemes.

By rearranging \( Q_G(e) \) by the right \( Q^r \)-translations, we deduce a morphism \( Q_{G;J}^{rat} \to \mathcal{X} \) as ind-schemes. This yields a continuous morphism \( h_{Q_{G;J}^{rat}} \to h_{\mathcal{X}} \). Therefore, \( h_{Q_{G;J}^{rat}} \) is an initial object in the category of sheaves on \( Zar_{\mathcal{X}} \) ind-representable by strict ind-schemes that admits a continuous morphism from \( Q \) as required.

\[ \square \]

Corollary 3.27. For each \( J \subset I \), the scheme \( Q_{G;J}^{rat} \) coarsely (ind-)represents the filtered sheaf \( Q_J \) defined by sheafifying

\[ \mathbb{A}^{EP}_{G;J} \ni R \mapsto Q_J(R) := G(R(\mathbb{z})) / (H(R) \cdot [P_2; P_3](R(\mathbb{z}))) \in \mathbb{Sets}. \]

\[ \text{Proof.} \] By construction, we have a continuous morphism of sheaves \( Q \to Q_J \) (by transplanting subsheaves \( Q_n \) to \( Q_J \) via this map). Thus, the coarse ind-representability of \( Q_{G;J}^{rat} \) implies that the maximal indscheme \( \mathcal{X} \) obtained by gluing points of \( Q_{G;J}^{rat} \) so that we have a continuous morphism \( h_{Q_{G;J}^{rat}} \to h_{\mathcal{X}} \) (coarsely) represents the filtered sheaf \( Q_J \). Every two rational functions on \( Q_G(w) \) \( (w \in W_{af}) \) are distinguished by some pair of \( K \)-valued points (Corollary 3.14). Since we have \( Q_J(K) = Q_{G;J}^{rat}(K) \), we conclude that \( X = Q_{G;J}^{rat}(v, w) \). \( \square \)

3.4 The properties of the schemes \( Q_J^r(v, w) \)

In the rest of this section, we assume \( \text{char } K \neq 2 \).

Lemma 3.28. For each \( \beta \in Q_J^r, J \subset I \), the set of \( K \)-valued points of \( Q_J^r(\beta, e)_K \) is in bijection with the collection \( \{u_{\lambda}(z)\}_{\lambda \in P_{\beta,+}} \) such that

- We have \( u_{\lambda}(z) \in V(\lambda)_K \otimes \bigoplus_{j=0}^{\infty} (-w_0^{\beta, \lambda}) K z^j \);

- For each \( \lambda, \mu \in P_{\beta} \), we have \( \eta_{\lambda, \mu}(u_{\lambda}(z) \otimes u_{\mu}(z)) = u_{\lambda + \mu}(z) \).

\[ \text{Proof.} \] The scheme \( Q_{G;J}^r(e) \) is the intersection of \( Q_{G;J}^{rat} \) with \( \prod_{i \in I \setminus J} P(V(\mathbb{w}_i)_K \otimes K[z]) \) by Theorem 3.13. By the symmetry of the construction of \( Q_J^r(\beta, e) \) in terms of \( \theta \), we conclude that

\[ Q_J^r(\beta, e) = Q_{G;J}^{rat} \cap \prod_{i \in I \setminus J} P(V(\mathbb{w}_i)_K \otimes K[z]) \cap \prod_{i \in I \setminus J} P(V(\mathbb{w}_i)_K \otimes K[z^{-1}] z^{-(\beta, w_0^{\mathbb{w}_i})}) \]

inside \( \prod_{i \in I \setminus J} P(V(\mathbb{w}_i)_K \otimes K[z, z^{-1}]) \), that is our degree bound. In view of this, it suffices to remember that the second condition is the same as the Pfucker relation that defines \( G(\mathbb{z}) / H(\mathbb{z}) \cdot [P(\mathbb{J}), P(\mathbb{J})]\langle \mathbb{z} \rangle \) in the last two paragraphs of the proof of Proposition 5.10. \( \square \)
We have a natural line bundle $O_{2J(v,w)}(\lambda)$ for each $\lambda \in P_3$. Thus, for each $\lambda \in P_3$, we have an isomorphism $H^0(J(v,w), O_{2J(v,w)}(\lambda))$.

This yields $O_{2J(v,w)}(\lambda)$ for each $\lambda \in P_3$ by tensor products.

**Lemma 3.29.** For each $w, v \in W_{af}$, the line bundle $O_{2J(v,w)}(\lambda)$ is very ample if $(\alpha_i^v, \lambda) > 0$ for every $i \in I \setminus J$.

**Proof.** We can assume that $2J(v,w) \neq \emptyset$ without the loss of generality.

By Lemma 3.3, we have

$$Q_J^r(v,w) \hookrightarrow \prod_{i \in I \setminus J} \mathbb{P}_\mathbb{Z}(R_{u_i}^r(2\mathbb{R}_v)).$$

From this and again by Lemma 3.3, we deduce that the following diagram is commutative (with $\rho_1 := \sum_{i \in I \setminus J} \mathbb{Z}_i$):

$$\begin{array}{ccc}
\pi_P \circ \mathbb{P}_\mathbb{Z}(\otimes_{i \in I \setminus J} R_{u_i}^r(2\mathbb{R}_v)) & \xrightarrow{\xi} & \mathbb{P}_\mathbb{Z}(\mathbb{R}_v^r(1, 2\mathbb{R}_v)) \\
\downarrow \mathbb{P}_\mathbb{Z}(2\mathbb{R}_v^r(1, 2\mathbb{R}_v)) & & \downarrow \mathbb{P}_\mathbb{Z}(2\mathbb{R}_v^r(1, 2\mathbb{R}_v)) \\
\mathbb{P}_\mathbb{Z}(R_{u_i}^r(2\mathbb{R}_v)) & \xrightarrow{\kappa} & \mathbb{P}_\mathbb{Z}(2\mathbb{R}_v^r(1, 2\mathbb{R}_v)) \\
Q_J^r(v,w) & & \end{array}$$

where the $\xi$ is the Veronese embedding and $\kappa$ is induced from the multiplication map. By $R_{u_i}^r(2\mathbb{R}_v^r(1, 2\mathbb{R}_v)) \subset H^0(Q_J^r(v,w), O_{2J(v,w)}(\rho_2))$, we conclude that $O_{2J(v,w)}(\lambda)$ is very ample. Since we have embedding

$$R_{u_i}^r(2\mathbb{R}_v^r(1, 2\mathbb{R}_v)) \subset H^0(Q_J^r(v,w), O_{2J(v,w)}(\lambda))$$

obtained through multiplications corresponding to the duals of extremal weight vectors (so that the image is base-point-free along a $(H \times \mathbb{G}_m)$-stable Zariski open neighbourhood of each $(H \times \mathbb{G}_m)$-fixed point of a projective variety), we conclude that $O_{2J(v,w)}(\lambda)$ is also very ample as required.

**Theorem 3.30.** For each $w = u \beta \in W_{af}$ ($u \in W, \beta' \in Q'$) and $\beta \in Q_J^r$, we have an isomorphism $Q'(\beta, w)_C \cong Q(\beta - \beta', u)$ as varieties. Moreover, $Q'(\beta, w)_C$ is irreducible and its dimension is given as

$$\dim Q'(\beta, w)_C = 2(\beta - \beta', \rho) + \dim \mathcal{B}(u).$$

**Proof.** By Lemma 3.8, we know that $(R_v^r)_C$ is a quotient of $(R_v^r)_K$ for $v \in W_{af}$ such that $v \leq \frac{\omega}{C}$. Hence, we have $Q'(\beta, w)_K \cap Q(\beta)_K = Q'(\beta, w)_K$. By Lemma 3.7, the scheme $Q'(\beta, w)_K \cap Q(\beta)_K$ is isomorphic to $Q'(\beta - \beta', u')_K \cap Q(\beta', u')_K$, where $v = u't_{\beta'}$ ($u' \in W, \beta' \in Q'$). By Lemma 3.28 and Theorem 3.8, we have:

- We have $(Q'(\beta, w)_K \cap Q(\beta)_K) = (Q'(\beta, v)_K \cap Q(\beta)_K)$ for each $v \in W_{af}$;
- The variety $(Q'(\beta, w)_K \cap Q(\beta)_K)$ is irreducible for every $v \in W_{af}$.

We have an equality

$$\dim (Q'(\beta, w)_C \cap Q(\beta)_C) = \dim (Q(\beta - \beta', v')_C) = 2(\beta - \beta', \rho) + \dim \mathcal{B}(u')$$

for $v = u't_{\beta'}$ ($u' \in W, \beta' \in Q'$) by Lemma 3.28 and Theorem 3.8. In addition, Lemma 3.8 implies

$$\dim \mathcal{B}(u) = \dim \mathcal{B}(u').$$
We have \( \text{dim } Q'(\beta, w)_C = \text{dim } Q'(\beta, w)_K \).

In particular, we have the desired dimension formula if \( Q'(\beta, w)_K \) is irreducible with its Zariski open dense subset \( (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \).

Since \( Q(\beta, w) \) and \( Q'(\beta, w)_C \) shares a same open subset and the former is irreducible, we have \( Q(\beta, w) = Q'(\beta, w)_C \) as closed subvarieties of \( (Q^*_K)_C \) if \( Q'(\beta, w)_C \) is irreducible.

Therefore, it suffices to prove that \( Q'(\beta, w)_K \) is irreducible (with its Zariski open dense subset \( (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \)). By Theorem 3.31, it suffices to prove that

\[
(\quad (Q'(\beta, w)_K \cap \mathcal{O}(sw)_K) \subset (Q'(\beta, w)_K \cap \mathcal{O}(w)_K)
\]

for every \( w \in W_{af} \) and every reflection \( s \in W_{af} \) such that \( \ell \mathcal{H}(sw) = \ell \mathcal{H}(w) + 1 \). Here \( Q_K(sw)_K \subset Q_K(v)_K \) is an irreducible component of the boundary by Theorem 3.31 (cf. Corollary 3.31). This boundary component is these cut out as (a part of) the zero of \( w_{w_{0,\lambda}} \in W_{w_{0,\lambda}}(\lambda) \) (\( \lambda \in P_+ \)). Thus, we deduce that \( (Q'(\beta, w)_K \cap \mathcal{O}(sw)_K) \) contains an irreducible component of \( (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \cap \{ f = 0 \} \)

for some single equation \( f \) (an instance of \( w_{w_{0,\lambda}} \)'s) if it is nonempty. By the comparison of dimensions, this forces

\[
(\quad (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \cap \{ f = 0 \} \cap \mathcal{O}(sw)_K \subset (Q'(\beta, w)_K \cap \mathcal{O}(sw)_K)
\]

to be an irreducible component if the LHS is nonempty. Since the \( (H \times G_m) \)-invariant curve that connects the fixed points in \( \mathcal{O}(sw)_K \) and \( \mathcal{O}(w)_K \) inside \( \mathcal{O}(sw)_K \cup \mathcal{O}(w)_K \) is contained in \( (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \), the LHS of (3.8) is nonempty. Hence, the irreducibility of \( (Q'(\beta, w)_K \cap \mathcal{O}(sw)_K) \)

forces

\[
(\quad (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \cap \{ f = 0 \} \cap \mathcal{O}(sw)_K = (Q'(\beta, w)_K \cap \mathcal{O}(sw)_K).
\]

Therefore, we conclude (3.31). This implies \( Q(\beta - \beta', u) = Q'(\beta, w)_C \) as an irreducible (reduced) closed subvariety of \( (Q^*_K)_C \), and \( Q'(\beta, w)_K \) is irreducible in general. Its dimension \( \text{dim } Q(\beta - \beta', u) \) comes from the dimension (of \( (Q'(\beta, w)_K \cap \mathcal{O}(w)_K) \)), that is a Zariski open dense subset of \( Q'(\beta, w)_K \).

\[
\text{Corollary 3.31. For each } w, v \in W_{af}, \text{ the dimension of an irreducible component of } Q'(v, w)_K \text{ is } \ell \mathcal{H}(v) - \ell \mathcal{H}(w). \text{ In particular, } Q'(v, w)_K \text{ is equidimensional.}
\]

Proof. The case \( v = w_{0,\beta} \) with \( \beta \in Q' \) follows from Theorem 3.31. The inequality \( \leq \) follows from this case since cutting by a hyperplane lowers the dimension at worst one. Every irreducible component of \( Q'(v, w)_K \) is \( (H \times G_m) \)-stable by construction. If an irreducible component of \( Q'(v, w)_K \) does not meet any \( \mathcal{L} \)-stable boundary divisor of \( Q_K(v) \) and any \( \theta \)-twists of the boundary divisor of \( Q_K(v_{w_0}) \), then such an irreducible component does not contain a \( (H \times G_m) \)-fixed point. This is a contradiction (to the properness of \( Q'(v, w)_K \)). If an irreducible component of \( Q'(v, w)_K \) is contained in \( Q'(v', w')_K \subset Q'(v', w')_K \) for \( v', w' \in W_{af} \), then it gives an irreducible component of \( Q'(v', w')_K \) with its dimension \( \geq \ell \mathcal{H}(v') - \ell \mathcal{H}(w') \). Thus, we can lower the dimension of an irreducible component of \( Q'(v, w)_K \) by intersecting with a divisor to raise \( v \) or lower \( w \) successively to reach to the case \( v = w \) (here we remark \( Q'(w, w)_K = \text{Spec } K \) for every \( w \in W_{af} \), and the resulting dimension is equal to 0 if and only if the dimension equalities hold for all intermediate steps. Therefore, the inequality \( > \) is impossible. This implies the equality as required.
Remark 3.32. The analogous assertions for Theorem 3.31 and Corollary 3.30 holds for the case $J \subseteq I$ (cf. Corollary 2.31 and Remark 2.32). In view of [11, §2], the proofs are verbatim.

Theorem 3.33. For each $w, v \in W_{af}$, $J \subseteq I$, and $\lambda \in P_{3,+}$, we have

$$H^i(Q'_J(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}((\lambda)) \cong \begin{cases} R^w_\lambda((\lambda))_{\mathbb{K}} & (i = 0, \lambda \in P_{3,+}) \\ \{0\} & (i \neq 0) \end{cases}.$$ 

Moreover, if $w', v' \in W_{af}$ satisfies $Q'_J(v', w') \subseteq Q'_J(v, w)$ and $\lambda \in P_{3,+}$, then the restriction map induces a surjection

$$H^0(Q'_J(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}((\lambda))) \twoheadrightarrow H^0(Q'_J(v', w')_{\mathbb{K}}, \mathcal{O}_{Q'_J(v', w')_{\mathbb{K}}}((\lambda))).$$

Remark 3.34. The strictly dominance condition in Theorem 3.31 cannot be removed naively. For $G = SL(3)$, $\mathbb{K} = \mathbb{C}$, and $w = e$, we have

$$\dim Q'_J(2\alpha_1^e, e)_{\mathbb{C}} = 8 > 7 = \dim Q'(2\alpha_1^e, e)_{\mathbb{C}}.$$ 

This results in the non-injectivity of the pullback map

$$H^0(Q'_J(2\alpha_1^e, e)_{\mathbb{C}}, \mathcal{O}_{Q'_J(2\alpha_1^e, e)_{\mathbb{C}}}(3\pi_1)) \twoheadrightarrow H^0(Q'(2\alpha_1^e, e)_{\mathbb{C}}, \mathcal{O}_{Q'(2\alpha_1^e, e)_{\mathbb{C}}}(3\pi_1)),$$

where the LHS is $R^{2\alpha_1^e}(3\pi_1)_{\mathbb{C}} = R^e_{2\alpha_1^e}(\{1\}, 3\pi_1)_{\mathbb{C}} \cong S^3(\mathbb{C}^3 \oplus \mathbb{C}^3z \oplus \mathbb{C}^3z^2)$, while the RHS is its quotient by the $G$-invariants of $\mathbb{C}^3 \oplus \mathbb{C}^3z \oplus \mathbb{C}^3z^2 \subseteq S^3(\mathbb{C}^3 \oplus \mathbb{C}^3z \oplus \mathbb{C}^3z^2)$. In fact, the ring $(R^e_{2\alpha_1^e})_{\mathbb{C}}$ is reduced but not irreducible, and the extra irreducible component is contained in the irrelevan loci.

Proof of Theorem 3.31. We first observe that it is enough to show the surjectivity of the assertion only for $v = v'$ case since the $w = w'$ case follows by applying $\theta$ (and the rest of the cases follow by a repeated applications of the two cases).

Since $Q(J, w)_{\mathbb{F}_p} \subset Q_{G, J}(w)_{\mathbb{F}_p}$ is affine, the union $Z$ of codimension one $I$-orbits contains the support of an ample divisor. The reduced union of the irreducible components of $Z$ affords a Weil divisor $D$ (by Corollary 2.31). Since our Frobenius splitting of $Q_{G, J}(w)_{\mathbb{F}_p}$ is $I$-canonical and it is compatible with the $I$-orbit closures (Corollary 2.31), we deduce that $Q_{G, J}(w)_{\mathbb{F}_p}$ is $D$-split in the sense of [11, Definition 3] (cf. [11, Definition 1.2]). Each irreducible component of $Z$ intersects non-trivially with $Q'_J(v', w)_{\mathbb{F}_p} \subseteq Q'_J(v, w)_{\mathbb{F}_p}$, for each $v' \in W_{af}$ by Corollary 2.31 (and Remark 2.32). Thus, we deduce that $Q'_J(v, w)_{\mathbb{F}_p}$ admits a Frobenius $D$-splitting compatible with $Q'_J(v, w')_{\mathbb{F}_p}$. Therefore, the cohomology vanishing part of the assertion and the surjectivity part of the assertion for $v' = v$ follow from [11, Proposition 1.13 (ii)]. The cohomology vanishing part of the assertion lifts to $\text{char} \mathbb{K} = 0$ by [11, Proposition 1.6.2]. The surjectivity part of the assertion lifts to $\text{char} \mathbb{K} = 0$ by [11, Corollary 1.6.3] and Lemma 3.32.

It remains to calculate $H^0(Q'_J(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}((\lambda)))$ for $\lambda \in P_{3,+}$. Here $H^0(Q'_J(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}((\lambda)))$ is obtained as the degree $\lambda$-part of the (graded) normalization of the ring $(R^w_\lambda(1))_{\mathbb{K}}$. For each prime $p$, our ring $(R^w_\lambda(1))_{\mathbb{F}_p}$ is weakly normal by Lemma 3.31. Let us consider the $P_{3,+}$-graded quotient

$$R' = \bigoplus_{\lambda \in P_{3,+}} R'((\lambda)) \otimes \mathbb{K}$$

that annihilates all the irrelevant irreducible components. Since $R^w_\lambda(1)_{\mathbb{F}_p}$ is a reduced ring, we have $R^w_\lambda(3, \lambda)_{\mathbb{F}_p} = R'(\lambda)$ for
\[ \lambda \in P_{3,++}. \] The ring \( R' \) is generated by \( \bigoplus_{i \in I \setminus J} R'(w_i) \) as \( R_w(J)_{\mathbb{F}_p} \) is so. We have
\[
H^0(Q'_J(v, w)_{\mathbb{F}_p}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{F}_p}}(\lambda)) \cong R_w(J, \lambda)_{\mathbb{F}_p}
\]
for sufficiently large \( \lambda \in P_{3,++} \) (see e.g. \textit{Exer. 5.9}). In other words, (3.33) hold for \( m \lambda \), where \( \lambda \in P_{3,++} \) is arbitrary and \( m \gg 0 \).

Let \( R^+ = H^0(Q'_J(v, w)_{\mathbb{F}_p}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{F}_p}}(\lambda)) \) for \( \lambda \in P_{3,++} \). We set \( R^+[\lambda] := \mathbb{F}_p \) and \( R'[\lambda] := \bigoplus_{m \geq 0} R^+(m \lambda) \subset R' \) for \( \lambda \in P_{3,++} \). Both are naturally rings. Then, we have a ring extension \( R'[\lambda] \subset R^+[\lambda] \) such that their degree \( m \gg 0 \)-parts are the same.

We prove \( R'(\lambda) = R^+[\lambda] \). The ring \( R'[\lambda] \) is weakly normal as we have an inclusion \( R'^+[\lambda] \hookrightarrow R_w(J)_{\mathbb{F}_p} \) and the reasoning that \( R_w(J)_{\mathbb{F}_p} \) is weakly normal equally applies to \( R'[\lambda] \). Since it is generated by \( R'(\lambda) \), we deduce that two rings \( R'[\lambda] \) and \( R^+ \) share the same spectrum (the irrelevant loci of \( R'[\lambda] \) is one point as it is a \( \mathbb{Z}_{\geq 0} \)-graded ring, and other points are just the same). This forces \( R'(\lambda) = R^+[\lambda] \) by the weak normality of \( R'[\lambda] \).

This yields the \( H^0 \)-part of the assertion for \( \text{char} \ \mathbb{K} > 0 \) through the extension of scalars. The \( H^0 \)-part of the assertion for \( \text{char} \ \mathbb{K} = 0 \) is obtained by taking the generic specialization of the base scheme \( \text{Spec} \ \mathbb{Z} \) of \( Q'_J(v, w) \).

**Corollary 3.35.** Let \( w, v \in W_{\text{af}} \) and \( J \subset I \). Assume that the map \( \Pi_J : Q'(v, w)_{\mathbb{K}} \to Q'_J(v, w)_{\mathbb{K}} \) defined through the projective coordinate ring is surjective. We have
\[
\mathbb{R}^0(\Pi_J)_! \mathcal{O}_{Q'(v, w)_{\mathbb{K}}} \cong \{0\} \quad \text{and} \quad \Pi_J^* \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}} \cong \mathcal{O}_{Q'(v, w)_{\mathbb{K}}}.
\]

**Remark 3.36.** Since the map \( Q'_G^{\text{rat}} \to Q'_G^{\text{rat}} \) is surjective, we can replace \( v \) with \( vt_\beta \) for some \( \beta \in \sum_{j \in J} \mathbb{Z}_{\geq 0} \alpha_j \) to obtain a surjection:
\[
\Pi_J : Q'(vt_\beta, w)_{\mathbb{K}} \to Q'_J(vt_\beta, w)_{\mathbb{K}} = Q'_J(v, w)_{\mathbb{K}}.
\]

**Proof of Corollary 3.35.** Thanks to \textit{[1], Lemma A.31} and Lemma 3.33, it suffices to prove
\[
H^0(Q'(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'(v, w)_{\mathbb{K}}}(\lambda)) \cong \{0\} \quad \text{and} \quad H^0(Q'_J(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}(\lambda)) \cong H^0(Q'_J(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}(\lambda))
\]
for each \( \lambda \in P_{3,++} \). The \( H^0 \)-part of the assertion follow from Theorem 3.33. The \( H^0 \)-part of the assertion follow as in the latter part of the proof of Theorem 3.33 since the surjectivity of \( \Pi_J \) guarantees that every non-zero element of \( R_w(J)_{\mathbb{F}_p} \) is supported outside of the irrelevant loci by pullback and (graded) normalization of the ring \( R_w(J)_{\mathbb{F}_p} \) is the degree \( P_{3,++} \)-part of the (graded) normalization of the ring \( R_w(J)_{\mathbb{F}_p} \) by the degree reasons.

**Corollary 3.37.** Let \( \lambda \in P_+ \) and \( w, v \in W_{\text{af}} \). We set \( J := \{ i \in I \mid \langle \alpha_i^\vee, \lambda \rangle = 0 \} \). Assume that \( \Pi_J(Q'(v, w)_{\mathbb{K}}) = Q'_J(v, w)_{\mathbb{K}} \). Then, we have
\[
H^0(Q'(v, w)_{\mathbb{K}}, \mathcal{O}_{Q'(v, w)_{\mathbb{K}}}(\lambda))^\vee = \mathcal{W}_{\text{rat}}(\lambda) \cap \theta(\mathcal{W}_v(-w_0 \lambda)).
\]

**Proof.** Combine Theorem 3.33 for \( Q'_J(v, w)_{\mathbb{K}} \) and Corollary 3.35.
Lemma 3.38. Let \( w, v \in W_{af} \) and \( J \subset \mathcal{I} \). For each \( i \in \mathcal{I}_{af} \) such that \( s_i w > v \) and \( s_i v > v \), the variety \( \Omega^1_j(v, w)_K \) is \( B_i \)-stable. In addition, we have a morphism
\[
\pi_i : SL(2, i) \times B_i \Omega^1_j(v, w)_K \to \Omega^1_j(v, s_i w)_K.
\]
The map \( \pi_i \) is a \( \mathbb{P}^1 \)-fibration if \( \Omega^1_j(v, w)_K \) is \( SL(2, i) \)-stable. In general, the fiber of \( \pi_i \) is either a point or \( \mathbb{P}^1 \) if it is non-empty.

Proof. For each \( \lambda \in P_{2,++} \), the module \( \mathbb{W}_w(-w_0 \lambda)_\mathbb{Z} \) is \( \mathcal{I}(i) \)-stable and the module \( \mathbb{W}_{ww_0}(\lambda)_\mathbb{Z} \) is \( \mathcal{I} \)-stable. It follows that the ring \( R^w_0(J) \) is \( B_i \)-stable. Hence the scheme \( \Omega^1_j(v, w)_K \) is \( B_i \)-stable.

The map \( \pi_i \) is a \( \mathbb{P}^1 \)-fibration if \( \Omega^1_j(v, w)_K \) is \( SL(2, i) \)-stable since we have \( \Omega^1_j(\beta, w)_K = \Omega^1_j(\beta, s_i w)_K \) (that in turn follows from \( \mathbb{W}_{s_i w_0}(\lambda) = \mathbb{W}_{ww_0}(\lambda) \) for \( \lambda \in P_{2,++} \)) in this case. The fiber of the map \( \pi_i \) is either a point or \( \mathbb{P}^1 \) if \( \Omega^1_j(v, w)_K \) is not \( SL(2, i) \)-stable as the corresponding statement holds true for \( \Omega^1_j(w)_K \subset \mathbb{Q}_{B_i}(s_i w)_K \) by a set-theoretic consideration (and it carries over to any \( B_i \)-stable locus). Therefore, we conclude the result.

Proposition 3.39. Let \( w, v \in W_{af} \) and \( J \subset \mathcal{I} \). For each \( i \in \mathcal{I}_{af} \) such that \( s_i w > v \) and \( s_i v > v \), we have a surjective map
\[
\pi_i : SL(2, i) \times B_i \Omega^1_j(v, w)_K \to \Omega^1_j(v, s_i w)_K
\]
such that \( (\pi_i)_* \mathcal{O}_{SL(2, i) \times B_i \Omega^1_j(v, w)_K} \cong \mathcal{O}_{\Omega^1_j(v, s_i w)_K} \) and
\[
R^3_{\pi_i}(\mathcal{O}_{SL(2, i) \times B_i \Omega^1_j(v, w)_K}) \cong \{0\}.
\]

Proof. For each \( \lambda \in P_{2,++} \), the \( \lambda \)-graded component
\[
R^w_0(J, \lambda)^{\vee} = \mathbb{W}_{ww_0}(\lambda)_\mathbb{Z} \cap \theta^*(\mathbb{W}_w(-w_0 \lambda)_\mathbb{Z}) \subset \mathcal{X}(\lambda)_\mathbb{Z}
\]
of the coordinate ring of \( \Omega^1_j(v, w)_K \) is obtained as the \( \mathbb{Z} \)-span of the positive global basis.

We set \( \Omega^1_j(v, w)_K := SL(2, i) \times B_i \Omega^1_j(v, w)_K \) for simplicity in the below.

In view of [HR Theorem 4.2.1], the basis element of \( \theta^*(\mathbb{W}_w(-w_0 \lambda)_\mathbb{Z}) \) corresponds to the disjoint union of \( \mathfrak{sl}(2, i) \)-crystals. By [HR, §2.8] and [HT Theorem 4.2.1], the character of \( \mathbb{W}_{ww_0}(\lambda)_\mathbb{Z} \) and \( \mathbb{W}_w(-w_0 \lambda)_\mathbb{Z} \) obeys the Demazure character formula (cf. [HT, Theorem 4.7]). Since the Demazure operator (corresponding to \( i \in \mathcal{I}_{af} \)) at the level of crystals preserves the \( \mathfrak{sl}(2, i) \)-crystal, so is the crystal elements corresponding to \( R^w_0(J, \lambda)^{\vee} \).

Therefore, we deduce
\[
\text{ch} H^0(\mathcal{O}_{\Omega^1_j(v, s_i w)_K}(\lambda)) = \text{ch} H^0(\mathcal{O}_{\Omega^1_j(v, w)_K}(\lambda)) - \text{ch} H^1(\mathcal{O}_{\Omega^1_j(v, w)_K}(\lambda)) \quad (3.11)
\]
for each \( \lambda \in P_{2,++} \) (since \( H^{>2} = \{0\} \) on the RHS by the dimensional reason).

Since \( \mathbb{W}_{s_i w_0}(\lambda) \) and \( \mathbb{W}_w(\lambda) \) are \( SL(2, i) \)-stable, so is \( R^w_{s_i}(\lambda)_K \). In particular, \( \Omega^1_j(\beta, s_i w)_K \) admits a \( SL(2, i) \)-action.

Hence, the inclusion \( R^w_{s_i}(J, \lambda)^{\vee} \subset R^w_{s_i}(J, \lambda)_K \) naturally induces an inclusion
\[
H^0(\Omega^1_j(v, s_i w)_K, \mathcal{O}_{\Omega^1_j(\beta, s_i w)_K}(\lambda)) \subset H^0(\Omega^1_j(v, w)_K, \mathcal{O}_{\Omega^1_j(v, w)_K}(\lambda)). \quad (3.12)
\]
This realizes the map \( \pi_i : \Omega^1_j(v, w)_K \to \Omega^1_j(v, s_i w)_K \) borrowed from Lemma C.38. In addition, it yields a surjection between generic points. Since \( \pi_i \) is a
map between projective varieties of finite type over the same defining field, we conclude that \( \pi_i \) is surjective. Since each fiber of \( \pi_i \) is connected (pt or \( \mathbb{P}^1 \)), the Stein factorization theorem [78, Theorem 03H0] and the weak normality [78, Definition 1.2.3] (of \( \mathcal{O}'_J(v,s,w)_K \) by Lemma 78.3) implies that \( (\pi_i)_*\mathcal{O}_{\mathcal{O}'_J(v,w)_K} \cong \mathcal{O}_{\mathcal{O}'_J(v,s,w)_K} \). Therefore, (6.14) is in fact an equality. In view of (6.11), we deduce \( \text{ch} H^1(\mathcal{O}_{\mathcal{O}'_J(v,w)_K}(\lambda)) = \{0\} \). Thus, Lemma 6.24 implies that

\[
(\pi_i)_*\mathcal{O}_{\mathcal{O}'_J(v,w)_K} \cong \mathcal{O}_{\mathcal{O}'_J(v,s,w)_K} \quad \text{and} \quad \mathbb{R}^1(\pi_i)_*\mathcal{O}_{\mathcal{O}'_J(v,w)_K} \cong \{0\}.
\]

Since we have \( \mathbb{R}^{\geq 2}(\pi_i)_*\mathcal{O}_{\mathcal{O}'_J(v,w)_K} \cong \{0\} \) by the dimension reason, we conclude the result.

A particular case of Proposition 6.3.2 is worth noting:

**Corollary 3.40.** Let \( \beta \in \mathcal{Q}'_r, w \in W, \) and \( J \subset I \). For each \( i \in I \) such that \( s_iw < w \), we have a surjective map

\[
\pi_i : P_i \times B \mathcal{Q}'_J(\beta, w)_K \to \mathcal{O}'_J(\beta, s_iw)_K
\]

such that \( \mathbb{R}^* (\pi_i)_*\mathcal{O}_{P_i \times B \mathcal{Q}'_J(\beta, w)_K} \cong \mathcal{O}_{\mathcal{O}'_J(\beta, s_iw)_K} \). \( \square \)

### 3.5 Lifting to/from characteristic zero

**Theorem 3.41.** Let \( \mathcal{X} \) be a Noetherian scheme flat over \( \mathbb{Z} \). If \( \mathcal{X}_{\bar{F}_p} \) is weakly normal for \( p \gg 0 \), then \( \mathcal{X}_{\mathbb{C}} \) is also weakly normal.

**Proof.** Since the weak normalization commutes with localization [77, Theorem IV.3], we can argue locally. Let \((S, \mathfrak{m})\) be a local ring of \( \mathcal{X}_{\mathbb{C}} \) and let \( S^- \) be the weak normalization of \( S \) ([78, Remark 1]). By the Noetherian hypothesis, we can invert finitely many primes and take a finite algebraic extension of \( \mathbb{Z} \) to obtain a ring \( A \) such that we have a commutative ring \( S_A \) over \( A \) and its ideal \( \mathfrak{m}_A \) with the following properties:

- We have \((S_A \otimes_A \overline{\mathbb{Q}}, \mathfrak{m}_A \otimes_A \overline{\mathbb{Q}}) \cong (S, \mathfrak{m})\);
- The \( A \)-modules \( S_A, \mathfrak{m}_A \), and \( S_A/\mathfrak{m}_A \) are torsion-free;
- The specialization of \( A \) to the algebraic closure of a finite field yields a weakly normal (local) ring ([77, Theorem V.2]).

As \( A \) is a Dedekind domain, we find that \( S_A, \mathfrak{m}_A, \) and \( S_A/\mathfrak{m}_A \) are flat over \( A \).

We have \( S^- = S[f_1, \ldots, f_n] \), where \( f_1, \ldots, f_n \) are integral elements. By multiplying with elements in \( \overline{\mathbb{Q}} \), we can assume that \( f_1, \ldots, f_n \) are integral over \( S_A \). By inverting additional primes in \( \mathbb{Z} \) if necessary (to assume that the denominator of \( f_i \) in \( \text{Frac}(S) \) does not vanish along specializations and achieve the conditions in the followings), we can further assume that \( S_A^- := S_A[f_1, \ldots, f_n] \) is flat over \( A \) and it is integral for every specialization of \( A \) to a field.

The ring \((S_A^-/\mathfrak{m}_AS_A^-) \otimes_A \mathbb{C}\) is a finite-dimensional local commutative \( \mathbb{C} \)-algebra by the weak normality assumption on \( S^- \) ([78, Remark 1]). In particular, the multiplication action of each element of \((S_A^-/\mathfrak{m}_AS_A^-) \otimes_A \mathbb{C}\) have a unique eigenvalue. Hence, if we present \((S_A^-/\mathfrak{m}_AS_A^-) = A[X_1, \ldots, X_m]/\sim\) (where \( \sim \) contains the minimal polynomials of the \( A \)-valued matrix \( X_i \)), then
the minimal polynomial of $X_i$ is of the form $(T - a_i)^{m_i}$ ($a_i \in A$). Therefore, we can assume that each $X_i$ is nilpotent by changing $X_i$ with $X_i + a_i$ if necessary. Hence, we conclude that $(X_1, \ldots, X_m) \subset (S_A/m_A S_A)$ have eigenvalues zero after specializing to $\mathbb{F}_p$.

By assumption, $(S_A) \otimes_A \mathbb{F}_p$ is weakly normal for every possible prime $p$ and every ring homomorphism $A \to \mathbb{F}_p$. Hence, the specialization $(S_A/m_A S_A) \otimes_A \mathbb{F}_p$ must contain $\mathbb{F}_p$ as its ring direct summand (if it is non-zero). This forces $\text{rank}(S_A/m_A S_A) = 1$ since we cannot have two linearly independent idempotents that have distinct eigenspaces by the previous paragraph.

Therefore, we deduce that $S_A = S_A$, that implies $(S,m)$ is itself weakly normal. In view of [37, Theorem V.2 and Corollary V.3], we conclude the assertion.

\textbf{Corollary 3.42.} For each $J \subset I$, $v, w \in W_{\mathfrak{at}}$, the scheme $Q'_J(v, w)_{\mathbb{C}}$ is weakly normal.

\textbf{Proof.} Apply Theorem 3.31 to Lemma 3.33. \hfill \square

\textbf{Proposition 3.43.} For each $J \subset I$, $v, w \in W$, the variety $Q'_J(v, w)_{\mathbb{Z}}$ is normal for $p \gg 0$ provided if $Q'_J(v, w)_{\mathbb{C}}$ is normal. The same is true for the irreducibility.

\textbf{Proof.} Since $Q'_J(v, w)$ is defined over $\mathbb{Z}$, the scheme $Q'_J(v, w)_{\mathbb{C}}$ is a scalar extension of $Q'_J(v, w)_{\mathbb{Z}}$. By [37, Lemma 038P], we deduce $Q'_J(v, w)_{\mathbb{Z}}$ is normal. We apply [37, Lemma 038E] to derive the irreducibility of $Q'_J(v, w)_{\mathbb{Z}}$. Now apply [37, Proposition 9.9.4 and Théorème 9.7.7] to $Q'_J(v, w)_{\mathbb{Z}} \to \text{Spec } \mathbb{Z}$. \hfill \square

\section{Normality of quasi-map spaces}

In this section, we continue to work under the setting of the previous section with an exception that $K = \mathbb{C}$. Also, a point of a scheme (over $\mathbb{C}$) means a closed point unless stated otherwise.

\subsection{Graph space resolution of $Q(\beta)$}

We refer to [37, 32, 4, 33, 39] for precise explanations of the material in this subsection. For each non-negative integer $n$ and $\beta \in Q^+_0$, we set $\mathcal{B}_{n, \beta}$ to be the space of stable maps of genus zero curves with $n$-parked points to $(\mathbb{P}^1 \times B)$ of bidegree $(1, \beta)$, that is also called the graph space of $B$. A point of $\mathcal{B}_{n, \beta}$ represents a genus zero curve $C$ with $n$-marked points, together with a map to $\mathbb{P}^1$ of degree one (if we forget a map to $B$). Hence, we have a unique $\mathbb{P}^1$-component of $C$ that maps isomorphically onto $\mathbb{P}^1$. We call this component the main component of $C$ and denote it by $C_0$. The space $\mathcal{B}_{n, \beta}$ is a normal projective variety by [32, Theorem 2] that have at worst quotient singularities arising from the automorphism of curves (and hence it is smooth as an orbifold).

The natural $(H \times G_m)$-action on $(\mathbb{P}^1 \times B)$ induces a natural $\mathfrak{g}$-action on $\mathcal{B}_{n, \beta}$.

We have a morphism $\pi_{n, \beta} : \mathcal{B}_{n, \beta} \to Q(\beta)$ that factors through $\mathcal{B}_{0, \beta}$ (Givental’s main lemma [39]; see [24, §8.3]). Let $e_j : B_{n, \beta} \to B$ ($1 \leq j \leq n$) be the evaluation at the $j$-th marked point, and let $ev_j : \mathcal{B}_{n, \beta} \to B$ be the $j$-th evaluation map to $\mathbb{P}^1 \times B$ composed with the second projection.
Since $\Omega(\beta)$ is irreducible (Theorem 1.4), [25, §8.3] asserts that $\mathcal{GB}_n,\beta$ is irreducible (as a special feature of flag varieties, see [32, §1.2] and [11]).

4.2 The variety $\mathcal{Q}(\beta, v, w)$

Let $\mathcal{GB}^b_{2,\beta}$ denote the subvariety of $\mathcal{GB}_{2,\beta}$ so that the first marked point projects to $0 \in \mathbb{P}^1$, and the second marked point projects to $\infty \in \mathbb{P}^1$ through the projection of quasi-stable curves $C$ to the main component $C_0 \cong \mathbb{P}^1$. Let us denote the restrictions of $\text{ev}_i$ $(i = 1, 2)$ and $\pi_{2,\beta}$ to $\mathcal{GB}^b_{2,\beta}$ by the same letter. By [11, 12], $\mathcal{GB}^b_{2,\beta}$ gives a resolution of singularities of $\mathcal{Q}(\beta)$ (in an orbifold sense).

Recall that each Schubert cell $\mathcal{O}_B(w)$ contains a unique $H$-fixed point $p_w$. For each $w \in W$, we set

$$\mathcal{O}^o_B(w) := N^-p_w \subset B,$$ and $\mathcal{B}^{op}(w) := \mathcal{O}^o_B(w) = N^-p_w \subset B$.

For $w, v \in W$, we define

$$\mathcal{GB}^b_{2,\beta}(w, v) := \text{ev}^{-1}_1(\mathcal{B}(w)) \cap \text{ev}^{-1}_2(\mathcal{B}^{op}(v)) \subset \mathcal{GB}^b_{2,\beta}$$

and

$$\mathcal{B}_{2,\beta}(w, v) := \text{e}^{-1}_1(\mathcal{B}(w)) \cap \text{e}^{-1}_2(\mathcal{B}^{op}(v)) \subset \mathcal{B}_{2,\beta}.$$

**Theorem 4.1** (Buch-Chaput-Mihalcea-Perrin [17]). The variety $\mathcal{GB}^b_{2,\beta}(w, v)$ is either empty or unirational (and hence connected and irreducible) variety that has rational singularities. The same is true for $\mathcal{B}_{2,\beta}(w, v)$. In particular, they are normal.

**Proof.** Since the both cases are parallel, we concentrate into the case $\mathcal{GB}^b_{2,\beta}(w, v)$.

As $\mathbb{P}^1 \times B$ is a homogeneous variety, Kim-Phandaripande [61, Theorem 2 and Theorem 3] applies and hence $\mathcal{GB}_{2,\beta}$ is a rational variety. Then, a pair of Schubert subvarieties (with respect to the a pair of opposite Borel subgroups of $SL(2) \times G$) of $\mathbb{P}^1 \times \mathcal{B}$ presented as $\{0\} \times \mathcal{B}(w)$ and $\{\infty\} \times \mathcal{B}^{op}(v)$ is used to define $\mathcal{GB}^b_{2,\beta}(w, v)$. Hence, [17, Proposition 3.2 c)] implies that $\mathcal{GB}^b_{2,\beta}(w, v)$ is either empty or unirational (and hence connected). Since the pair of Schubert varieties with respect to the opposite Borel subgroups forms the dense subset of the pair of translations of Schubert varieties by applying the $G$-action, it must contain a pair of Schubert varieties in general position. Therefore, [17, Corollary 3.1] implies that $\mathcal{GB}^b_{2,\beta}(w, v)$ has rational singularity. The last assertion is a well-known property of rational singularities [52, Definition 5.8].

**Proposition 4.2.** For each $v, w \in W$ and $\beta \in Q^+_\vee$, the variety $\mathcal{Q}(vt, \beta, w)_C$ is irreducible.

**Proof.** In view of the proof of Theorem 4.1, it suffices to prove that the intersection $(\mathcal{Q}(vt, \beta, w)_C \cap \mathcal{O}(u)_C)$ is irreducible with dimension $\ell(\mathcal{Q}(vt, \beta)) - \ell(\mathcal{Q}(u))$ for each $u \in W_d$ such that $vt, \beta \leq \infty u$ (see Lemma 4.1). By swapping the roles of $z$ with $z^{-1}$, we further deduce that it suffices to prove that

$$\mathcal{Q}(vt, \beta, w)_C := \{ f \in \mathcal{Q}(\beta)_C | f(0) \in \mathcal{O}_B(w), f(\infty) \in \mathcal{O}^{op}_B(v) \}$$

is irreducible and has dimension

$$\ell(\mathcal{Q}(vt, \beta)) - \ell(\mathcal{Q}(w)) = \ell(v) - \ell(w) + 2(\beta, p).$$
Since the evaluation maps $\text{ev}_i$ ($i = 1, 2$) descend to $\hat{\Omega}(vt_\beta, w)_\mathbb{C}$ from $\mathfrak{g}\mathfrak{B}_{2, \beta}^\mathbb{B}$, Proposition 3.2 forces $\hat{\Omega}(vt_\beta, w)_\mathbb{C}$ to be irreducible (if it is nonempty).

The case $v = w_0$ follows from Theorem 4.4A. We can use the automorphism $z \rightarrow z^{-1}$ and the Chevalley involution of $G$ to swap the roles of $vt_\beta$ and $w$ in $\hat{\Omega}(vt_\beta, w)_\mathbb{C}$ (this corresponds to multiplying $w_0$ from the right in view of Lemma 4.4B). Taking intersection with boundary divisors (that is codimension one also in $Q_G(e)$) reduces the dimension of $\hat{\Omega}(vt_\beta, w)_\mathbb{C}$ at most one. Since the case $\beta = 0$ is known as being Richardson varieties of $\mathbb{B}$ (131), the induction with respect to $\leq \bar{\rho}$ forces the exact dimension estimate since $\hat{\Omega}(vt_\beta, w)_\mathbb{C}$ cannot be proper unless it is 0-dimensional by Theorem 4.1 (cf. Corollary 4.4A). □

In view of Proposition 4.4. we set $\Omega_J(\beta, v, w) := \Omega_J^*(vt_\beta, w)_\mathbb{C}$ for each $J \subset \mathcal{I}$, $\beta \in Q^*_+ \cap v$, and $v, w \in W$ in the below (see also Corollary 4.4A). We have $\Omega(\beta, w_0, w) = \Omega(\beta, w)$ by Theorem 4.4A.

**Corollary 4.3** (of proof of Proposition 4.4A). For each $\beta \in Q^*_+$ and $v, w \in W$, the scheme $\Omega(\beta, v, w)$ is nonempty if and only if $vt_\beta \leq w$. If it is nonempty, then it has dimension $\ell(v) - \ell(w) + 2(\beta, \rho)$. Moreover, the space $\Omega(\beta, v, w)$ (viewed as a subspace of $\Omega(\beta)$) contains a quasi-map that has no defects at 0 and $\infty$.

**Remark 4.4.** Corollary 4.3 removes the condition $\gamma \gg 0$ in [11, Lemma 8.5.2].

Thanks to Proposition 4.4A and Lemma 4.4B, we deduce that the map $\pi_{2, \beta}$ restricts to a $(H \times \mathbb{G}_m)$-equivariant birational proper map

$$
\pi_{2, \beta, w, v} : \mathfrak{g}\mathfrak{B}_{2, \beta}^\mathbb{B}(w, v) \rightarrow \Omega(\beta, v, w)
$$

by inspection.

### 4.3 From Givental’s main lemma

For each $w, v \in W$, we define subvarieties of $\mathfrak{B}_{2, \beta}$ as:

$$
\mathfrak{B}_{2, \beta}[w] := \mathfrak{e}_1^{-1}(p_w) \quad \text{and} \quad \mathfrak{B}_{2, \beta}[w, v] := \mathfrak{e}_1^{-1}(p_w) \cap \mathfrak{e}_2^{-1}(p_v).
$$

Similarly, we set

$$
\mathfrak{g}\mathfrak{B}_{2, \beta}[w, v] := \mathfrak{e}_1^{-1}(p_w) \cap \mathfrak{e}_2^{-1}(p_v) \subset \mathfrak{g}\mathfrak{B}_{2, \beta}^\mathbb{B} \quad \text{and} \quad \mathfrak{B}_{1, \beta}[w] := \mathfrak{e}_1^{-1}(p_w) \subset \mathfrak{B}_{1, \beta}[w].
$$

**Lemma 4.5.** For each $x, y \in \mathbb{B}$ and $\beta \in Q^*_+$, there exists $w \in W$ such that

$$
\mathfrak{e}_1^{-1}(x) \cap \mathfrak{e}_2^{-1}(y) \cong \mathfrak{B}_{2, \beta}[w, w_0].
$$

The same is true for $\mathfrak{ev}$ and $\mathfrak{g}\mathfrak{B}$.

**Proof.** We consider only the case of $\mathfrak{e}$ and $\mathfrak{B}$ as the other case is completely parallel. Since $(x, y) \in \mathfrak{B} \times \mathfrak{B}$ and the $G$-action on $\mathfrak{B}$ is transitive, we can assume $y = p_{w_0}$. Since we have $\text{Stab}_G y = B$, we can further rearrange $x = p_w$ for some $w \in W$ by (4.3). □
Theorem 4.6 (Givental’s main lemma [33], see [23] §8). Let \((f, D) \in \Omega(\beta)\) be a quasi-map with its defect \(D = \sum_{x \in \mathbb{P}^1(\mathbb{C})} \beta_x \otimes [x]\). Then, we have
\[
\pi_{2,\beta}^{-1}(f, D) \cong B_{2,\beta_0}[w] \times B_{2,\beta'}[w] \times \prod_{x \in \mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\}} B_{1,\beta_x}[w] \subset \mathcal{GB}^2_{2,\beta}.
\]
In particular, the map \(\pi_{2,\beta}\) is birational along the loci with \(D = 0\).

Remark 4.7. In Theorem 4.6, the first marked points of (the stable maps in) \(\mathcal{GB}^2_{2,\beta}\) (the marked points at \(0 \in \mathbb{P}^1\); i.e. the image of \(\text{ev}^1\)) is identified with the second marked points of (the stable maps in) \(B_{2,\beta_0}[w]\) (i.e. the image of \(\mathbf{e}_2\)), and the second marked points of \(\mathcal{GB}^2_{2,\beta}\) (the marked points at \(\infty \in \mathbb{P}^1\)) is identified with the second marked points of \(B_{2,\beta_\infty}[w]\). (Other marked points are used to glue pieces of stable maps together.)

Lemma 4.8 ([33] Lemma 8.5.1). For each \(\beta \in Q^+_+\) such that \((\beta, \alpha_i) \geq 1\) for each \(i \in \mathbb{I}\), the evaluation map
\[
\text{ev} := (\text{ev}_1 \times \text{ev}_2) : \mathcal{GB}^2_{2,\beta} \to \mathcal{B} \times \mathcal{B}
\]
is surjective.

Proof. Taking into account the fact that \(\mathcal{GB}^2_{2,\beta}\) is projective, it suffices to prove that the tangent map associated to \(\text{ev}\) is surjective on a dense open subset of \(\mathcal{GB}^2_{2,\beta}\).

Since the map \(\pi_{2,\beta}\) is birational by Theorem 4.6, we replace the problem with the case of a genuine map \(f : \mathbb{P}^1 \to \mathcal{B}\). Thanks to [33], Proposition 3.5, \(\Omega(\beta)\) and hence \(\mathcal{GB}^2_{2,\beta}\) are smooth at \(f\). Moreover, its tangent space is described as
\[
H^0(\mathbb{P}^1, f^*T\mathcal{B}), \quad H^1(\mathbb{P}^1, f^*T\mathcal{B}) = \{0\}
\]
and the filtration of \(T\mathcal{B}\) as \(G\)-equivariant line bundles yields the following associated graded
\[
\bigoplus_{\alpha \in \Delta_+} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1)((\beta, \alpha))}.
\]
(Here we used \((\beta, \alpha_i) \geq -1\) for each \(i \in \mathbb{I}\).) In particular, we have
\[
\dim H^0(\mathbb{P}^1, f^*T\mathcal{B}) = \Sigma_{\alpha \in \Delta_+} \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((\beta, \alpha))).
\]
The effect of fixing the image of two points \(0, \infty \in \mathbb{P}^1\) corresponds to imposing divisor twist by \(\mathcal{O}_{\mathbb{P}^1}([-0] - [\infty])\). We have
\[
\dim H^0(\mathbb{P}^1, f^*T\mathcal{B} \otimes \mathcal{O}_{\mathbb{P}^1}([-0] - [\infty])) \leq \dim \bigoplus_{\alpha \in \Delta_+} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((\beta, \alpha) - 2))
\]
\[
= \dim \bigoplus_{\alpha \in \Delta_+} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((\beta, \alpha)) - 2|\Delta_+|)
\]
\[
= \dim \Omega(\beta) - 2 \dim \mathcal{B}
\]
\[
= \dim \mathcal{GB}^2_{2,\beta} - 2 \dim \mathcal{B}.
\]
Here the first inequality comes from the short exact sequences, the second equality is \((\beta, \alpha_i) \geq 1\) for each \(i \in \mathbb{I}\), the third one is the the smoothness of \(\Omega(\beta)\) at \(f\) and \(|\Delta_+| = \dim \mathcal{B}\), and the fourth one is the birationality of \(\pi_{2,\beta}\) (restricted to \(\mathcal{GB}^2_{2,\beta} \subset \mathcal{GB}^2_{2,\beta}\)).

From this, we deduce that \(d\pi_{2,\beta}\) is generically surjective as required. \(\square\)
Proposition 4.9. For each $\beta \in Q^+_\Sigma$ such that $\langle \beta, \alpha_i \rangle \geq 1$ for each $i \in I$ and each $w, v \in W$, the scheme $\mathfrak{H}_{2,\beta}[w, v]$ is connected and nonempty.

Proof. The map $ev$ borrowed from Lemma 4.3 is $G$-equivariant, and $(B \times B)$ admits an open dense $G$-orbit $O$ by the Bruhat decomposition. Since $\mathfrak{H}_{2,\beta}$ is an irreducible variety, so is its Zariski open set $ev^{-1}(O)$. If we consider $x \in O$, then the irreducible components of $ev^{-1}(x)$ (that must be finite as we consider varieties, that is finite type) must be permuted by $\text{Stab}_G x \cong H$. Since $H$ is connected, we cannot have a non-trivial action. Therefore, the irreducible component of $ev^{-1}(x)$ must be unique. Thanks to the Stein factorization theorem [10, Theorem 05H1], the map $ev$ factors through the normalization $Y$ of $(B \times B)$ inside (the function field of) $\mathfrak{H}_{2,\beta}$ and the map $\mathfrak{H}_{2,\beta} \to Y$ has connected fiber. Since the general fiber of $ev$ is connected, so is the map $Y \to (B \times B)$. It follows that $Y \to (B \times B)$ is a birational map (that is also finite by the Stein factorization theorem). This implies that $Y$ and $(B \times B)$ share the same function field. Thus, we conclude that $Y \cong (B \times B)$ by the normality of $(B \times B)$. This particularly means that every fiber of $ev$ is connected. Therefore, the assertion follows by choosing $(H \times H)$-fixed points in $(B \times B)$ as particular cases. \qed

Proposition 4.10. For each $\beta \in Q^+_\Sigma$ such that $\langle \beta, \alpha_i \rangle \geq 1$ for each $i \in I$ and $w, v \in W$, the scheme $B_{2,\beta}[w, v]$ is connected and nonempty. For each $\beta \in Q^+_\Sigma$ and $w, v \in W$, the schemes $B_{1,\beta}[w]$ and $B_{2,\beta}[w]$ are connected and nonempty.

Proof. We prove the first assertion. We have a rational map $\mathfrak{H}_{2,\beta}[w, v] \to B_{2,\beta}[w, v]$ obtained by forgetting the map to $\mathbb{P}^1$. Moreover, the loci this map is not defined is that the main component is degree 0 and has only two marked points. By modifying the universal family by adjoining such two marked points in such a degree 0 component, we conclude that $\mathfrak{H}_{2,\beta}[w, v] \to B_{2,\beta}[w, v]$ extends to a map of topological spaces. By examining the condition to be a stable map [62, §1.1], we deduce that this map is surjective onto the image. Therefore, the connectedness and the nonemptiness of $\mathfrak{H}_{2,\beta}[w, v]$ implies these of $B_{2,\beta}[w, v]$, that is the first assertion.

The second assertion is straight-forward from the irreducibility of $B_{n,\beta}$, together with the fact that $e_1$ is $G$-equivariant fiber bundle over $B$ (see the proof of Proposition 4.9). \qed

Lemma 4.11. Let $(f, D) \in Q(\beta)$ be a quasi-map with its defect $D$ with the following properties:

- $D = \sum_{x \in F_1(C)} \beta_x \otimes [x]$;
- $\langle \beta_0, \alpha_i \rangle \geq 1$ and $\langle \beta_\infty, \alpha_i \rangle \geq 1$ for each $i \in I$.

Then, $\pi^{-1}_{3, w, v}(f, D)$ is connected for every $w, v \in W$.

Proof. By Proposition 4.10 and Theorem 4.18, we can forget about the contribution of $B_{1,\beta}[w]$ when $x \neq 0, \infty$. By our assumption and Proposition 4.11, we know that $B_{2,\beta_0}[u, w'] \neq \emptyset$ and $B_{2,\beta_\infty}[u, v'] \neq \emptyset$

and are nonempty and connected for each $u, w', v' \in W$. Since $B_{2,\beta_0}$ and $B_{2,\beta_\infty}$ are proper, we always find a limit point with respect to the $H$-action. It follows
that $\mathcal{B}_{2,\beta_0}[u] \cap \mathcal{e}_2^{-1}(\mathcal{B}(u'))$ (resp. $\mathcal{B}_{2,\beta_0}[u] \cap \mathcal{e}_2^{-1}(\mathcal{B}^{op}(u'))$) is connected for each $u, u' \in W$ as we can connect every two points by appropriately sending to/from $H$-limit points that are contained in a connected component of the form $\mathcal{B}_{2,\beta_0}[u, w']$ $(w' \in W)$. Thanks to Theorem 4.10, we conclude the assertion. 

\section{Normality of $\Omega(\beta, v, w)$}

Let $\Omega^+(\beta, v, w)$ be the normalization of $\Omega(\beta, v, w)$ for each $\beta \in Q^+_2$ and $v, w \in W$. We denote the normalization map by $\eta_{\beta, v, w} : \Omega^+(\beta, v, w) \rightarrow \Omega(\beta, v, w)$.

\begin{proposition}
For each $\beta \in Q^+_2$ and $v, w \in W$, the variety $\Omega(\beta, v, w)$ is normal if and only if every fiber of $\pi_{\beta, v, w}$ is connected.
\end{proposition}

\begin{proof}
As $\mathcal{B}_{2,\beta}(v, w)$ is normal and $\pi_{\beta, v, w}$ is proper, we know that

$$
(\pi_{\beta, v, w})^{*}\mathcal{O}_{\mathcal{B}_{2,\beta}(v, w)} \cong \mathcal{O}_{\Omega^+(\beta, v, w)}.
$$

The properness of $\pi_{\beta, v, w}$ also implies that $\mathcal{O}_{\Omega^+(\beta, v, w)}$ is a coherent sheaf on $\Omega(\beta, v, w)$. For each closed point $x$ of $\Omega(\beta, v, w)$, we set

$$
\Theta(x) := \dim_{\mathbb{C}} \mathcal{O}_{\Omega^+(\beta, v, w)} \otimes_{\mathcal{O}_{\Omega(\beta, v, w)}} C_x. \tag{4.1}
$$

By the Stein factorization theorem, the map $\eta_{\beta, v, w}$ is finite. By Corollary 4.2 and 4.3, we know that the variety $\Omega(\beta, v, w)$ is weakly normal. From this, we deduce that $\Theta(x) = \# \eta_{\beta, v, w}^{-1}(x)$ (cf. [28, Remark 1]). Moreover, it counts the number of irreducible components of the fiber of $\eta_{\beta, v, w}$.

The coherence of $\mathcal{O}_{\Omega^+(\beta, v, w)}$ implies that the RHS of (4.1) is an upper-semicontinuous function on $\Omega(\beta, v, w)$, and hence so is $\Theta$.

If we have $\Theta \equiv 1$ on $\Omega(\beta, v, w)$, then we have $\Omega^+(\beta, v, w) = \Omega(\beta, v, w)$ by the weak normality of the latter (cf. [28, Remark 1]). Therefore, the if part of the assertion follows.

If we have $\Theta \not\equiv 1$ on $\Omega(\beta, v, w)$, then we have $\Omega^+(\beta, v, w) \neq \Omega(\beta, v, w)$. Hence, the only if part of the assertion follows.

These complete the proof of Proposition 4.12. \hfill \square

\begin{corollary}
(Braverman-Finkelberg). For each $\beta \in Q^+_2$, the variety $\Omega(\beta)$ is normal.
\end{corollary}

\begin{remark}
Our proof of Corollary 4.12 is independent of [11] (however based on common former papers [11, 28]). Hence, we obtain a new proof of the normality of $\Omega(\beta)$ and $Z(\beta, w_0)$. Together with Theorem 4.6, Corollary 4.12 also makes the contents in [22] logically independent of [11] (cf. Appendix A).
\end{remark}

\begin{proof}[Proof of Corollary 4.12]
Recall that $\Omega(\beta) = \Omega(\beta, w_0, e)$. We borrow the upper semi-continuous function $\Theta$ that counts the number of connected components of the fiber of $\eta_{\beta, w_0, e}$ from [11] in the proof of Proposition 4.12.

By Proposition 4.12, it suffices to prove that $\Theta \equiv 1$ on $\Omega(\beta)$. In other words, it suffices to prove that the fiber $\eta_{\beta, w_0, e}^{-1}(x)$ is connected for each $x \in \Omega(\beta)$.

By Theorem 4.12, we deduce that the set of connected components of $\eta_{\beta, w_0, e}^{-1}(x)$ is in bijection with the set of connected components of $\prod_{y \in Z(\beta, w_0)} \mathcal{B}_{2,\beta}[w]$. By Proposition 4.11, this latter space is connected.

Therefore, we conclude the result. \hfill \square
Theorem 4.15. For each $\beta \in Q^+_e$ and $w \in W$, the varieties $\mathcal{Q}(\beta, w_0, w_0)$ and $\mathcal{Q}(\beta, e, w_0)$ are normal.

Proof. We set $(v, w) = (w_0, w_0)$ or $(e, w_0)$. We borrow the upper semi-continuous function $\Theta$ that counts the number of connected components of the fiber of $\eta_{\beta,v,w}$ from (4.11) in the proof of Proposition 4.11.

By Proposition 4.14 it suffices to prove that $\Theta \equiv 1$ on $\mathcal{Q}(\beta, v, w)$ by assuming the contrary to deduce contradiction. For each $x \in \mathcal{Q}(\beta, v, w)$ such that $\Theta(x) \geq 2$, the fiber $\eta_{\beta,v,w}^{-1}(x)$ is disconnected.

By our choice of $(v, w)$ and Theorem 4.10 (cf. Proposition 4.12 and Corollary 4.13), we deduce that the set of connected components of $\eta_{\beta,v,w}^{-1}(x)$ is in bijection with the set of connected components of $\mathcal{B}_{2,\beta_0}[u, w_0]$ or $\mathcal{B}_{2,\beta_0}[u, w_0] \times \mathcal{B}_{2,\beta_0}[w', w_0]$ for some $u, u' \in W$.

To see whether this is the case, we specialize to the case of $(v, w) = (w_0, w_0)$ (to guarantee that the contribution at $\infty \in \mathbb{P}^1$ in Theorem 4.10 is the same as the points in $\mathbb{C}^* \subset \mathbb{P}^1$, that is connected by Proposition 4.14). We can choose $\beta_0 < \beta' \in Q^+_e$ such that $(\alpha'_i, \beta_0') \geq 1$ for every $i \in I$. By Proposition 4.14, we deduce that the fiber of the loci $\mathcal{Z}$ of $\mathcal{Q}(\beta, w_0)$ that have defect $\beta'_0$ along $0$ is connected for every $\beta \in Q^+$. Hence we have $\Theta(y) = 1$ for each $y \in \mathcal{Z}$.

For each $\tilde{x} \in \mathcal{B}_{2,\beta}$ such that $x = \pi_{2,\beta}(\tilde{x}) \in \mathcal{Q}(\beta, w_0)$ has defect $\beta_0$ at $0$, we can replace $\beta$ with $\beta + \beta'_0 - \beta_0$ and add additional irreducible components $C'$ to the main component $\mathbb{P}^1$ of $\tilde{x}$ (as quasi-stable curves) outside of $0 \in \mathbb{P}^1$ (whose images to $\mathcal{B}$ have their degrees sum up to $(\beta'_0 - \beta_0)$). (This does not change the defect of $x$ at $0$, and also does not change $\Theta(x)$.) Then, we shrink all the inserting points of $C'$ on $\mathbb{P}^1$ to $0$ to obtain $\tilde{x}' \in \mathcal{B}_{2,\beta}$ in the limit, that exists by the valuative criterion of properness as $\mathcal{B}_{2,\beta}$ is projective. By examining the images of this family on $\mathcal{Q}(\beta, w_0)$ via $\pi_{2,\beta}$, we deduce $y = \pi_{2,\beta}(\tilde{x}') \in \mathcal{Z}$. Therefore, the semi-continuity of $\Theta$ implies that $\Theta(x) \leq \Theta(y) = 1$, that is $\Theta(x) = 1$. Hence, $\mathcal{B}_{2,\beta_0}[w, w_0]$ must be connected. This is a contradiction, and we conclude that $\Theta \equiv 1$ (for general $(v, w) \in \{(e, w_0), (w_0, w_0)\}$ by the two paragraphs ahead).

Therefore, Proposition 4.14 implies the result.

Corollary 4.16. Let $\beta \in Q^+_e$ and $w \in W$. For each $i \in I$ such that $s_i w < w$, we have a surjective map

$$\pi_i : P_i \times^B \mathcal{Q}(\beta, w) \to \mathcal{Q}(\beta, s_i w)$$

such that $(\pi_i)_* \mathcal{O}_{P_i \times \mathcal{Q}(\beta, w)} \cong \mathcal{O}_{\mathcal{Q}(\beta, s_i w)}$ and $\mathbb{R}_{>0}(\pi_i)_* \mathcal{O}_{P_i \times \mathcal{Q}(\beta, w)} \cong \{0\}$.

Proof. Combine Corollary 4.10 with Theorem 4.11 (and take the generic localizations over $\mathbb{Z}$).

Corollary 4.17. For each $\beta \in Q^+_e$ and $w \in W$, the variety $\mathcal{Q}(\beta, w_0, w_0)$ is normal.

Proof. The case $w = w_0$ is in Theorem 4.15. Assume that the assertion holds for $w$. Let $i \in I$ such that $s_i w < w$. Then, Corollary 4.10 implies that $\mathcal{O}_{\mathcal{Q}(\beta, s_i w)}$ is isomorphic to the normal sheaf of rings $(\pi_i)_* \mathcal{O}_{P_i \times \mathcal{Q}(\beta, w)}$. Hence, the assertion holds for $s_i w < w$. This proceeds the induction and we conclude the result.
Corollary 4.18. For each $\beta \in Q_+^\vee$ and $w, v \in W$, the subspace

$$e_1^{-1}(p_v) \cap e_2^{-1}(B(w)) \subset B_{2, \beta}$$

is connected.

Proof. This space appears in the fiber of $\pi_{\beta, w_0, w}$ along the constant quasimap $\mathbb{P}^1 \to \{p_v\} \subset B$ with its defect concentrated in $0 \in \mathbb{P}^1$. Hence, the assertion follows from Corollary 4.17 and Proposition 3.4.12. \qed

Theorem 4.19. For each $\beta \in Q_+^\vee$ and $w, v \in W$, the scheme $Q(\beta, v, w)$ is connected.

Proof. The combination of Theorem 4.6 and Corollary 4.18 implies that every fiber of $\pi_{\beta, v, w}$ is connected. Thus, Proposition 4.12 implies the result. \qed

Corollary 4.20. For each $J \subset I$, $\beta \in Q_+^\vee$, $w, v \in W$, the variety $Q_J(\beta, v, w)$ is irreducible and normal.

Proof. By Corollary 5.13 and Remark 5.14, we can rearrange the map $\Pi_J$ to be surjective with connected fibers. Hence, Proposition 3.3 implies the irreducibility of $Q_J(\beta, v, w)$. By the Stein factorization theorem applied to the composition map $\Pi_J \circ \pi_{\beta, v, w}$, we deduce that the normalization of $Q_J(\beta, v, w)$ is bijective to $Q_J(\beta, v, w)$. Therefore, the weak normality of $Q_J(\beta, v, w)$ (Corollary 3.42) implies the result. \qed

We set $W_J := W/w_0W_Jw_0$, and identify it with the set of their minimal length representatives in $W$. We set $\mathfrak{s}_J$ to be the sum of the positive root that belongs to the unipotent radical of $P(J)$.

Corollary 4.21. For each $J \subset I$, $\beta \in Q_+^\vee$, $w \in W$, the variety $Q_J(\beta, 0, w)$ is non-empty and has dimension $\dim G/P(J) - 2 \langle \lambda, \rho_J \rangle - \ell(w)$. In addition, $Q_J(\beta, w, 0)$ is the collection of DP-data as in Definition 1.6 with the data and compatibility conditions only for $\mathfrak{s}_J \in P(J)$. The proof of Corollary 5.2 implies that $Q_J(\beta, v, w)$ is irreducible and has expected dimension if it is non-empty.

Proof. The collection of DP-data in Definition 1.6 with the compatibility condition imposed only for $\lambda \in P(J)$ (without non-defining points) represents the (closure of the) space of maps $\mathbb{P}^1 \to G/P(J)$ such that the image of $[\mathbb{P}^1]$ in $\mathfrak{B}$ is $-w_0\beta \in H_2(B, \mathbb{Z})$. The dimension of the space of such maps is given by $\dim G/P(J) - 2 \langle w_0\beta, \rho_J \rangle$ since the first Chern class of the tangent bundle of $G/P(J)$ is $2\rho_J$ (cf. [31, Proposition 3.5]). Since the defect condition always lowers the dimension by one, we conclude the case of $w \neq 0$ from Corollary 4.21. The general case follows since $Q_J(\beta, w_0, e)$ is $G$-stable and restricting the evaluation at one point to a codimension $\ell(w)$ locus lowers the dimension by $\ell(w)$. \qed

Corollary 4.22. For each $J \subset I$ and $w, v \in W_{af}$, the scheme $Q'_J(v, w)$ is irreducible and normal for $p \gg 0$.

Proof. Apply Proposition 5.22 to Corollary 5.21 (cf. Lemma 5.1). \qed

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Corollary 4.24. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 or $p \gg 0$. For each $J \subseteq \mathbb{I}$ and $w, v \in W_{af}$, the scheme $Q'_J(v, w)_{\mathbb{K}}$ is projectively normal with respect to a line bundle $\mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}(\lambda)$ $(\lambda \in P_{J, +})$.

Proof. In view of Lemma 3.23, Lemma 3.27, and Theorem 3.24, the multiplication of the section ring afforded by $\{\mathcal{O}_{Q'_J(v, w)_{\mathbb{K}}}(m\lambda)\}_{m \geq 0}$ is surjective. Therefore, Corollary 2.20 and Corollary 3.24 implies the result. \qed

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Appendix A An analogue of the Kempf vanishing theorem

We work in the setting of §3.2. Let $\mathbb{K}$ be an algebraically closed field of characteristic $\neq 2$. The case $\mathbb{K} = \mathbb{C}$ is treated in §3.2. The aim of this appendix is to show that our scheme $Q_{G, J}(w)$ is (projectively) normal, and present an analogue of the Kempf vanishing theorem [26] in our setting.

Proposition A.1. For each $w \in W_{af}$ and $J \subseteq \mathbb{I}$, the ring $R_w(J)_{\mathbb{K}}$ is normal.

Remark A.2. The proof of Proposition A.1 presented here gives a new proof of [26, Theorem 4.26] in characteristic zero in the presence of Theorem 3.24 along the lines of [12, §5].

Proof of Proposition A.1. We first prove that case $w = e$ and $J = \emptyset$. Let $\hat{Q}_G$ denote the open $G[z]$-orbit of $Q_G(e)$ obtained by the $G$-translation of Lemma 3.23.

We have an inclusion

$$\mathbb{W}(\lambda)^\vee \subset \Gamma(\hat{Q}_G, \mathcal{O}_{Q_G(e)}(\lambda))$$

(A.2)

since $R_\mathbb{K}$ is integral (Corollary 3.24). We also have an inclusion

$$\Gamma(\hat{Q}_G, \mathcal{O}_{Q_G(e)}(\lambda)) \hookrightarrow \Gamma(\mathcal{O}(e), \mathcal{O}_{Q_G(e)}(\lambda)) \quad \lambda \in P_+.$$ Thanks to Lemma 3.23 and its proof, we deduce

$$\Gamma(\mathcal{O}(e), \mathcal{O}_{Q_G(e)}(\lambda)) \cong \mathbb{K}[I/(I \cap HN(z))] \otimes_{\mathbb{K}} \mathbb{K}_\lambda,$$

that is cocyclic as a $U_{\mathbb{K}}^+$-module. Since the $G$-action on $\Gamma(\hat{Q}_G, \mathcal{O}_{Q_G(e)}(\lambda))$ is algebraic, we deduce that

$$\mathbb{K}[I/(I \cap HN(z))] \otimes_{\mathbb{K}} \mathbb{K}_\lambda \hookrightarrow \Gamma(\hat{Q}_G, \mathcal{O}_{Q_G(e)}(\lambda))^\vee$$

is a $U_{\mathbb{K}}^+$-integrable quotient. By Proposition A.3, we conclude a surjection

$$\mathbb{W}(\lambda)_{\mathbb{K}} \twoheadrightarrow \Gamma(\hat{Q}_G, \mathcal{O}_{Q_G(e)}(\lambda))^\vee.$$ Compared with (A.2), we conclude the isomorphism

$$\Gamma(\hat{Q}_G, \mathcal{O}_{Q_G(e)}(\lambda)) \cong \mathbb{K}[I/(I \cap HN(z))] \otimes_{\mathbb{K}} \mathbb{K}_\lambda.$$ since the space of sections supported on a dense open Zarsiki subset must be larger than (or equal to) the space of sections supported on the whole space for an integral scheme. In other words, $R_\mathbb{K}$ is the maximal $U_{\mathbb{K}}^+$-integrable $U_{\mathbb{K}}^+$-stable subring of

$$S_\mathbb{K} := \bigoplus_{\lambda \in P_+} \Gamma(\mathcal{O}(e), \mathcal{O}_{Q_G(e)}(\lambda)) = \bigoplus_{\lambda \in P_+} \mathbb{K}[I/(I \cap HN(z))] \otimes_{\mathbb{K}} \mathbb{K}_\lambda.$$
The scheme into account, the proof in $2.14$ and Corollary $5.1$. For each $w; v \in P_{l+}$, let $H^0(Q_{G; J}(w)_K, \mathcal{O}_{Q_{G; J}(w)_K}(\lambda)) = \mathbb{W}_{w, w_0}(\lambda)_K$.

**Proof.** As finding $H^0$ from $R_w(J)_K$ can be seen as finding graded pieces of the normalization, the assertion follows from Proposition $5.1$.

**Proposition A.5** ([52] Proposition 5.1). Assume that $\text{char } K \neq 2$. Let $w \in W$ and $J \subset I$. Then, an $I$-equivariant line bundle on $Q_{G; J}(w)_K$ is a character twist of $(Q_{G; J}(w)_K(\lambda))_{\lambda \in \mathbb{P}}$.

**Proof.** Taking Corollary $4.4$ and Corollary $4.5$ into account, the proof in [52, Proposition 5.1] works in this setting.

**Theorem A.6.** Let $w \in W$ and $J \subset I$. For each $\lambda \in \mathbb{P}$, we have

\[ H^{>0}(Q_{G; J}(w)_K, \mathcal{O}_{Q_{G; J}(w)_K}(\lambda)) = \{0\}. \]

**Proof.** By the application of the Demazure functors (see the proof of Proposition $4.4$ cf. [52, §4] and [52, §5.1.3]), we deduce that $\mathbb{W}_w(\lambda)_K$ is free over a polynomial ring if and only if $\mathbb{W}(\lambda)_K$ is. In view of the normality of $R_w(J)_K$ and the freeness of $\mathbb{W}(\lambda)_K$, the same reasoning as in [52, Theorem 4.28 and Theorem 4.29] yields the assertion (with the PBW bases replaced with those in [15] or [53]).

**Remark A.7.** Our claim here is that the reasoning of the higher cohomology vanishing of $Q_{G; J}(w)$ is totally different from that in finite case, and uniform across characteristic.

**Appendix B** An application of the Pieri-Chevalley formula

We work in the setting of $\S 4.2$. The aim of this appendix is to present a method to describe the global sections of nef line bundles on $Q'(v, w)_K$ for $v, w \in W_{af}$ and an algebraically closed field $K$ of characteristic $\neq 2$.

For each $\mu \in P_{l+}$, we have an extremal weight module $X(\mu)_K$ and its global base $B(\mu)$ borrowed from Theorem $5.3$.

**Lemma B.1.** Let $\mu \in P_{l+}$. A subset of $B(\mu)$ spans

\[ H^0(Q_{G}(w)_K, \mathcal{O}_{Q_{G}(w)_K}(\mu))^\vee = H^0(Q_{G; J}(w)_K, \mathcal{O}_{Q_{G; J}(w)_K}(\mu))^\vee \subset X(\lambda)_K \quad \text{and} \]

\[ \theta(H^0(Q_{G; J}(v)_K, \mathcal{O}_{Q_{G; J}(v)_K}(-w_0\mu))^\vee) = \theta(H^0(Q_{G; J}(v)_K, \mathcal{O}_{Q_{G; J}(v)_K}(-w_0\mu))^\vee) \subset X(\mu)_K \]

for each $w, v \in W_{af}$.

**Proof.** Taking Theorem $5.3$ into account, the assertion follows from Theorem $5.3$. 

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Theorem A.4, combined with Theorem A.3, yields
\[ H^0(\Omega'(v, w)_X, \mathcal{O}_{\Omega'(v, w)}(\mu))^\vee \hookrightarrow H^0(\mathcal{Q}_G(w)_X, \mathcal{O}_{\mathcal{Q}_G(w)}(\mu))^\vee \subset \chi(\mu) \quad (B.3) \]
for each \( w, v \in W_{ad} \).

Lemma B.2. Let \( \mu \in P_{++} \). For each \( b \in B(\mu) \), there exist unique elements \( \kappa(b), \iota(b) \in W_{ad} \) with the following properties:

1. We have \( b \in H^0(\mathcal{Q}_G(w)_X, \mathcal{O}_{\mathcal{Q}_G(w)}(\mu))^\vee \) for \( w \in W_{ad} \) if and only if \( \kappa(b) \leq \frac{w}{w} \);
2. We have \( b \in \theta(H^0(\mathcal{Q}_G(vw_0)_X, \mathcal{O}_{\mathcal{Q}_G(vw_0)}(-w_0\mu))) \) for \( v \in W_{ad} \) if and only if \( \iota(b) \leq \frac{w}{w} \).

**Proof.** In view of Theorem A.3, the first assertion is a rephrasement of Lemma A.2. The second assertion is obtained from the first assertion in view of Lemma A.3 and [B1 (1.5.1)].

**Corollary B.3.** The functions \( \kappa \) and \( \iota \) play the same role as the same named functions in [B2 (2.17)] (with opposite convention on the order \( \leq \frac{w}{w} \)).

**Proof.** Compare Lemma A.4 with [B2, Theorem 2.8].

**Corollary B.4.** Let \( \mu \in P_{++} \) and \( w, v \in W_{ad} \). The space \( H^0(\Omega'(v, w)_X, \mathcal{O}_{\Omega'(v, w)}(\mu))^\vee \) is spanned by the subset of \( b \in B(\mu) \) that satisfies \( w \geq \frac{v}{v} \) \( \kappa(b) \geq \frac{v}{v} \) \( \iota(b) \geq \frac{v}{v} \).

**Proof.** This is a rephrasement of Corollary A.3 for \( J = 0 \).

**Theorem B.5** ([B2, Theorem 5.8 and its proof]. Let \( w \in W_{ad} \) and \( \lambda \in P_{+} \). There exists a collection of elements \( a^w_\lambda(\lambda) \in \mathbb{Z}[q^{-1}][H] \) \( (u \in W_{ad}) \) such that
\[ a^w_\lambda(\lambda) = e^{-uw_0} ; \quad a^w_\lambda(\lambda) = 0 \quad \text{if} \quad u \not\leq \frac{w}{w} \] and
\[ \text{gch} \Gamma(\mathcal{Q}_G(w), \mathcal{O}_{\mathcal{Q}_G(w)}(\lambda + \mu)) = \sum_{u \in W_{ad}} a^w_\lambda(\lambda) \text{gch} \Gamma(\mathcal{Q}_G(u), \mathcal{O}_{\mathcal{Q}_G(u)}(\mu)) \]
for every \( \mu \in P_{++} \).

The goal of this Appendix is to prove the following:

**Theorem B.6.** Let \( w, v \in W_{ad} \) and \( \lambda \in P_{+} \). Let \( \{a^w_\lambda(\lambda)\}_{v \in W_{ad}} \) be the collection of elements in Theorem B.5. Then, we have
\[ \text{gch} \Gamma(\Omega'(v, w), \mathcal{O}_{\Omega'(v, w)}(\lambda)) = \sum_{u \geq \frac{w}{w}} a^w_\lambda(\lambda). \]

**Proof.** The proof of the numerical part of Theorem B.5 is [B2, Theorem 3.5] (proved in [B2, §8.1]) and it counts the elements of \( \mathcal{B}(\lambda + \mu) \) that contributes \( \Gamma(\mathcal{Q}_G(w), \mathcal{O}_{\mathcal{Q}_G(w)}(\lambda + \mu)) = \mathbb{W}_{w_0}(\lambda + \mu) \) in two ways. In particular, we can additionally impose the condition \( \kappa(\bullet) \geq \frac{w}{w} \) to deduce that
\[ \text{gch} \Gamma(\Omega'(v, w), \mathcal{O}_{\Omega'(v, w)}(\lambda + \mu)) = \sum_{u \leq \frac{w}{w}} a^w_\lambda(\lambda) \text{gch} \Gamma(\Omega'(v, w), \mathcal{O}_{\Omega'(v, w)}(\mu)) \]
for every \( \mu \in P_{++} \) in view of Corollary B.5. The Euler characteristic of \( \mathcal{O}_{\Omega'(v, w)}(\mu) \) \( (\mu \in P) \) is a rational function on the characters of \( H \), and we can specialize to \( \mu = 0 \). Now we apply Theorem B.5 to deduce
\[ \chi(\Omega'(v, w), \mathcal{O}_{\Omega'(v, w)}(\mu)) = \text{gch} \Gamma(\Omega'(v, w), \mathcal{O}_{\Omega'(v, w)}(\mu)) \]
for every \( \mu \in P_{+} \). This implies the desired equality.
Remark B.7. 1) In view of Corollary 3.35 and Remark 3.36, Theorem B.6 describes the space of global sections of $\mathcal{O}_{\mathfrak{g}(v,w)}(\lambda)$ for every $w, v \in W_{af}$, $J \subseteq I$, and $\lambda \in P_{\lambda, +}$. 2) In conjunction with Theorem 3.33, Theorem B.6 can be seen as an analogue of Lakshmibai-Littelmann [43, Theorem 34] for semi-infinite flag manifolds.

References


