Loop structure on equivariant K-theory of semi-infinite flag manifolds*

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Abstract

We explain that the Pontryagin product structure on the equivariant K-group of an affine Grassmannian considered in [Lam-Schilling-Shimozono, Compos. Math. **146** (2010)] coincides with the tensor structure on the equivariant K-group of a semi-infinite flag manifold considered in [K-Naito-Sagaki, Duke Math. **169** (2020)]. Then, we construct an explicit isomorphism between the equivariant K-group of a semi-infinite flag manifold and a suitably localized equivariant quantum K-group of the corresponding flag manifold. These exhibit a new framework to understand the ring structure of equivariant quantum K-groups and the Peterson isomorphism.

Introduction

Let G be a simply connected simple algebraic group over $\mathbb{C}$ with a maximal torus H. Let Gr denote its affine Grassmannian and let $\mathcal{B}$ be its flag variety.

Following the seminal work of Peterson [42] (on the quantum cohomology), there were many efforts to understand the (small) quantum K-group $qK(\mathcal{B})$ of $\mathcal{B}$ in terms of the K-group $K(\mathrm{Gr})$ of affine Grassmannians (see [35, 34] and the references therein). One of its form, borrowed from Lam-Li-Mihalcea-Shimozono [34], is a (conjectural) ring isomorphism:

$$K_H(\mathrm{Gr})_{\mathrm{loc}} \cong qK_H(\mathcal{B})_{\mathrm{loc}}, \tag{0.1}$$

where subscript H indicate the H-equivariant version and the subscript loc denote certain localizations. Here the multiplication in $K_H(\mathrm{Gr})_{\mathrm{loc}}$ is the *Pontryagin product*, that differs from the usual product, while the multiplication of $qK_H(\mathcal{B})_{\mathrm{loc}}$ is standard in quantum K-theory [19, 36].

On the other hand, we have another version $\mathbf{Q}_G^{\mathrm{rat}}$ of affine flag variety of G, called the semi-infinite flag variety ([13, 16, 12]). Almost from the beginning [18], it is expected that $\mathbf{Q}_G^{\mathrm{rat}}$ have some relation with the quantum cohomology of $\mathcal{B}$. In fact, we can calculate the equivariant K-theoretic J-function of $\mathcal{B}$ using $\mathbf{Q}_G^{\mathrm{rat}}$ ([20, 6]), and the reconstruction theorem [37, 24] tells us that they essentially recover the ring structure of the (big) quantum K-group of $\mathcal{B}$.

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In [28], we have defined and calculated the equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$, that is also expected to have some relation to $qK_H(\mathcal{B})$, and hence also to $K_H(\mathrm{Gr})$. The goal of this paper is to tell the exact relations as follows:

Theorem A ($\doteq$ Theorem 2.1)**.** *The both of $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$ and $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ admits actions of a variant $\mathcal{H}$ of the double affine Hecke algebra and the coroot lattice $Q^\vee$ of G. It gives rise to a dense embedding*

$$\Phi : K_H(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

of $(\mathcal{H}, Q^\vee)$-bimodules that sends the Pontryagin product on the LHS to the tensor product on the RHS.

Here we note that the topology of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ arises from the Schubert stratification of $\mathbf{Q}_G^{\mathrm{rat}}$, and its role in Theorem A is minor. By transplanting the path model of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$, Theorem A yields multiplication formulas of the classes in $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$ ([28, 41]).

Our strategy to prove Theorem A is as follows: the $\mathcal{H} \otimes \mathbb{C}Q^\vee$-module $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$ is cyclic. Hence, its $\mathcal{H} \otimes \mathbb{C}Q^\vee$-endomorphism is determined by the image of a cyclic vector. Moreover, the tensor product action of an equivariant line bundle on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ yields a $\mathcal{H} \otimes \mathbb{C}Q^\vee$-endomorphism. These make it possible to identify important parts of the Pontryagin action on the LHS that gives a $\mathcal{H} \otimes \mathbb{C}Q^\vee$-endomorphism with the tensor product action on the RHS.

The other part of the exact relation we exhibit is:

Theorem B ($\doteq$ Theorem 3.1 and Theorem 4.1)**.** *We have a $K_H(\mathrm{pt})$-module isomorphism*

$$\Psi : qK_H(\mathcal{B})_{\mathrm{loc}} \xrightarrow{\cong} K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends the quantum product of a primitive anti-nef line bundle to the tensor product of the corresponding line bundle.
Moreover, Ψ sends a Schubert class of the LHS to a Schubert class in the RHS, and intertwines the Novikov variable twist in the LHS to the right translation of the Schubert classes in the RHS. In particular, the topology of $qK_H(\mathcal{B})_{\mathrm{loc}}$ with respect to the Novikov variables is compatible with the topology of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ through Ψ.

Here we remark that a priori $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ is not a ring (see Remark 1.15).

Unlike Theorem A, Theorem B is best understood by its completed topological form as the inverse quantum multiplication of an anti-nef line bundle corresponds to the tensor product of a nef line bundle through Ψ. The latter tensor product action on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ is quite natural, and its structure constants with respect to the Schubert classes are positive ([28, Theorem 5.11]). However, it lives genuinely in the completions in general (as the sum is infinite; see §2.1).

Such tensor products (with infinitely many nonzero structure constants) play a central role in our proof of Theorem B. To analyze them, we need to through-in an extra q-variable that is responsible for the $\mathbb{G}_m$-action on a curve $\mathbb{P}^1$ in the both sides. In particular, our proof of Theorem B is in fact the $q = 1$ specialization of an isomorphism

$$\Psi_q : \mathbb{C}[q^{\pm 1}] \otimes qK_H(\mathcal{B})_{\mathrm{loc}} \cong K_{H \times \mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}}),$$

that intertwines shift operators (of line bundles on $\mathcal{B}$) and line bundle twists (on $\mathbf{Q}_G^{\mathrm{rat}}$). Combining Theorems A and B, we conclude:

Corollary C ($\doteq$ Corollary 3.2)**.** *We have a commutative diagram, whose bottom arrow is a natural embedding of rings:*

$$\begin{tikzcd}
 & K_H(\mathbf{Q}_G^{\mathrm{rat}}) & \\
K_H(\mathrm{Gr})_{\mathrm{loc}} \arrow[ur, hook, "\Phi"] \arrow[rr, hook, "(0.1)"] & & qK_H(\mathcal{B})_{\mathrm{loc}} \arrow[ul, "\Psi"', "\cong"]
\end{tikzcd} .$$

Here the uncompleted version of $qK_H(\mathcal{B})_{\mathrm{loc}}$ *is isomorphic to* $K_H(\mathrm{Gr})_{\mathrm{loc}}$*, the map* Φ *is an injective* $K_H(\mathrm{pt}) \otimes \mathbb{C}Q^\vee$*-algebra homomorphism, and* $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ *acquires the structure of a ring from* $K_H(\mathrm{Gr})$ *or* $qK_H(\mathcal{B})$*.*

The explicit nature of Corollary C verifies conjectures in [34] (Corollary 3.2). In the same vein, we find that the structure constants of quantum multiplications, as well as the shift operator actions, on $qK_H(\mathcal{B})$ are finite (Corollary 3.3 and Corollary 4.3; see also Anderson-Chen-Tseng [2]). Therefore, this paper also provides an indispensable step in the proof [2] of the finiteness of the multiplication of quantum K-groups of partial flag manifolds, that stood as a fundamental problem in the quantum Schubert calculus from the beginning.

The idea of the construction of Ψ in Theorem B is to compare the structure of the both sides via the asymptotic behavior of the cohomology of quasi-map spaces with respect to the degree of curves. Adapting the technicality on the topology and the q-variables discussed above, it is rather natural to consider such a thing if we know the "cohomological invariance" between two models of semi-infinite flag manifolds proved in [7, 28], the reconstruction theorem in the form of [24], and the J-function calculations in [20, 6]. In order to show that it respects products (Theorem 4.1), we need to analyze the geometry of graph spaces and quasi-map spaces. Our analysis includes a proof that the Zastava space closures have rational singularities and are Cohen-Macauley (Theorem 4.8), based on its normality[1] that we prove in [26], which might be of its own interest. We note that Theorem A, and hence Corollary C, also have $\mathbb{G}_m$-equivariant versions by supplementing cosmetic arguments to the results presented in this paper that we exhibit in [27] together with its representation-theoretic consequences.

Note that $\mathbf{Q}_G^{\mathrm{rat}}$ is the reduced indscheme associated to the formal loop space of $\mathcal{B}$ ([26] see also [6, 28]). Hence, it is tempting to spell out the following, that unifies the proposals by Givental [18, §4] (cf. Iritani [23]), Peterson [42] (cf. [34]), and Arkhipov-Kapranov [3, §6.2]:

Conjecture D. *Let* X *be a smooth projective convex variety* (*see* [*31*]) *with an action of an algebraic group* H*. Let* $\mathcal{L}X$ *be the formal loop space of* X (*see* [*3*])*. Then, we have an inclusion that intertwines the quantum product and tensor product of primitive anti-nef line bundles:*

$$\Psi_X : qK_H(X) \hookrightarrow K_H((\mathcal{L}X)_{\mathrm{red}}),$$

where $K_H((\mathcal{L}X)_{\mathrm{red}})$ *denotes the* H*-equivariant* K*-group of the reduced counterpart of* $\mathcal{L}X$ (*defined as a straightforward generalization of* [*28*])*.*

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[1] Previous versions of this paper contained proofs of Theorem 4.1 with gaps. To clarify the whole point, the author decided to separate out the proof of the normality and other related technical results into [26] (see Theorem 4.7).

Here we point out that taking reduced part played an essential role in the explicit calculations when $X = \mathcal{B}$ (cf. [40, 14]).

The organization of this paper is as follows: In section one, we recall some basic results from previous works (needed to formulate Theorems A and B), with some complementary results. In section two, we formulate the precise version of Theorem A, exhibit its $SL(2)$-example, and prove Theorem A. In section three, we formulate the precise version of Theorem B, explain why it solves conjectures in [34] (Corollary 3.2), make recollections on quasi-map spaces and J-functions, and construct the map Ψ following ideas of [20, 6, 7, 24] using results from [28]. This proves the main portion of Theorem B, and also Corollary C. In section four, we first state Theorem 4.1 about identification of bases under the map Ψ that completes the proof of Corollary 3.2. Then, we recall basic materials on graph spaces, prove that Zastava space closures have rational singularities by a detailed analysis using the results from [16, 26], and prove Theorem 4.1.

Finally, a word of caution is in order. The equivariant K-groups dealt in this paper are *not* identical to these dealt in [35] and [28] in the sense that both groups are just dense subset (or intersects with a dense subset) in the original K-groups (the both groups are suitably topologized). The author does not try to complete this point as he believes it not essential.

1 Preliminaries

A vector space is always a $\mathbb{C}$-vector space, and a graded vector space refers to a $\mathbb{Z}$-graded vector space whose graded pieces are finite-dimensional and its grading is bounded from the above. Tensor products are taken over $\mathbb{C}$ unless stated otherwise. We define the graded dimension of a graded vector space as

$$\mathrm{gdim}\, M := \sum_{i \in \mathbb{Z}} q^i \dim_{\mathbb{C}} M_i \in \mathbb{Q}(\!(q^{-1})\!).$$

For a (possibly operator-valued) rational function $f(q)$ on q, we set $\overline{f(q)} := f(q^{-1})$. In this paper, a variety is a separated integral scheme of finite type over $\mathbb{C}$. We sometimes identify a commutative local ring with its spectrum.

1.1 Groups, root systems, and Weyl groups

Basically, material presented in this subsection can be found in [11, 33].

Let G be a connected, simply connected simple algebraic group of rank r over $\mathbb{C}$, and let B and H be a Borel subgroup and a maximal torus of G such that $H \subset B$. We set N $(= [B, B])$ to be the unipotent radical of B and let N^- be the opposite unipotent subgroup of N with respect to H. We set $B^- := HN^-$. We denote the Lie algebra of an algebraic group by the corresponding German small letter. We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group E, we denote its set of $\mathbb{C}[z]$-valued points by $E[z]$, its set of $\mathbb{C}[\![z]\!]$-valued points by $E[\![z]\!]$, and its set of $\mathbb{C}(z)$-valued points by $E(z)$. Let $\mathbf{I} \subset G[\![z]\!]$ be the preimage of $B \subset G$ via the evaluation at $z = 0$ (the Iwahori subgroup of $G[\![z]\!]$).

Let $P := \mathrm{Hom}_{gr}(H, \mathbb{G}_m)$ be the weight lattice of H, let $\Delta \subset P$ be the set of roots, let $\Delta_+ \subset \Delta$ be the set of roots that yield root subspaces in $\mathfrak{b}$, and let $\Pi \subset \Delta_+$ be the set of simple roots. We set $\Delta_- := -\Delta_+$. Let $Q^\vee$ be the dual lattice of P with a natural pairing $\langle \bullet, \bullet \rangle : Q^\vee \times P \to \mathbb{Z}$. We define

$\Pi^\vee \subset Q^\vee$ to be the set of positive simple coroots, and let $Q^\vee_+ \subset Q^\vee$ be the set of non-negative integer span of $\Pi^\vee$. For $\beta, \gamma \in Q^\vee$, we define $\beta \geq \gamma$ if and only if $\beta - \gamma \in Q^\vee_+$. We set $P_+ := \{\lambda \in P \mid \langle \alpha^\vee, \lambda \rangle \geq 0, \ \forall \alpha^\vee \in \Pi^\vee\}$. Let $\mathtt{I} := \{1, 2, \ldots, r\}$. We fix bijections $\mathtt{I} \cong \Pi \cong \Pi^\vee$ such that $i \in \mathtt{I}$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, and a simple reflection $s_i \in W$ corresponding to α_i. We also have a reflection $s_\alpha \in W$ corresponding to $\alpha \in \Delta_+$. Let $\{\varpi_i\}_{i \in \mathtt{I}} \subset P_+$ be the set of fundamental weights (i.e. $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{i,j}$) and we set $\rho := \sum_{i \in \mathtt{I}} \varpi_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in P_+$.

Let $\Delta_{\mathrm{af}} := \Delta \times \mathbb{Z}\delta \cup \{m\delta\}_{m \neq 0}$ be the untwisted affine root system of Δ with its positive part $\Delta_+ \subset \Delta_{\mathrm{af},+}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{\mathrm{af}} := \Pi \cup \{\alpha_0\}$, and $\mathtt{I}_{\mathrm{af}} := \mathtt{I} \cup \{0\}$, where ϑ is the highest root of Δ_+. We set $W_{\mathrm{af}} := W \ltimes Q^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \mathtt{I}_{\mathrm{af}}\}$, where s_0 is the reflection with respect to α_0. Let $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ be the length function and let $w_0 \in W$ be the longest element in $W \subset W_{\mathrm{af}}$. Together with the normalization $t_{-\vartheta^\vee} := s_\vartheta s_0$ (for the coroot $\vartheta^\vee$ of ϑ), we introduce the translation element $t_\beta \in W_{\mathrm{af}}$ for each $\beta \in Q^\vee$.

For each $i \in \mathtt{I}_{\mathrm{af}}$, we have a subgroup $SL(2, i) \subset G(\!(z)\!)$ that is isomorphic to $SL(2, \mathbb{C})$ corresponding to $\alpha_i \in \mathtt{I}_{\mathrm{af}}$. We set $B_i := SL(2, i) \cap \mathbf{I}$, that is a Borel subgroup of $SL(2, i)$. For each $i \in \mathtt{I}$, we denote the minimal parabolic subgroup of G corresponding to $i \in \mathtt{I}$ by P_i. For each $w \in W$ or $w \in W_{\mathrm{af}}$, we find a representative $\dot{w}$ in $N_G(H)$ or $N_{G(\!(z)\!)}(H(\!(z)\!))$, respectively.

Let W_{af}^- denote the set of minimal length representatives of W_{af}/W in W_{af}. We set

$$Q^\vee_< := \{\beta \in Q^\vee \mid \langle \beta, \alpha_i \rangle < 0, \forall i \in \mathtt{I}\}.$$

Let $\leq$ be the Bruhat order of W_{af}. In other words, $w \leq v$ holds if and only if a subexpression of a reduced decomposition of v yields a reduced decomposition of w (see [4]). We define the generic (semi-infinite) Bruhat order $\leq_{\frac{\infty}{2}}$ as:

$$w \leq_{\frac{\infty}{2}} v \Leftrightarrow wt_\beta \leq vt_\beta \qquad \text{for every } \beta \in Q^\vee \text{ such that } \langle \beta, \alpha_i \rangle \ll 0 \text{ for } i \in \mathtt{I}. \tag{1.1}$$

By [38], this defines a preorder on W_{af}. Here we remark that $w \leq v$ if and only if $w \geq_{\frac{\infty}{2}} v$ for $w, v \in W$. See also [28, §2.2].

For each $\lambda \in P_+$, we denote a finite-dimensional simple G-module with a B-eigenvector with its H-weight λ by $L(\lambda)$. Let $R(G)$ be the (complexified) representation ring of G. We have an identification $R(G) = (\mathbb{C}P)^W \subset \mathbb{C}P$ by taking characters. For a semi-simple H-module V, we set

$$\mathrm{ch}\, V := \sum_{\lambda \in P} e^\lambda \cdot \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathbb{C}_\lambda, V).$$

If V is a $\mathbb{Z}$-graded H-module in addition, then we set

$$\mathrm{gch}\, V := \sum_{\lambda \in P, n \in \mathbb{Z}} q^n e^\lambda \cdot \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathbb{C}_\lambda, V_n).$$

Let $\mathcal{B} := G/B$ and call it the flag manifold of G. It is equipped with the Bruhat decomposition

$$\mathcal{B} = \bigsqcup_{w \in W} \mathbb{O}_{\mathcal{B}}(w)$$

into B-orbits such that $\dim \mathbb{O}_{\mathcal{B}}(w) = \ell(w_0) - \ell(w)$ for each $w \in W \subset W_{\mathrm{af}}$. Namely, we have $\mathbb{O}_{\mathcal{B}}(w) = B\dot{w}\dot{w}_0 B/B$. We set $\mathcal{B}(w) := \overline{\mathbb{O}_{\mathcal{B}}(w)} \subset \mathcal{B}$. We also define $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(w) := B^- \dot{w}\dot{w}_0 B/B$ and $\mathcal{B}^{\mathrm{op}}(w) := \overline{\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(w)}$.

For each $\lambda \in P$, we have a line bundle $\mathcal{O}_{\mathcal{B}}(\lambda)$ such that

$$\operatorname{ch} H^0(\mathcal{B}, \mathcal{O}_{\mathcal{B}}(\lambda)) = \operatorname{ch} L(\lambda), \quad \mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(-\mu) \cong \mathcal{O}_{\mathcal{B}}(\lambda - \mu) \qquad \lambda, \mu \in P_+.$$

The line bundle $\mathcal{O}_{\mathcal{B}}(\lambda)$ is usually referred to as $G \times^B (w_0\lambda)$.

We have a notion of H-equivariant K-group $K_H(\mathcal{B})$ of $\mathcal{B}$ with coefficients in $\mathbb{C}$ (see e.g. [32]). Explicitly, we have

$$K_H(\mathcal{B}) = \bigoplus_{w \in W} \mathbb{C}P\,[\mathcal{O}_{\mathcal{B}(w)}] = \mathbb{C}P \otimes_{R(G)} \bigoplus_{\lambda \in P} \mathbb{C}[\mathcal{O}_{\mathcal{B}}(\lambda)]. \tag{1.2}$$

The map ch extends to a $\mathbb{C}P$-linear map

$$\chi : K_H(\mathcal{B}) \to \mathbb{C}P,$$

that we call the H-equivariant Euler-Poincaré characteristic. The group $K_H(\mathcal{B})$ is equipped with the product structure $\cdot$ induced by the tensor product of line bundles. For each $i \in \mathtt{I}$, we have

$$[\mathcal{O}_{\mathcal{B}(s_i)}] = [\mathcal{O}_{\mathcal{B}}] - e^{\varpi_i}[\mathcal{O}_{\mathcal{B}}(-\varpi_i)] \in K_H(\mathcal{B}) \tag{1.3}$$

coming from the short exact sequence

$$0 \to \mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathcal{B}}(-\varpi_i) \to \mathcal{O}_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}(s_i)} \to 0. \tag{1.4}$$

1.2 Level zero nil-DAHA

Definition 1.1. The level zero nil-DAHA $\mathcal{H}$ of type G is a $\mathbb{C}$-algebra generated by $\{e^\lambda\}_{\lambda \in P} \cup \{D_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ subject to the following relations:

1. $e^{\lambda+\mu} = e^\lambda \cdot e^\mu$ for $\lambda, \mu \in P$;
2. $D_i^2 = D_i$ for each $i \in \mathtt{I}_{\mathrm{af}}$;
3. For each distinct $i, j \in \mathtt{I}_{\mathrm{af}}$, we set $m_{i,j} \in \mathbb{Z}_{>0}$ as the minimum number such that $(s_i s_j)^{m_{i,j}} = 1$. Then, we have
$$\overbrace{D_i D_j \cdots}^{m_{i,j}\text{-terms}} = \overbrace{D_j D_i \cdots}^{m_{i,j}\text{-terms}};$$
4. For each $\lambda \in P$ and $i \in \mathtt{I}$, we have
$$D_i e^\lambda - e^{s_i\lambda} D_i = \frac{e^\lambda - e^{s_i\lambda}}{1 - e^{\alpha_i}};$$
5. For each $\lambda \in P$, we have
$$D_0 e^\lambda - e^{s_\vartheta\lambda} D_0 = \frac{e^\lambda - e^{s_\vartheta\lambda}}{1 - e^{-\vartheta}}.$$

Let $\mathcal{S} := \mathbb{C}P \otimes \mathbb{C}W_{\mathrm{af}}$ be the smash product algebra, whose multiplication reads as:

$$(e^{\lambda} \otimes w)(e^{\mu} \otimes v) = e^{\lambda + w\mu} \otimes wv \qquad \lambda, \mu \in P, w, v \in W_{\mathrm{af}},$$

where s_0 acts on P as s_{ϑ}. Let $\mathbb{C}(P)$ denote the fraction field of (the Laurant polynomial algebra) $\mathbb{C}P$. We have a scalar extension

$$\mathcal{A} := \mathbb{C}(P) \otimes_{\mathbb{C}P} \mathcal{S} = \mathbb{C}(P) \otimes \mathbb{C}W_{\mathrm{af}}.$$

Theorem 1.2 ([35] §2.2)**.** *We have an embedding of algebras $\imath^* : \mathcal{H} \hookrightarrow \mathcal{A}$:*

$$e^{\lambda} \mapsto e^{\lambda} \otimes 1,\ D_i \mapsto \frac{1}{1 - e^{\alpha_i}} \otimes 1 - \frac{e^{\alpha_i}}{1 - e^{\alpha_i}} \otimes s_i, \qquad \lambda \in P, i \in \mathtt{I}$$
$$D_0 \mapsto \frac{1}{1 - e^{-\vartheta}} \otimes 1 - \frac{e^{-\vartheta}}{1 - e^{-\vartheta}} \otimes s_0.$$

Since we have a natural action of $\mathcal{A}$ on $\mathbb{C}(P)$, we obtain an action of $\mathcal{H}$ on $\mathbb{C}(P)$, that we call the polynomial representation.

For $w \in W_{\mathrm{af}}$, we find a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ $(i_1, \ldots, i_\ell \in \mathtt{I}_{\mathrm{af}})$ and set

$$D_w := D_{s_{i_1}} D_{s_{i_2}} \cdots D_{s_{i_\ell}} \in \mathcal{H}.$$

By Definition 1.1 3), the element D_w is independent of the choice of a reduced expression. By Definition 1.1 2), we have $D_i D_{w_0} = D_{w_0}$ for each $i \in \mathtt{I}$, and hence $D_{w_0}^2 = D_{w_0}$. We have an explicit form

$$D_{w_0} = 1 \otimes \left(\sum_{w \in W} w \right) \cdot \frac{e^{-\rho}}{\prod_{\alpha \in \Delta^+} (e^{-\alpha/2} - e^{\alpha/2})} \otimes 1 \in \mathcal{A} \tag{1.5}$$

obtained from the (left W-invariance of the) Weyl character formula.

1.3 Affine Grassmannians

We define our (thin) affine Grassmannian and (thin) affine flag manifold by

$$\mathrm{Gr}_G := G(\!(z)\!)/G[\![z]\!] \quad \text{and} \quad X := G(\!(z)\!)/\mathbf{I},$$

respectively. We have a natural fibration map $\pi : X \to \mathrm{Gr}_G$ whose fiber is isomorphic to $\mathcal{B}$. For each $w \in W_{\mathrm{af}}$, we set $\mathbb{O}_w = \mathbf{I}\dot{w}\mathbf{I}/\mathbf{I}$. For each $\beta \in Q^{\vee}$, we find $w \in t_{\beta} W$ and set $\mathbb{O}^{\mathrm{Gr}}_{\beta} := \pi(\mathbb{O}_w)$. The set $\mathbb{O}^{\mathrm{Gr}}_{\beta}$ does not depend on the choice since we have $\dot{w} \in G[\![z]\!]$ for $w \in W$.

Theorem 1.3 (Bruhat decomposition, [33] Corollary 6.1.20)**.** *We have* $\mathbf{I}$*-orbit decompositions*

$$\mathrm{Gr} = \bigsqcup_{\beta \in Q^{\vee}} \mathbb{O}^{\mathrm{Gr}}_{\beta} \quad \textit{and} \quad X = \bigsqcup_{w \in W_{\mathrm{af}}} \mathbb{O}_w$$

with the following properties:

1. *we have $\mathbb{O}_v \subset \overline{\mathbb{O}_w}$ if and only if $v \leq w$.* □

Let us set $\mathrm{Gr}_\beta := \overline{\mathbb{O}^{\mathrm{Gr}}_\beta}$ and $X_w := \overline{\mathbb{O}_w}$ for $\beta \in Q^\vee$ and $w \in W_{\mathrm{af}}$. For $w \in W_{\mathrm{af}}^-$, we also set $\mathrm{Gr}_w := \mathrm{Gr}_\beta$ for a unique $\beta \in Q^\vee$ such that $w \in t_\beta W$.

We set

$$K_H(\mathrm{Gr}) := \bigoplus_{\beta \in Q^\vee} \mathbb{C}P\,[\mathcal{O}_{\mathrm{Gr}_\beta}] \quad \text{and} \quad K_H(X) := \bigoplus_{w \in W_{\mathrm{af}}} \mathbb{C}P\,[\mathcal{O}_{X_w}].$$

Theorem 1.4 (Kostant-Kumar [32])**.** *The vector space $K_H(X)$ affords a regular representation of $\mathcal{H}$ such that:*

1. *the subalgebra $\mathbb{C}P \subset \mathcal{H}$ acts by the multiplication as $\mathbb{C}P$-modules;*
2. *we have $D_i[\mathcal{O}_{X_w}] = [\mathcal{O}_{X_{s_i w}}]$ $(s_i w > w)$ or $[\mathcal{O}_{X_w}]$ $(s_i w < w)$.* □

Being a regular representation, we sometimes identify $K_H(X)$ with $\mathcal{H}$ (through $e^\lambda[\mathcal{O}_{X_w}] \leftrightarrow e^\lambda D_w$ for $\lambda \in P, w \in W_{\mathrm{af}}$) and consider product of two elements in $\mathcal{H} \cup K_H(X)$, that results in an element of $K_H(X) \cong \mathcal{H} \subset \mathcal{A}$.

Theorem 1.5 (Kostant-Kumar [32])**.** *The pullback defines a map $\pi^* : K_H(\mathrm{Gr}_G) \hookrightarrow K_H(X)$ such that*

$$\pi^*[\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{X_{t_\beta}}]D_{w_0} = D_{t_\beta} D_{w_0} \qquad \beta \in Q^\vee.$$

In particular, $\mathrm{Im}\,\pi^* = \mathcal{H}D_{w_0}$ *is a $\mathcal{H}$-submodule.* □

Let $\mathcal{C} := \mathbb{C}(P) \otimes \mathbb{C}Q^\vee \subset \mathcal{A}$ be a subalgebra generated by elements of the form $f \otimes t_\beta$ $(f \in \mathbb{C}(P),\, \beta \in Q^\vee)$. By our convention on the W_{af}-action on P, we deduce that $\mathcal{C}$ is commutative. We have a natural projection map

$$\mathsf{pr} : \mathcal{A} = \mathbb{C}(P) \otimes \mathbb{C}W_{\mathrm{af}} \longrightarrow \mathbb{C}(P) \otimes \mathbb{C}Q^\vee = \mathcal{C}$$

defined as $\mathsf{pr}(f \otimes t_\beta w) = f \otimes t_\beta$ for each $f \in \mathbb{C}(P), w \in W, \beta \in Q^\vee$.

Theorem 1.6 (Lam-Schilling-Shimozono)**.** *The composition map* $\mathsf{pr} \circ \imath^* \circ \pi^*$ *defines an embedding*

$$K_H(\mathrm{Gr}) \hookrightarrow K_H(X) \to \mathcal{C} \quad (\subset \mathcal{A})$$

whose image is equal to $K_H(X) \cap \mathcal{C}$. It descends to a $\mathbb{C}P$-module isomorphism

$$r^* : K_H(\mathrm{Gr}) \hookrightarrow K_H(X) \cap \mathcal{C} \quad (\subset \mathcal{A}).$$

This equips $K_H(\mathrm{Gr})$ a subalgebra structure of a commutative algebra $\mathcal{C}$.

Proof. By [34, Proposition 2], we deduce that the image of D_v under the map pr is the same for each $v \in t_\beta W$. Therefore, the assertion follows from the description of [35, §5.2]. □

Thanks to Theorem 1.6, we obtain a commutative product structure of $K_H(\mathrm{Gr})$ inherited from $\mathcal{C}$, that we denote by $\odot$. We call it the *Pontryagin product*. This the same product as in [35, §5.2], and its relation with the Pontryagin product in the topological K-group of the based loop space of the maximal compact subgroup of G is explained in [35, §5.1].

Below, we might think of an element of $K_H(\mathrm{Gr})$ as an element of $K_H(X)$ through π^*, an element of $\mathcal{A}$ through $\imath^* \circ \pi^*$, and as an element of $\mathcal{C}$ through r^* interchangeably. The next result is probably well-known to experts, but so far the author is unable to find an appropriate reference.

Theorem 1.7. *Let $w \in W_{\mathrm{af}}^-$ and let $\beta \in Q^\vee_<$. We have*

$$[\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}].$$

Proof. By our assumption on β, we have $\ell(t_\beta) = \ell(w_0) + \ell(w_0 t_\beta)$ (see [39, (2.4.1)]). In particular, the element $[\mathcal{O}_{\mathrm{Gr}_\beta}]$, viewed as an element of $\mathcal{A}$ through $\imath^* \circ \pi^*$, is of the form $(\sum_{v \in W} v)\xi$ for some $\xi \in \mathcal{A}$ by (1.5). Hence, it is invariant by the left action of W. Since the effect of the map pr is to twist by elements of W from the right in a term by term fashion, we deduce the equality

$$[\mathcal{O}_{\mathrm{Gr}_w}][\mathcal{O}_{\mathrm{Gr}_\beta}] = \mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}])[\mathcal{O}_{\mathrm{Gr}_\beta}]$$

of multiplications in $\mathcal{A}$ (multiplication in a non-commutative algebra). By examining the definition of pr, we further deduce

$$\mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}][\mathcal{O}_{\mathrm{Gr}_\beta}]) = \mathsf{pr}(\mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}])[\mathcal{O}_{\mathrm{Gr}_\beta}]) = \mathsf{pr}([\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]). \tag{1.6}$$

Since $w \in W_{\mathrm{af}}^-$, we have $\ell(w) + \ell(t_\beta) = \ell(wt_\beta)$ (see [42, Lecture 8, page12]). Consequently, we have $D_{wt_\beta} = D_w D_{t_\beta}$. Therefore, (1.6) and Theorem 1.6 implies that

$$[\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] = [\mathcal{O}_{\mathrm{Gr}_w}][\mathcal{O}_{\mathrm{Gr}_\beta}] = [\mathcal{O}_{\mathrm{Gr}_w}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}] \in K_H(\mathrm{Gr})$$

as required. □

Since $t_\beta \in W_{\mathrm{af}}^-$ for each $\beta \in Q^\vee_<$, Theorem 1.7 implies that the set

$$\{[\mathcal{O}_{\mathrm{Gr}_\beta}] \mid \beta \in Q^\vee_<\} \subset (K_H(\mathrm{Gr}), \odot)$$

forms a multiplicative system. We denote by $K_H(\mathrm{Gr})_{\mathrm{loc}}$ its localization. The action of an element $[\mathcal{O}_{\mathrm{Gr}_\beta}]$ on $K_H(\mathrm{Gr})$ in Theorem 1.7 is torsion-free, and hence we have an embedding $K_H(\mathrm{Gr}) \hookrightarrow K_H(\mathrm{Gr})_{\mathrm{loc}}$.

Corollary 1.8. *Let $i \in \mathtt{I}$. For $\beta \in Q^\vee_<$, we set*

$$\mathbf{h}_i := [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]^{-1}.$$

Then, the element $\mathbf{h}_i$ is independent of the choice of β.

Proof. By Theorem 1.7, we have

$$\begin{aligned}
[\mathcal{O}_{\mathrm{Gr}_{s_i t_{\gamma+\beta}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{\gamma+\beta}}]^{-1} &= [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{-1} \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]^{-1} \\
&= [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\beta}]^{-1}
\end{aligned}$$

for $\gamma \in Q^\vee_<$. Hence, we conclude the assertion. □

For each $\gamma \in Q^\vee$, we can write $\gamma = \beta_1 - \beta_2$, where $\beta_1, \beta_2 \in Q^\vee_<$. Using this, we define an element

$$\mathtt{t}_\gamma := [\mathcal{O}_{\mathrm{Gr}_{\beta_1}}] \odot [\mathcal{O}_{\mathrm{Gr}_{\beta_2}}]^{-1}.$$

Lemma 1.9. *For each $\gamma \in Q^\vee$, the element $\mathtt{t}_\gamma \in K_H(\mathrm{Gr})_{\mathrm{loc}}$ is independent of the choices involved.*

Proof. Similar to the proof of Corollary 1.8. The details are left to the reader. □

1.4 Semi-infinite flag manifolds

We define the semi-infinite flag manifold as the reduced scheme associated to:

$$\mathbf{Q}_G^{\mathrm{rat}} := G(\!(z)\!)/(H \cdot N(\!(z)\!)).$$

This is a pure ind-scheme of ind-infinite type [26]. Note that the group $Q^\vee \subset H(\!(z)\!)/H$ acts on $\mathbf{Q}_G^{\mathrm{rat}}$ from the right. The indscheme $\mathbf{Q}_G^{\mathrm{rat}}$ is equipped with a $G(\!(z)\!)$-equivariant line bundle $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$ for each $\lambda \in P$. Here we normalized in such a way that the restricted dual of $\Gamma(\mathbf{Q}_G^{\mathrm{rat}}, \mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda))$ is generated by its H-weight $(-\lambda)$-part as a $B(\!(z)\!)$-module. We warn that this convention is twisted by $-w_0$ from that of [28]. For $w \in W_{\mathrm{af}}$, we set $\mathbb{O}(w) := \mathbf{I}\dot{w}HN(\!(z)\!)/HN(\!(z)\!)$ and $\mathbf{Q}_G(w) := \overline{\mathbb{O}(w)}$. We refer $\mathbf{Q}_G(w)$ as a Schubert variety of $\mathbf{Q}_G^{\mathrm{rat}}$.

Theorem 1.10 ([16, 12, 28])**.** *We have an $\mathbf{I}$-orbit decomposition*

$$\mathbf{Q}_G^{\mathrm{rat}} = \bigsqcup_{w \in W_{\mathrm{af}}} \mathbb{O}(w)$$

with the following properties:

1. *each $\mathbb{O}(w)$ has infinite dimension and infinite codimension in $\mathbf{Q}_G^{\mathrm{rat}}$;*
2. *the right action of $\gamma \in Q^\vee$ on $\mathbf{Q}_G^{\mathrm{rat}}$ yields the translation $\mathbb{O}(w) \mapsto \mathbb{O}(wt_\gamma)$;*
3. *we have $\mathbb{O}(w) \subset \overline{\mathbb{O}(v)}$ if and only if $w \le_{\frac{\infty}{2}} v$.* □

We define a $\mathbb{C}P$-module $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ as:

$$K_H(\mathbf{Q}_G^{\mathrm{rat}}) := \{ \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}P, \, \exists \beta_0 \in Q^\vee \text{ s.t. } a_{ut_\beta} = 0, \, \forall u \in W, \beta \not> \beta_0 \},$$

where the sum in the definition is understood to be formal. We define its subset

$$K_H(\mathbf{Q}_G(e)) := \{ \sum_{w \in W_{\mathrm{af}}} a_w [\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}P \text{ s.t. } a_{ut_\beta} = 0, \, \forall u \in W, \beta \not\geq 0 \}.$$

The right $Q^\vee$-translations of $K_H(\mathbf{Q}_G(e))$ equips $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ a(n open base of a linear) topology. In addition, we have:

Lemma 1.11. *$K_H(\mathbf{Q}_G(e))$ are $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ are free module over $\mathbb{C}P \otimes \mathbb{C}[\![Q_+^\vee]\!]$ and $\mathbb{C}P \otimes (\mathbb{C}Q^\vee \otimes_{\mathbb{C}Q_+^\vee} \mathbb{C}[\![Q_+^\vee]\!])$, respectively. Moreover, their ranks are $|W|$.*

Proof. We have an explicit basis $\{[\mathcal{O}_{\mathbf{Q}_G(w)}]\}_{w \in W}$ in the both cases. □

We remark that our $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ and $K_H(\mathbf{Q}_G(e))$ are $q = 1$ specializations of certain subsets of the equivariant K-groups $K_{\mathbf{I} \ltimes \mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$ and $K_{\mathbf{I} \ltimes \mathbb{G}_m}(\mathbf{Q}_G(e))$ considered in [28]. To this end, we need to verity that the natural actions of the Demazure operators and the tensor product action in [28] yield the corresponding actions on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. The first one is immediate from the expression:

Theorem 1.12 ([28] §6 and [25] Theorem A)**.** *The vector space $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ affords a representation of $\mathcal{H}$ with the following properties:*

1. *the subalgebra $\mathbb{C}P \subset \mathcal{H}$ acts by the multiplication as $\mathbb{C}P$-modules;*

2. we have

$$D_i([\mathcal{O}_{\mathbf{Q}_G(w)}]) = \begin{cases} [\mathcal{O}_{\mathbf{Q}_G(s_i w)}] & (s_i w >_{\frac{\infty}{2}} w) \\ [\mathcal{O}_{\mathbf{Q}_G(w)}] & (s_i w <_{\frac{\infty}{2}} w) \end{cases}.$$

By Theorem 1.12, we deduce that the right translation $Q^\vee$-action arising from Theorem 1.10 2) yield $\mathcal{H}$-module endomorphisms of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$.

Theorem 1.13 (cf. [28] Theorem 6.5 see also [25])**.** *For each $\lambda \in P$, the $\mathbb{C}P$-linear extension of the assignment*

$$[\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)] \in K_H(\mathbf{Q}_G^{\mathrm{rat}}) \quad w \in W_{\mathrm{af}}$$

defines a $\mathcal{H}$-module automorphism (that we call $\Xi(\lambda)$) which commutes with the right $Q^\vee$-action. Moreover, we have $\Xi(\lambda) \circ \Xi(\mu) = \Xi(\lambda + \mu)$ for $\lambda, \mu \in P$.

Proof. The latter assertion is automatic provided if the former assertion holds as $\Xi(\lambda)$ is induced by the tensor product with $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$. Hence, we concentrate into the first assertion. The (main) difference between here and [28, Theorem 6.5] is the lack of the $\mathbb{G}_m$-action. Thus, it suffices to see that the tensor product action yields a well-defined automorphism of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ by forgetting the q-grading.

Since $\{\Xi(\lambda)\}_\lambda$ must be commutative to each other, it further reduces to prove that $\Xi(\pm\varpi_i)$ ($i \in \mathtt{I}$) actually define an automorphism of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. We have $[\mathcal{O}_{\mathbf{Q}_G(w)}(\varpi_i)] \in K_H(\mathbf{Q}_G^{\mathrm{rat}})$ for $i \in \mathtt{I}$ by the Pieri-Chevalley rule [28, Theorem 5.13] as the set of paths with fixed initial/final directions are finite. (This latter reasoning in turn follows as the q^{-1}-degrees of paths whose initial/final directions are bounded from ut_β and vt_γ ($u, v \in W, \beta, \gamma \in Q^\vee$) must belong to $[\langle \beta, \varpi_i \rangle, \langle \gamma, \varpi_i \rangle]$ by our count of q-degrees in [28].) This implies that $\Xi(\varpi_i)$ defines a well-defined automorphism of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ for each $i \in \mathtt{I}$.

Moreover, the set of paths with the same initial/final direction is unique (see [28, Definition 2.6]), and hence the transition matrix between $\{[\mathcal{O}_{\mathbf{Q}_G(w)}(\varpi_i)]\}_{w \in W_{\mathrm{af}}}$ and $\{[\mathcal{O}_{\mathbf{Q}_G(w)}]\}_{w \in W_{\mathrm{af}}}$ is unitriangular (up to diagonal matrix consisting of characters in P) with respect to $\le_{\frac{\infty}{2}}$. Therefore, we can invert this matrix to obtain $[\mathcal{O}_{\mathbf{Q}_G(w)}(-\varpi_i)] \in K_H(\mathbf{Q}_G^{\mathrm{rat}})$ for $i \in \mathtt{I}$. This implies that $\Xi(-\varpi_i)$ defines a well-defined automorphism of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ for each $i \in \mathtt{I}$ as required. $\square$

Lemma 1.14. *For each $i \in \mathtt{I}$, we have an equality*

$$[\mathcal{O}_{\mathbf{Q}_G(s_i)}] = [\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)]$$

obtained from the short exact sequence

$$0 \to \mathbb{C}_{\varpi_i} \otimes \mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i) \to \mathcal{O}_{\mathbf{Q}_G(e)} \to \mathcal{O}_{\mathbf{Q}_G(s_i)} \to 0 \qquad (1.7)$$

that is $\mathbf{I} \ltimes \mathbb{G}_m$-equivariant.

Proof. Since $\mathbf{Q}_G(e)$ is a normal scheme (see [28, Theorem 4.26]), a line bundle on it is completely determined via its restriction to an open subscheme whose codimension is at least two. The dense open $G[\![z]\!]$-orbit $\mathbb{O}$ in $\mathbf{Q}_G(e)$ has codimension two boundary since the boundary is $\bigcup_{i \in \mathtt{I}} \mathbf{Q}_G(t_{\alpha_i^\vee})$, and we have $t_{\alpha_i^\vee} <_{\frac{\infty}{2}} s_i <_{\frac{\infty}{2}} e$ by inspection. Hence, $\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)$ is determined by its restriction to $\mathbb{O}$, that is an (uncountable dimensional) affine fibration over $\mathcal{B}$. Here $\mathcal{O}_{\mathbb{O}}(-\varpi_i)$ is the pullback of $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ (cf. [28, Proof of Proposition 5.1]).

For $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$, we have (1.4) and it is known that $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ is a B-divisor twist of $\mathcal{O}_{\mathcal{B}}$. Thus, the corresponding statement prolongs to the whole $\mathbf{Q}_G(e)$ from $\mathbb{O}$. As a consequence, the short exact sequence (1.4) yields the short exact sequence (1.7). The resulting short exact sequence is $\mathbf{I} \ltimes \mathbb{G}_m$-equivariant as the natural surjection $\mathcal{O}_{\mathbf{Q}_G(e)} \to \mathcal{O}_{\mathbf{Q}_G(s_i)}$ is. □

Remark 1.15. Lemma 1.14 implies $[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)] \in K_H(\mathbf{Q}_G(e)) \subset K_H(\mathbf{Q}_G^{\mathrm{rat}})$. However, this does not imply $[\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(-\varpi_i)] \in K_H(\mathbf{Q}_G^{\mathrm{rat}})$ as $[\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}] \not\in K_H(\mathbf{Q}_G^{\mathrm{rat}})$. By the same reason, $\Xi(\lambda)$ in Theorem 1.13 (= tensor product with $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$) is the multiplication by an element in the ring $K_H(\mathbf{Q}_G(e))$, but not in $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. In fact, the definition of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ does not equip it a ring structure. This stems from the fact that $\mathbf{Q}_G^{\mathrm{rat}}$ is a(n infinite) union of Schubert varieties (whose dimensions are also infinite), but not a Schubert variety by itself.

Motivated by Lemma 1.14, we consider a $\mathbb{C}P$-module endomorphism H_i ($i \in \mathtt{I}$) of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ as:

$$H_i : [\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(w)}] - e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}_G(w)}(-\varpi_i)] \qquad w \in W_{\mathrm{af}}.$$

Proposition 1.16. *The space $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ is topologically generated by $[\mathcal{O}_{\mathbf{Q}(e)}(\lambda)]$ ($\lambda \in P$), together with the $\mathbb{C}P$-multiplications and the right $Q^\vee$-actions.*

Proof. Let $K_H(\mathbf{Q}_G(e))_+$ be the (formal) $\mathbb{C}P$-span of $\{[\mathcal{O}_{\mathbf{Q}_G(wt_\beta)}]\}_{w \in W, 0 \neq \beta \in Q^\vee_+}$ in $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. In view of (1.2), we have $K_H(\mathbf{Q}_G(e))/K_H(\mathbf{Q}_G(e))_+ \cong K_H(\mathcal{B})$ as $\mathbb{C}P$-modules that sends $[\mathcal{O}_{\mathbf{Q}_G(w)}]$ to $[\mathcal{O}_{\mathcal{B}(w)}]$ ($w \in W$). By Theorem 1.12 and Theorem 1.4, this intertwines the action of D_i ($i \in \mathtt{I}$). By the Pieri-Chevalley formula [28, Theorem 5.13], we see that

$$[\mathcal{O}_{\mathbf{Q}(w_0)}(\lambda)] \mod K_H(\mathbf{Q}_G(e))_+ = e^{w_0\lambda}[\mathcal{O}_{\mathcal{B}(w_0)}] = [\mathcal{O}_{\mathcal{B}(w_0)}(\lambda)] \qquad \lambda \in P.$$

By the Demazure character formulas ([25] and [33, VIII]), we conclude that

$$[\mathcal{O}_{\mathbf{Q}(w)}(\lambda)] \mod K_H(\mathbf{Q}_G(e))_+ = [\mathcal{O}_{\mathcal{B}(w)}(\lambda)] \qquad w \in W, \lambda \in P.$$

Therefore, the first two actions generate $K_H(\mathbf{Q}_G(e))/K_H(\mathbf{Q}_G(e))_+ \cong K_H(\mathcal{B})$ from $[\mathcal{O}_{\mathbf{Q}_G(e)}]$. Now we use the right $Q^\vee$-action to conclude the result. □

1.5 Equivariant quantum K-group of $\mathcal{B}$

We introduce a polynomial ring $\mathbb{C}Q^\vee_+$ and the formal power series ring $\mathbb{C}[\![Q^\vee_+]\!]$ with its variables $Q_i = Q^{\alpha_i^\vee}$ ($i \in \mathtt{I}$). We set $Q^\beta := \prod_{i \in \mathtt{I}} Q_i^{\langle \beta, \varpi_i \rangle}$ for each $\beta \in Q^\vee$. We define the H-equivariant (small) quantum K-group of $\mathcal{B}$ as:

$$qK_H(\mathcal{B}) := K_H(\mathcal{B}) \otimes \mathbb{C}[\![Q^\vee_+]\!]. \tag{1.8}$$

Thanks to (the H-equivariant versions of) [19, 36], it is equipped with the commutative and associative product $\star$ (called the quantum multiplication) with the following properties:

1. the element $[\mathcal{O}_{\mathcal{B}}] \otimes 1 \in qK_H(\mathcal{B})$ is the identity (with respect to $\cdot$ and $\star$);
2. the map $Q^\beta \star$ ($\beta \in Q^\vee_+$) is the multiplication of Q^β in the RHS of (1.8);
3. we have $\xi \star \eta \equiv \xi \cdot \eta \mod (Q_i; i \in \mathtt{I})$ for every $\xi, \eta \in K_H(\mathcal{B}) \otimes 1$.

From the above properties, we can localize $qK_H(\mathcal{B})$ with respect to the multiplicative system $\{Q^\beta\}_{\beta \in Q_+^\vee}$ to obtain a ring $qK_H(\mathcal{B})_{\mathrm{loc}}$.

We set

$$qK_{H\times\mathbb{G}_m}(\mathcal{B}) := K_H(\mathcal{B}) \otimes \mathbb{C}(\!(q^{-1})\!)[\![Q_+^\vee]\!].$$

We sometimes identify $K_H(\mathcal{B})$ with the submodule $K_H(\mathcal{B}) \otimes 1$ of $qK_H(\mathcal{B})$ or $qK_{H\times\mathbb{G}_m}(\mathcal{B})$. We set $p_i := [\mathcal{O}_{\mathcal{B}}(\varpi_i)]$ for $i \in \mathtt{I}$, and we consider it as an endomorphism of $qK_H(\mathcal{B})$ or $qK_{H\times\mathbb{G}_m}(\mathcal{B})$ through the scalar extension of the product of $K_H(\mathcal{B})$ (i.e. the classical product; we always understand that $p_i^{\pm 1}$ acts via the classical product). For each $i \in \mathtt{I}$, let $q^{Q_i \partial_{Q_i}}$ denote the $(\mathbb{C}P)(\!(q^{-1})\!)$-endomorphism of $qK_{H\times\mathbb{G}_m}(\mathcal{B})$ such that

$$q^{Q_i \partial_{Q_i}}(\xi \otimes Q^\beta) = q^{\langle \beta, \varpi_i \rangle} \xi \otimes Q^\beta \qquad \xi \in K_H(\mathcal{B}), \beta \in Q_+^\vee.$$

Following [24, §2.4], we consider the operator $T \in \mathrm{End}_{(\mathbb{C}P)(\!(q^{-1})\!)}\, qK_{H\times\mathbb{G}_m}(\mathcal{B})$ (obtained from the same named operator in [24] by setting $0 = t \in K(\mathcal{B})$). Then, we have the shift operator (also obtained from an operator $A_i(q,t)$ in [24] by setting $t = 0$) defined by

$$A_i(q) = T^{-1} \circ p_i^{-1} q^{Q_i \partial_{Q_i}} \circ T \in \mathrm{End}\, qK_{H\times\mathbb{G}_m}(\mathcal{B}) \qquad i \in \mathtt{I}. \tag{1.9}$$

An element $J(Q,q) := T([\mathcal{O}_{\mathcal{B}}]) \in qK_{H\times\mathbb{G}_m}(\mathcal{B})$ is called the (equivariant K-theoretic) small quantum J-function, and is computed in [20, 6] (cf. Theorem 3.8).

Theorem 1.17 (Reconstruction theorem [24] Proposition 2.20)**.** *For each*

$$f(q, x_1, \ldots, x_r, Q) \in \mathbb{C}P[q^{\pm 1}, x_1, \ldots, x_r][\![Q_+^\vee]\!],$$

we have the following equivalence:

$$\begin{aligned} f(q, p_1^{-1} q^{Q_1 \partial_{Q_1}}, \ldots, p_r^{-1} q^{Q_r \partial_{Q_r}}, Q) J(Q,q) = 0 \in qK_{H\times\mathbb{G}_m}(\mathcal{B}) \\ \Leftrightarrow f(q, A_1(q), \ldots, A_r(q), Q)[\mathcal{O}_{\mathcal{B}}] = 0 \in qK_{H\times\mathbb{G}_m}(\mathcal{B}). \end{aligned}$$

Remark 1.18. The original form of Theorem 1.17 is about big quantum K-group. We have made the specialization $t = 0$ to deduce our form. It should be noted that **1)** this equivariant setting is automatic from the construction, and **2)** we state Theorem 1.17 for unmodified quantum J-function instead of the modified one employed in [24, Proposition 2.20].

For each $i \in \mathtt{I}$, we set $a_i := A(1)$ (thanks to [24, Remark 2.14]).

Theorem 1.19 ([24] Corollary 2.9)**.** *For $i \in \mathtt{I}$, the operator a_i defines the quantum multiplication by $a_i([\mathcal{O}_{\mathcal{B}}])$ in $qK_H(\mathcal{B})$.*

Proof. By [24, Corollary 2.9], the set $\{a_i\}_{i \in \mathtt{I}}$ defines mutually commutative endomorphisms of $qK_H(\mathcal{B})$ that commutes with the $\star$-multiplication. Since $\mathrm{End}_R R \cong R$ for every ring R, we conclude the assertion. $\square$

Theorem 1.20 (Anderson-Chen-Tseng [2] Lemma 6, see also [1])**.** *For each $i \in \mathtt{I}$, we have $A_i(q)([\mathcal{O}_{\mathcal{B}}]) = [\mathcal{O}_{\mathcal{B}}(-\varpi_i)] \in qK_H(\mathcal{B}) \subset qK_{H\times\mathbb{G}_m}(\mathcal{B})$.*

We give an alternative proof of Theorem 1.20 just after Theorem 4.1.

2 Relation with affine Grassmannians

We work in the same settings as in the previous section.

Theorem 2.1. *We have a $\mathcal{H}$-module embedding*

$$\Phi : K_H(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends the Pontryagin product on the LHS to the tensor product on the RHS. More precisely, we have: For each $i \in \mathtt{I}$ and $\xi \in K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$, it holds

$$\Phi(\mathbf{h}_i \odot \xi) = H_i(\xi).$$

In addition, the image of Φ is precisely the set of finite linear combinations of Schubert classes, that forms a dense subset of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$.

Remark 2.2. **1)** It is known that $\{\mathbf{h}_i\}_{i\in\mathtt{I}}$, $\mathbb{C}P$, and $\{\mathtt{t}_\beta\}_{\beta\in Q^\vee}$ generates the ring $K_H(\mathrm{Gr}_G)_{\mathrm{loc}}$. One way to prove it is to compare $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ with its original definition in [28, §5]; **2)** We add extra $\mathbb{G}_m$-action on $K_H(\mathrm{Gr}_G)$ and prove an analogue of Theorem 2.1 in [27] on the basis of the results presented here.

2.1 Example: $SL(2)$-case

Assume that $G = SL(2)$. We make an identification $P_+ = \mathbb{Z}_{\geq 0}\varpi$, and $Q_+^\vee = \mathbb{Z}_{\geq 0}\{\alpha^\vee = \alpha\}$. We have $W = \{e, s\}$. Let $\mathtt{t}$ denote the right $Q^\vee$-action on $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ (or $K_{\mathbf{I}\ltimes\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$) corresponding to $\alpha^\vee$, and let q denote the character of $\mathbb{G}_m$ that acts on the variable z (in $G(\!(z)\!)$) by degree one character (so-called the loop rotation action).

The Pieri-Chevalley rule for ϖ ([28, Theorem 5.13]) yields the equations:

$$[\mathcal{O}_{\mathbf{Q}_G(e)}(\varpi)] = \frac{1}{1-q^{-1}\mathtt{t}}(e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}])$$
$$[\mathcal{O}_{\mathbf{Q}_G(s)}(\varpi)] = \frac{1}{1-q^{-1}\mathtt{t}}(q^{-1}e^{\varpi}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]).$$

Forgetting the extra $\mathbb{G}_m$-action yield:

$$[\mathcal{O}_{\mathbf{Q}_G(e)}(\varpi)] = \frac{1}{1-\mathtt{t}}(e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}])$$
$$[\mathcal{O}_{\mathbf{Q}_G(s)}(\varpi)] = \frac{1}{1-\mathtt{t}}(e^{\varpi}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]).$$

Inverting this equation yields that

$$[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi)] = e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{-\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}]$$
$$[\mathcal{O}_{\mathbf{Q}_G(s)}(-\varpi)] = -e^{\varpi}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}].$$

Therefore, we obtain

$$[\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi)] = [\mathcal{O}_{\mathbf{Q}_G(s)}]$$
$$[\mathcal{O}_{\mathbf{Q}_G(s)}] - e^{\varpi}[\mathcal{O}_{\mathbf{Q}_G(s)}(-\varpi)] = e^{\alpha}\mathtt{t}[\mathcal{O}_{\mathbf{Q}_G(e)}] + (1-e^{\alpha})[\mathcal{O}_{\mathbf{Q}_G(s)}].$$

By Theorem 2.1, this transplants to

$$[\mathcal{O}_{\mathrm{Gr}_{st_{-\alpha}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{t_{-m\alpha}}}] = [\mathcal{O}_{\mathrm{Gr}_{st_{-(m+1)\alpha}}}]$$
$$[\mathcal{O}_{\mathrm{Gr}_{st_{-\alpha}}}] \odot [\mathcal{O}_{\mathrm{Gr}_{st_{-m\alpha}}}] = e^{\alpha}[\mathcal{O}_{\mathrm{Gr}_{t_{-m\alpha}}}] + (1-e^{\alpha})[\mathcal{O}_{\mathrm{Gr}_{st_{-(m+1)\alpha}}}].$$

for $m > 0$. This coincides with the calculation in [34, (17)].

2.2 Transporting the $\mathcal{H}$-action to $\mathcal{C}$

Proposition 2.3. *The $\mathcal{H}$-action of $K_H(\mathrm{Gr})$ induces a $\mathcal{H}$-action on $\mathcal{C}$ as:*

$$\begin{aligned} D_0(f \otimes t_\beta) &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes t_{s_\vartheta(\beta-\vartheta^\vee)} & f \in \mathbb{C}(P) \\ D_i(f \otimes t_\beta) &= \frac{f}{1-e^{\alpha_i}} \otimes t_\beta - \frac{e^{\alpha_i} s_i(f)}{1-e^{\alpha_i}} \otimes t_{s_i\beta} & i \in \mathtt{I}, \beta \in Q^\vee \\ e^\mu(f \otimes t_\beta) &= e^\mu f \otimes t_\beta & \mu \in P. \end{aligned}$$

Proof. For $i \in \mathtt{I}_{\mathrm{af}}$, the action of D_i on $\mathcal{A}$ is the left multiplication of $\frac{1}{1-e^{\alpha_i}} \otimes 1 - \frac{e^{\alpha_i}}{1-e^{\alpha_i}} \otimes s_i$ (if we understand $\alpha_0 = -\vartheta$). Applying to an element $f \otimes t_\beta u \in \mathcal{A}$ ($f \in \mathbb{C}(P), \beta \in Q^\vee, u \in W$), we deduce

$$\begin{aligned} D_i(f \otimes t_\beta u) &= \frac{f}{1-e^{\alpha_i}} \otimes t_\beta u - \frac{e^{\alpha_i} s_i(f)}{1-e^{\alpha_i}} \otimes t_{s_i\beta} s_i u \quad i \neq 0 \\ D_0(f \otimes t_\beta u) &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta u - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes s_0 t_\beta u \\ &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta u - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes s_\vartheta t_{-\vartheta^\vee} t_\beta u \\ &= \frac{f}{1-e^{-\vartheta}} \otimes t_\beta u - \frac{e^{-\vartheta} s_\vartheta(f)}{1-e^{-\vartheta}} \otimes t_{s_\vartheta(\beta-\vartheta^\vee)} s_\vartheta u. \end{aligned}$$

Hence, applying pr yields the desired formula on D_i for $i \in \mathtt{I}_{\mathrm{af}}$. Together with the left multiplication of $e^\lambda \otimes 1$, these formula transplants the $\mathcal{H}$-action from $K_H(\mathrm{Gr})$ to $K_H(\mathrm{Gr}) \cap \mathcal{C}$.

Since $K_H(\mathrm{Gr}) = \mathcal{C} \cap K_H(X)$, we have $\mathbb{C}(P) \otimes_{\mathbb{C}P} K_H(\mathrm{Gr}) \subset \mathcal{C}$. By comparing the leading terms of $\{[\mathrm{Gr}_\beta]\}_{\beta \in Q^\vee} \subset \mathcal{C}$ with respect to the Bruhat order (on the second component of $\mathcal{C} \subset \mathcal{A} = \mathbb{C}(P) \otimes \mathbb{C}W_{\mathrm{af}}$), we derive $\mathcal{C} \subset \mathbb{C}(P) \otimes_{\mathbb{C}P} K_H(\mathrm{Gr})$. It follows that $\mathcal{C} = \mathbb{C}(P) \otimes_{\mathbb{C}P} K_H(\mathrm{Gr})$. Hence, the above formulas define the $\mathcal{H}$-action on $\mathcal{C}$ as the scalar extension of that on $K_H(\mathrm{Gr}) \subset \mathcal{C}$ as required (one can also directly check the relations of $\mathcal{H}$). □

Below, we may write the action of D_i on $\mathcal{C}$ by $D_i^\#$ to distinguish with the action on $K_H(X)$ or $\mathcal{A}$.

Corollary 2.4. *Let $i \in \mathtt{I}$. Let $\xi \in \mathcal{C}$ be an element such that $D_i^\#(\xi) = \xi$. Then, ξ is a $\mathbb{C}$-linear combination of*

$$f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta} \qquad f \in \mathbb{C}(P), \beta \in Q^\vee.$$

Proof. By Proposition 2.3, the action of $D_i^\#$ preserves $\mathbb{C}(P) \otimes t_\beta + \mathbb{C}(P) \otimes t_{s_i\beta}$ for each $i \in \mathtt{I}$ and $\beta \in Q^\vee$. Hence, it suffices to find a condition that $a \otimes t_\beta + b \otimes t_{s_i\beta}$ ($a, b \in \mathbb{C}(P)$) is stable by the action of $D_i^\#$. It reads as:

$$\begin{aligned} D_i^\#(a \otimes t_\beta + b \otimes t_{s_i\beta}) &= \frac{a - e^{\alpha_i} s_i(b)}{1-e^{\alpha_i}} \otimes t_\beta + \frac{b - e^{\alpha_i} s_i(a)}{1-e^{\alpha_i}} \otimes t_{s_i\beta} \\ &= a \otimes t_\beta + b \otimes t_{s_i\beta}. \end{aligned}$$

This is equivalent to $b = s_i(a)$ (or $s_i(a+b) = a+b$ in the case of $s_i\beta = \beta$) as required. □

Corollary 2.5. *Let $i \in \mathtt{I}$. Let $\xi, \xi' \in \mathcal{C}$ be elements such that $D_i^{\#}(\xi) = \xi$. We have*

$$D_i^{\#}(\xi\xi') = \xi D_i^{\#}(\xi').$$

Proof. By Corollary 2.4, it suffices to prove

$$D_i^{\#}((f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})g \otimes t_\gamma) = (f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})D_i^{\#}(g \otimes t_\gamma)$$

for every $f, g \in \mathbb{C}(P)$ and $\beta, \gamma \in Q^\vee$. We derive as:

$$\begin{aligned} D_i^{\#}((f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})g \otimes t_\gamma) &= D_i^{\#}(fg \otimes t_{\beta+\gamma} + s_i(f)g \otimes t_{s_i\beta+\gamma}) \\ &= \frac{fg}{1-e^{\alpha_i}} \otimes t_{\beta+\gamma} - \frac{e^{\alpha_i}s_i(fg)}{1-e^{\alpha_i}} \otimes t_{s_i\beta+s_i\gamma} \\ &+ \frac{s_i(f)g}{1-e^{\alpha_i}} \otimes t_{s_i\beta+\gamma} - \frac{e^{\alpha_i}fs_i(g)}{1-e^{\alpha_i}} \otimes t_{\beta+s_i\gamma} \\ &= (f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})(\frac{g}{1-e^{\alpha_i}} \otimes t_\gamma - \frac{e^{\alpha_i}s_i(g)}{1-e^{\alpha_i}} \otimes t_{s_i\gamma}) \\ &= (f \otimes t_\beta + s_i(f) \otimes t_{s_i\beta})D_i^{\#}(g \otimes t_\gamma). \end{aligned}$$

This completes the proof. □

Lemma 2.6. *Let $\xi, \xi' \in \mathcal{C}$ be elements such that $D_i^{\#}(\xi) = \xi$ for every $i \in \mathtt{I}$. We have*

$$D_0^{\#}(\xi\xi') = \xi D_0^{\#}(\xi').$$

Proof. By Corollary 2.4, we deduce $w\xi w^{-1} = \xi \in \mathcal{A}$ for every $w \in W$. In particular, we have $s_\vartheta \xi s_\vartheta = \xi$.

Therefore, it suffices to prove

$$D_0^{\#}((f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})g \otimes t_\gamma) = (f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})D_0^{\#}(g \otimes t_\gamma)$$

for every $f, g \in \mathbb{C}(P)$ and $\beta, \gamma \in Q^\vee$. We derive as:

$$\begin{aligned} D_0^{\#}((f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})g \otimes t_\gamma) &= D_0^{\#}(fg \otimes t_{\beta+\gamma} + s_\vartheta(f)g \otimes t_{s_\vartheta\beta+\gamma}) \\ &= \frac{fg}{1-e^{-\vartheta}} \otimes t_{\beta+\gamma} - \frac{e^{-\vartheta}s_\vartheta(fg)}{1-e^{-\vartheta}} \otimes t_{s_\vartheta\beta+s_\vartheta(\gamma-\vartheta^\vee)} \\ &+ \frac{s_\vartheta(f)g}{1-e^{-\vartheta}} \otimes t_{s_\vartheta\beta+\gamma} - \frac{e^{-\vartheta}fs_\vartheta(g)}{1-e^{-\vartheta}} \otimes t_{\beta+s_\vartheta(\gamma-\vartheta^\vee)} \\ &= (f \otimes t_\beta + s_\vartheta(f) \otimes t_{s_\vartheta\beta})D_0^{\#}(g \otimes t_\gamma). \end{aligned}$$

This completes the proof. □

Theorem 2.7. *For each $\beta \in Q^\vee_<$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have*

$$D_i([\mathcal{O}_{\mathrm{Gr}_\beta}] \odot \xi) = [\mathcal{O}_{\mathrm{Gr}_\beta}] \odot D_i(\xi) \qquad \xi \in K_H(\mathrm{Gr}).$$

Proof. By construction, we have

$$\pi^*([\mathcal{O}_{\mathrm{Gr}_\beta}]) = D_{t_\beta}D_{w_0} = D_iD_{t_\beta}D_{w_0} \qquad i \in \mathtt{I},$$

where the second identity follows from $\ell(t_\beta) = \ell(s_it_\beta) + 1$. By Proposition 2.3, we deduce that $r^*([\mathcal{O}_{\mathrm{Gr}_\beta}])$ satisfies the $D_i^{\#}$-invariance for each $i \in \mathtt{I}$. Therefore, Corollaries 2.5 and 2.6 imply the result. □

Corollary 2.8. *For $\beta \in Q^\vee_<$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have $D_i = \mathtt{t}_{-\beta} \circ D_i \circ \mathtt{t}_\beta$. In particular, we have a natural extension of the $\mathcal{H}$-action from $K_H(\mathrm{Gr})$ to $K_H(\mathrm{Gr})_{\mathrm{loc}}$.*

Proof. The first assertion is a direct consequence of Theorem 2.7. As we have $K_H(\mathrm{Gr})_{\mathrm{loc}} = K_H(\mathrm{Gr})[\mathtt{t}_\beta \mid \beta \in Q^\vee_<]$, the latter assertion follows. $\square$

2.3 Inclusion as $\mathcal{H}$-modules

Lemma 2.9. *Let $i \in \mathtt{I}_{\mathrm{af}}$. For each $w \in W^-_{\mathrm{af}}$, we have*

$$D_i([\mathcal{O}_{\mathrm{Gr}_w}]) = \begin{cases} [\mathcal{O}_{\mathrm{Gr}_{s_i w}}] & (s_i w >_{\frac{\infty}{2}} w) \\ [\mathcal{O}_{\mathrm{Gr}_w}] & (s_i w <_{\frac{\infty}{2}} w) \end{cases}.$$

Proof. By Theorem 1.7 and Corollary 2.8, we can replace $[\mathcal{O}_{\mathrm{Gr}_w}]$ with $[\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}]$ for $\beta \in Q^\vee$ such that $\langle \beta, \varpi_i \rangle \ll 0$ for all $i \in \mathtt{I}$. Therefore, the assertion is a rephrasement of Theorem 1.4 and Theorem 1.5 as $s_i w >_{\frac{\infty}{2}} w$ is equivalent to $s_i w t_\beta > w t_\beta$ (see (1.1)). $\square$

Lemma 2.10. *The vector space $K_H(\mathrm{Gr})_{\mathrm{loc}}$ is a cyclic module with respect to the action of $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$ with its cyclic vector $[\mathcal{O}_{\mathrm{Gr}_0}]$.*

Proof. By construction, it suffices to find every $\{[\mathcal{O}_{\mathrm{Gr}_\beta}]\}_{\beta \in Q^\vee}$ in the linear span of $\mathcal{H} \cdot \{\mathtt{t}_\gamma \odot [\mathcal{O}_{\mathrm{Gr}_0}]\}_{\gamma \in Q^\vee}$. This follows from a repeated application of the actions of $\{D_i\}_{i \in \mathtt{I}_{\mathrm{af}}}$ and Theorem 1.7 (cf. [25, Theorem 4.6]). $\square$

Corollary 2.11. *An endomorphism ψ of $K_H(\mathrm{Gr})_{\mathrm{loc}}$ as a $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$-module is completely determined by the image of $[\mathcal{O}_{\mathrm{Gr}_0}]$.* $\square$

Proposition 2.12. *By sending $[\mathcal{O}_{\mathrm{Gr}_0}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(e)}]$, we have a unique injective $\mathcal{H}$-module morphism*

$$K_H(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

such that twisting by $\mathtt{t}_\beta$ corresponds to the right action of $\beta \in Q^\vee$. This map particularly gives

$$[\mathcal{O}_{\mathrm{Gr}_{ut_\beta}}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(ut_\beta)}] \qquad u \in W, \beta \in Q^\vee_<.$$

Proof. The comparison of the D_i-actions on the basis elements in Lemma 2.9 and Theorem 1.12 implies that we indeed obtain a $\mathcal{H}$-module inclusion, by enhancing the assignment $[\mathcal{O}_{\mathrm{Gr}_{ut_\beta}}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(ut_\beta)}]$ for $u \in W, \beta \in Q^\vee_<$ into a $\mathbb{C}P$-module homomorphism. We know the actions of $\mathtt{t}_\beta$ and β on the both sides by Theorem 1.7 and Theorem 1.10, that coincide on elements that generates $K_H(\mathrm{Gr})_{\mathrm{loc}}$ by the actions of $\mathbb{C}P$ and $\{\mathtt{t}_\beta\}_{\beta \in Q^\vee}$. Hence, we deduce a $\mathcal{H}$-module embedding $K_H(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$ that intertwines the $\mathtt{t}_\beta$-action to the right β-action. Such an embedding must be unique by Corollary 2.11. $\square$

Remark 2.13. Lemma 2.9 is a purely combinatorial statement about the comparison of two orders on W_{af}. As a consequence, we obtain an embedding

$$K_{\mathbf{I} \ltimes \mathbb{G}_m}(\mathrm{Gr}_G) \hookrightarrow K_{\mathbf{I} \ltimes \mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$$

of nil-DAHA modules (with a parameter q) that sends $[\mathcal{O}_{\mathrm{Gr}_w}]$ to $[\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}(w)}]$ for each $w \in W^-_{\mathrm{af}}$ (cf. [32, 35, 28]; see [27] for its further consequences).

2.4 Proof of Theorem 2.1

This subsection is entirely devoted to the proof of Theorem 2.1. The embedding part of Theorem 2.1 is already proved in Proposition 2.12. It also implies that the image of this embedding is the set of finite linear combinations of Schubert classes.

Let $i \in \mathtt{I}$. We have an endomorphism $\Xi(-\varpi_i)$ of $K_H(\mathbf{Q}_G^{\mathrm{rat}})$ that commutes with the right $Q^\vee$-action and the left $\mathcal{H}$-action. By Lemma 1.14, the image of $[\mathcal{O}_{\mathbf{Q}_G(e)}]$ under $\Xi(-\varpi_i)$ belongs to the image of $K_H(\mathrm{Gr})_{\mathrm{loc}}$. In particular, $\Xi(-\varpi_i)$ induces a $\mathcal{H}$-module endomorphism of $K_H(\mathrm{Gr})_{\mathrm{loc}}$.

In order to identify the endomorphisms $\mathbf{h}_i\odot$ and H_i, it suffices to compare some linear combination with the well-understood element, namely id. Therefore, we compare the endomorphisms of $K_H(\mathrm{Gr})_{\mathrm{loc}}$ (as $\mathbb{C}P$-modules) induced by

$$\Theta_i := e^{-\varpi_i}(\mathrm{id} - \mathbf{h}_i\odot)$$

and

$$\Xi(-\varpi_i) = e^{-\varpi_i}(\mathrm{id} - H_i).$$

The both endomorphisms send $[\mathcal{O}_{\mathrm{Gr}_0}]$ to

$$e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \odot [\mathcal{O}_{\mathrm{Gr}_{t_\beta}}]^{-1} \qquad (\beta \in Q^\vee_<)$$

by Proposition 2.12, Lemma 1.8, and Lemma 1.14.

We prove that the both of Θ_i and $\Xi(-\varpi_i)$ commute with the $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$-action. It is Theorem 1.13 for $\Xi(-\varpi_i)$. Hence, we concentrate on the action of Θ_i.

The action of Θ_i commutes with $\mathbb{C}P \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$ as $(K_H(\mathrm{Gr})_{\mathrm{loc}}, \odot)$ is a commutative ring. Thus, Corollaries 2.5 and 2.6 (and Theorem 2.7) reduces the problem to

$$D_j(e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}])) = e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \qquad j \in \mathtt{I}, \beta \in Q^\vee_<.$$

If $j \neq i$, then we have $s_j s_i t_\beta < s_i t_\beta$ and $s_j t_\beta < t_\beta$. Moreover, we have $D_j(e^{-\varpi_i}\bullet) = e^{-\varpi_i} D_j(\bullet)$. It follows that

$$\begin{aligned} D_j(e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}])) &= e^{-\varpi_i} D_j([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &= e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]). \end{aligned}$$

If $j = i$, then we compute as

$$\begin{aligned} D_i(e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}])) &= e^{-\varpi_i+\alpha_i} D_i([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &\quad + \frac{e^{-\varpi_i} - e^{-\varpi_i+\alpha_i}}{1 - e^{\alpha_i}}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &= e^{-\varpi_i+\alpha_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{t_\beta}}]) \\ &\quad + e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]) \\ &= e^{-\varpi_i}([\mathcal{O}_{\mathrm{Gr}_\beta}] - [\mathcal{O}_{\mathrm{Gr}_{s_i t_\beta}}]). \end{aligned}$$

Hence, Θ_i defines an endomorphism of $K_H(\mathrm{Gr})$ that commutes with the $\mathcal{H} \otimes \mathbb{C}[\mathtt{t}_\gamma \mid \gamma \in Q^\vee]$-action.

Therefore, Corollary 2.11 guarantees $\Theta_i = \Xi(-\varpi_i) \in \mathrm{End}(K_H(\mathrm{Gr})_{\mathrm{loc}})$. From this, we also deduce $\mathbf{h}_i\odot = H_i \in \mathrm{End}(K_H(\mathrm{Gr})_{\mathrm{loc}})$ as required.

3 Relation with quantum K-group

We continue to work in the setting of the previous section.

Theorem 3.1. *We have a $\mathbb{C}P$-module isomorphism*

$$\Psi : qK_H(\mathcal{B})_{\mathrm{loc}} \xrightarrow{\cong} K_H(\mathbf{Q}_G^{\mathrm{rat}}),$$

that sends $[\mathcal{O}_{\mathcal{B}}]$ to $[\mathcal{O}_{\mathbf{Q}_G(e)}]$, quantum product of a line bundle $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ ($i \in \mathtt{I}$) to the tensor product of $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(-\varpi_i)$, and the multiplication by Q^β to the right $Q^\vee$-action of β for each $\beta \in Q^\vee$.

Proof. Combine Theorem 3.11 and Theorem 1.20 (cf. Theorem 1.19). □

Corollary 3.2 (Conjectured by Lam-Li-Mihalcea-Shimozono [34])**.** *We have a natural $\mathbb{C}P$-algebra dense embedding*

$$\Psi^{-1} \circ \Phi : K_H(\mathrm{Gr})_{\mathrm{loc}} \hookrightarrow qK_H(\mathcal{B})_{\mathrm{loc}},$$

such that

$$\Psi^{-1} \circ \Phi([\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}][\mathcal{O}_{\mathrm{Gr}_\gamma}]^{-1}) = [\mathcal{O}_{\mathcal{B}(w)}]Q^{\beta-\gamma} \quad w \in W \tag{3.1}$$

holds for every $\beta, \gamma \in Q^\vee_<$.

Proof. For the first assertion, combine Theorem 2.1 and Theorem 3.1 to obtain the map $\Psi^{-1} \circ \Phi$, that have dense image. Note that the both sides are rings and the identity $[\mathcal{O}_{\mathrm{Gr}_0}]$ goes to the identity $[\mathcal{O}_{\mathcal{B}}]$. The map $\Psi^{-1} \circ \Phi$ commutes with the natural $Q^\vee$-actions given by $\mathtt{t}_\gamma$ and Q^γ for each $\gamma \in Q^\vee$ by Proposition 2.12 and Theorem 3.1. Moreover, the action of Θ_i (see §2.4) and the quantum multiplication by $[\mathcal{O}_{\mathcal{B}}(-\varpi_i)]$ corresponds for each $i \in \mathtt{I}$ (by Theorem 2.1 and Theorem 3.1). Therefore, the $\odot$-multiplication by the element $\mathbf{h}_i$ and $\star$-multiplication by $[\mathcal{O}_{\mathcal{B}(s_i)}] = ([\mathcal{O}_{\mathcal{B}}] - e^{\varpi_i}[\mathcal{O}_{\mathcal{B}}(-\varpi_i)])$ coincide for each $i \in \mathtt{I}$. Since the ring $K_H(\mathrm{Gr})_{\mathrm{loc}}$ is generated by $\{\mathbf{h}_i\}_{i\in\mathtt{I}}$ up to the $\mathbb{C}P$-action and $\{\mathtt{t}_\gamma\}_\gamma$-action (Remark 2.2), we conclude that $\Psi^{-1} \circ \Phi$ is a ring embedding.

For the second assertion, note that the Proposition 2.12 asserts that

$$\Phi([\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{\pm 1}) = [\mathcal{O}_{\mathbf{Q}_G(wt_{\beta\pm\gamma})}] \quad w \in W$$

for each $\beta, \gamma \in Q^\vee_<$ (cf. Lemma 1.9). From this, we derive

$$\begin{aligned}\Psi^{-1} \circ \Phi([\mathcal{O}_{\mathrm{Gr}_{wt_\beta}}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}]^{\pm 1}) &= \Psi^{-1}([\mathcal{O}_{\mathbf{Q}_G(w)}])Q^{\beta\pm\gamma} && \text{by Theorem 3.1}\\ &= [\mathcal{O}_{\mathcal{B}(w)}]Q^{\beta\pm\gamma} && \text{by Theorem 4.1.}\end{aligned}$$

This yields the desired equality. □

In view of [34], we obtain another proof of the finiteness of quantum K-theory of $\mathcal{B}$ originally proved in Anderson-Chen-Tseng [1, 2]. We reproduce the reasoning here for the sake of reference:

Corollary 3.3 (Anderson-Chen-Tseng [1, 2])**.** *For each $w, v \in W$, we have*

$$[\mathcal{O}_{\mathcal{B}(w)}] \star [\mathcal{O}_{\mathcal{B}(v)}] \in \bigoplus_{\beta \in Q^\vee_+, u \in W} \mathbb{C}P[\mathcal{O}_{\mathcal{B}(u)}]Q^\beta.$$

In other words, the multiplication rule of $qK_H(\mathcal{B})$ is finite.

Proof of Corollary 3.3 due to Lam-Li-Mihalcea-Shimozono [34]. By Corollary 3.2 (cf. Theorem 1.7), the assertion follows from

$$[\mathcal{O}_{\mathrm{Gr}_\beta}] \odot [\mathcal{O}_{\mathrm{Gr}_\gamma}] \in \bigoplus_{\kappa \in Q^\vee} \mathbb{C}P[\mathcal{O}_{\mathrm{Gr}_\kappa}] \quad \forall \beta, \gamma \in Q^\vee. \tag{3.2}$$

By definition, the LHS of (3.2) is a product inside the ring $\mathcal{C}$ that has $\{[\mathcal{O}_{\mathrm{Gr}_\kappa}]\}_\kappa$ as its $\mathbb{C}P$-basis (Theorem 1.6). Hence, the assertion follows. □

3.1 Quasi-map spaces

Here we recall basics of quasi-map spaces from [16, 12].

We have W-equivariant isomorphisms $H^2(\mathcal{B}, \mathbb{Z}) \cong P$ and $H_2(\mathcal{B}, \mathbb{Z}) \cong Q^\vee$. This identifies the (integral points of the) nef cone of $\mathcal{B}$ with $P_+ \subset P$ and the effective cone of $\mathcal{B}$ with $Q^\vee_+$. A quasi-map (f, D) is a map $f : \mathbb{P}^1 \to \mathcal{B}$ together with a $\Pi^\vee$-colored effective divisor

$$D = \sum_{\alpha \in \Pi^\vee, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha^\vee) \alpha^\vee \otimes [x] \in Q^\vee \otimes_{\mathbb{Z}} \mathrm{Div}\, \mathbb{P}^1 \quad \text{with} \quad m_x(\alpha^\vee) \in \mathbb{Z}_{\geq 0}. \tag{3.3}$$

We call D the defect of (f, D), and $\sum_{\alpha \in \Pi^\vee} m_x(\alpha^\vee)\alpha^\vee$ the defect of (f, D) at $x \in \mathbb{P}^1(\mathbb{C})$. Here we define the total defect of (f, D) by

$$|D| := \sum_{\alpha \in \Pi^\vee, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha^\vee)\alpha^\vee \in Q^\vee_+.$$

For each $\beta \in Q^\vee_+$, we set

$$\mathcal{Q}(\mathcal{B}, \beta) := \{f : \mathbb{P}^1 \to X \mid \text{ quasi-map s.t. } f_*[\mathbb{P}^1] + |D| = \beta\},$$

where $f_*[\mathbb{P}^1]$ is the class of the image of $\mathbb{P}^1$ multiplied by the degree of $\mathbb{P}^1 \to \mathrm{Im}\, f$. We denote $\mathcal{Q}(\mathcal{B}, \beta)$ by $\mathcal{Q}(\beta)$ in case there is no danger of confusion.

Definition 3.4 (Drinfeld-Plücker data)**.** Consider a collection $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in P_+}$ of inclusions $\psi_\lambda : \mathcal{L}^\lambda \hookrightarrow L(\lambda) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ of line bundles $\mathcal{L}^\lambda$ (as coherent subsheaves) over $\mathbb{P}^1$. The data $\mathcal{L}$ is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of G-modules

$$\eta_{\lambda,\mu} : L(\lambda + \mu) \hookrightarrow L(\lambda) \otimes L(\mu)$$

induces an isomorphism

$$\eta_{\lambda,\mu} \otimes \mathrm{id} : \psi_{\lambda+\mu}(\mathcal{L}^{\lambda+\mu}) \xrightarrow{\cong} \psi_\lambda(\mathcal{L}^\lambda) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \psi_\mu(\mathcal{L}^\mu)$$

for every $\lambda, \mu \in P_+$.

Theorem 3.5 (Drinfeld, see Finkelberg-Mirković [16])**.** *The variety $\mathcal{Q}(\beta)$ is isomorphic to the variety formed by isomorphism classes of the DP-data $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in P_+}$ such that* $\deg \mathcal{L}^\lambda = -\langle \beta, \lambda \rangle$.

For each $w \in W$, let $\mathcal{Q}(\beta, w) \subset \mathcal{Q}(\beta)$ be the closure of the set formed by quasi-maps that are defined at $z = 0$ (i.e. have no defect at 0), and their values at $z = 0$ are contained in $\mathcal{B}(w) \subset \mathcal{B}$. (Hence, we have $\mathcal{Q}(\beta) = \mathcal{Q}(\beta, e)$.) If $\mathcal{Q}(\beta, w) \neq \emptyset$, then we have

$$\dim \mathcal{Q}(\beta, w) = 2 \langle \beta, \rho \rangle + \dim \mathcal{B}(w) \tag{3.4}$$

by [16, Proposition 3.5]. The irreducibility of $\mathcal{Q}(\beta, w)$ is implicitly contained in [25], and is stated in [26, Theorem B].

For each $\lambda \in P$, $w \in W$, and $\beta \in Q_+^\vee$, we have a G-equivariant line bundle $\mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)$ obtained by the (tensor product of the) pull-backs $\mathcal{O}_{\mathcal{Q}(\beta,w)}(\varpi_i)$ of the i-th $\mathcal{O}(1)$ via the embedding

$$\mathcal{Q}(\beta, w) \hookrightarrow \prod_{i \in \mathtt{I}} \mathbb{P}(L(\varpi_i)^* \otimes_{\mathbb{C}} \mathbb{C}[z]_{\leq \langle \beta, \varpi_i \rangle}). \tag{3.5}$$

Using this, we set

$$\chi(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}}(\lambda)) := \sum_{i \geq 0} \mathrm{gch}\, H^i(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) \in \mathbb{C}[P][q^{-1}] \quad \beta \in Q^\vee, \lambda \in P_+,$$

where the grading q is understood to count the degree of z detected by the $\mathbb{G}_m$-action. Here we understand that $\chi(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) = 0$ if $\beta \notin Q_+^\vee$.

We have $\mathcal{B} = \mathcal{Q}(0)$ by the Plücker embedding. By expanding the map $\mathbb{P}^1 \to \mathcal{B} \to \mathbb{P}(L(\lambda)^*)$ ($\lambda \in P_+$) into a collection of formal power series $L(\lambda)^* \otimes \mathbb{C}[\![z]\!]$, we find an embedding $\mathcal{Q}(\beta) \subset \mathbf{Q}_G(e)$ (the best reference is [26, §3.4]). These result in embeddings $\mathcal{B} \subset \mathcal{Q}(\beta) \subset \mathbf{Q}_G(e)$ such that the line bundles $\mathcal{O}(\lambda)$ ($\lambda \in P$) corresponds to each other by restrictions ([7, 25, 28]). The intersection of $\mathcal{Q}(\beta) \cap \mathbb{O}(w) \subset \mathcal{Q}(\beta, w)$ defines an open dense subset for each $w \in W$.

Here we temporarily switch to the complex analytic topology in order to state Theorem 3.6. (Although the whole results are of algebraic nature as recorded in [5, 9], it looks simpler to present them by their analytic counterparts.) For each $\beta \in Q_+^\vee$, we set $\mathcal{Z}(\beta) := \mathcal{Q}(\beta) \cap \mathbb{O}(w_0)$ and call it the Zastava space (of degree β). We set

$$C^{(\beta)} := \prod_{i \in \mathtt{I}} \left(C^{\langle \beta, \varpi_i \rangle} / \mathfrak{S}_{\langle \beta, \varpi_i \rangle} \right)$$

for a Riemann surface C (or a finite set of points). For each $i \in \mathtt{I}$, a point $(f, D) \in \mathcal{Z}(\beta)$ defines an element of $u_i(f, D) \in L(\varpi_i)^* \otimes_{\mathbb{C}} \mathbb{C}[z]$ through (3.5) for each $i \in \mathtt{I}$ (up to a scalar multiple), that also yields a polynomial $\phi_i(f, D; z) \in \mathbb{C}[z]$ by pairing with the lowest weight vector of $L(\varpi_i)$. By examining the roots of $\phi_i(f, D; z)$ (the multiplicity at ∞ is understood as $\langle \beta, \varpi_i \rangle - \deg \phi_i(f, D; z)$), we obtain the factorization morphism

$$\pi^\beta : \mathcal{Z}(\beta) \longrightarrow (\mathbb{P}^1 \setminus \{0\})^{(\beta)}$$

since $0 \in \mathbb{P}^1$ is never a root of such polynomials (see e.g. [16, §5.2.2]). By construction, the point $\pi^\beta(f, D) \in (\mathbb{P}^1)^{(\beta)}$ contains at least $\langle \beta_x, \varpi_i \rangle$-copies of the point $x \in \mathbb{P}^1$ in the i-th configuration for $(f, D) \in \mathcal{Z}(\beta)$ with $D = \sum_{x \in \mathbb{P}^1} \beta_x \otimes [x]$.

Theorem 3.6 (Finkelberg-Mirković [16] §6.3.2)**.** *Let β, β_1, β_2 be elements in $Q_+^\vee$ such that $\beta = \beta_1 + \beta_2$, and let $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{P}^1 \setminus \{0\}$ be a pair of disjoint complex analytic open subsets. We have an isomorphism of complex analytic sets*

$$(\pi^\beta)^{-1}(\mathcal{U}_1^{(\beta_1)} \times \mathcal{U}_2^{(\beta_2)}) \cong (\pi^{\beta_1})^{-1}(\mathcal{U}_1^{(\beta_1)}) \times (\pi^{\beta_2})^{-1}(\mathcal{U}_2^{(\beta_2)}),$$

where $\mathcal{U}_1^{(\beta_1)} \times \mathcal{U}_2^{(\beta_2)} \subset (\mathbb{P}^1)^{(\beta)}$ is a natural inclusion.

Remark 3.7. **1)** Theorem 3.6 implies that $\mathcal{Z}(\beta)$, and hence also $\mathcal{Q}(\beta)$, must be irreducible; **2)** In [6, 8], it is established that $\mathcal{Z}(\beta)$ is a normal variety and the morphism π^β is flat (in the course of their proof of Theorem 4.5).

3.2 Quantum J-functions and generating functions

In this subsection, we reformulate results provided in Givental-Lee [20] and Braverman-Finkelberg [6]. Hence, both of the "theorems" in this subsection are understood as blends of their results, and their "proofs" are just explanations on how they work.

Theorem 3.8. *There exists an element $J'(Q,q) \in \mathbb{C}P(q)[\![Q^\vee_+]\!]$ with the following properties:*

1. *the composition of maps*

$$\mathbb{C}P[\![q]\!][\![Q^\vee_+]\!] \cong K_G(\mathcal{B})[\![q]\!][\![Q^\vee_+]\!] \subset K_H(\mathcal{B})[\![q]\!][\![Q^\vee_+]\!]$$

sends $J'(Q,q)$ to $J(Q,q)$;

2. *for each $\lambda \in P$, we have an identity in $\mathbb{C}P[q^{-1}][\![Q^\vee_+]\!]$:*

$$D_{w_0}(J'(Qq^\lambda,q)e^{w_0\lambda}J'(Q,q^{-1})) = \sum_{\beta \in Q^\vee_+} \chi(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}}(\lambda))Q^\beta,$$

where we understand that Qq^λ sends Q^β to $Q^\beta q^{-\langle \beta,\lambda\rangle}$ for each $\lambda \in P$.

Proof. For the first assertion, it is actually (the individual Q^β-coefficients of) $J'(Q,q)$ that is calculated as the graded characters of the rings of regular functions of Zastava spaces in [6, 8]. As explained in [6, §1.3], these graded character counts the push-forward of the (corresponding part of the) denominator of the localization formula for the corresponding Kontsevich graph spaces. This can be understood to be a definition of the (K-theoretic) J-function as in [20, §2.2] (since the push-forward of the structure sheaf does not depend on the choice of a resolution; cf. Remark 4.4).

The second assertion is equivalent to [7, Lemma 5]. □

For $\vec{n} = (n_1,\ldots,n_r) \in \mathbb{Z}^r_{\geq 0}$, we set $x^{\vec{n}} := x_1^{n_1}\cdots x_r^{n_r}$. For $\lambda \in P$, we set $\lambda[\vec{n}] := \lambda - \sum_{i=1}^r n_i\varpi_i$.

Theorem 3.9. *For each $\sum_{\beta \in Q^\vee_+, \vec{n}\in\mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(q)x^{\vec{n}}Q^\beta \in \mathbb{C}P[q^{\pm 1}, x_1,\ldots,x_r][\![Q^\vee_+]\!]$ such that*

$$\sum_{\beta \in Q^\vee_+, \vec{n}\in\mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(q) \otimes_{R(G)[q^{\pm1}]} \left(\prod_{j=1}^r (p_i^{-1}q^{Q_i\partial_{Q_i}})^{n_i}\right) Q^\beta J(Q,q) = 0, \qquad (3.6)$$

we have the following equalities:

$$\sum_{\beta \in Q^\vee_+, \vec{n}\in\mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(q)q^{-\langle\beta,\lambda[\vec{n}]\rangle}\chi(\mathcal{Q}(\gamma-\beta), \mathcal{O}_{\mathcal{Q}}(\lambda[\vec{n}])) = 0 \quad \lambda \in P_+, \gamma \in Q^\vee_+.$$

Proof. The assertion is [20, §4.2] (see also [7, Lemma 5] and [8, §5]), that employs the localization theorem applied to the graph spaces with no marked points (we refer to §4.1 for notation).

Here we demonstrate an alternative proof (it depends on the argument in the previous paragraph through Theorem 3.8, though). We can substitute Q with Qq^{λ} in (3.6) multiplied with $e^{w_0\lambda}$. By factoring out the effect of additional powers of q coming from $q^{Q_i\partial_{Q_i}}$'s, we derive a formula

$$\sum_{\beta\in Q_+^\vee,\vec{n}\in\mathbb{Z}_{\geq 0}^r} f_{\beta,\vec{n}}(q)\otimes_{R(G)[q^{\pm 1}]} q^{-\langle\beta,\lambda[\vec{n}]\rangle}Q^{\beta}J'(Qq^{\lambda[\vec{n}]},q)e^{w_0(\lambda[\vec{n}])}=0.$$

Applying Theorem 3.8 2), we conclude the desired equation. $\square$

3.3 Identification of defining equations

Proposition 3.10. *For each $\lambda\in P$, we have*

$$\lim_{\beta\to\infty}\chi(\mathcal{Q}(\beta),\mathcal{O}_{\mathcal{Q}(\beta)}(\lambda))=\operatorname{gch} H^0(\mathbf{Q}_G(e),\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda))\in\mathbb{C}[\![q^{-1}]\!][P]. \tag{3.7}$$

Moreover, we have

$$H^{>0}(\mathbf{Q}_G(e),\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda))=\{0\}.$$

Proof. The limit in (3.7) exists, and it gives the character of the (dual of the) global Weyl module by [7, §4.2] and [8] (here we use Theorem 3.8 2)). $\square$

Theorem 3.11. *We have a well-defined $\mathbb{C}P$-linear isomorphism*

$$\Psi: qK_H(\mathcal{B})_{\mathrm{loc}}\longrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends $[\mathcal{O}_{\mathcal{B}}]$ to $[\mathcal{O}_{\mathbf{Q}_G(e)}]$, the quantum multiplication by a_i to the endomorphism $\Xi(-\varpi_i)$ ($i\in\mathtt{I}$), and the multiplication by Q^β to the right $Q^\vee$-action of β for each $\beta\in Q^\vee$.

Remark 3.12. Our proof of Theorem 3.11 says that Ψ is actually the $q=1$ specialization of the isomorphism

$$\Psi_q:\mathbb{C}[q^{\pm 1}]\otimes_{\mathbb{C}} qK_H(\mathcal{B})_{\mathrm{loc}}\longrightarrow K_{\mathbf{I}\ltimes\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$$

such that $\Psi_q\circ A_i(q)=\Xi_q(-\varpi_i)\circ\Psi_q$ ($i\in\mathtt{I}$), where $\Xi_q(-\varpi_i)$ denotes the tensor product of $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(-\varpi_i)$ in the $\mathbb{G}_m$-equivariant setting (see [28, §5 and §6]).

Proof of Theorem 3.11. By Theorem 1.17, it suffices to start from

$$f=\sum_{\beta\in Q_+^\vee,\vec{n}\in\mathbb{Z}_{\geq 0}^r} f_{\beta,\vec{n}}(q)x^{\vec{n}}Q^{\beta}\in\operatorname{Ann}_{\mathbb{C}P[q^{\pm 1},x_1,\ldots,x_r][\![Q_+^\vee]\!]}J(Q,q)$$

(where x_i acts on $J(Q,q)$ as $p_i^{-1}q^{Q_i\partial_{Q_i}}$ for each $i\in\mathtt{I}$) and find the corresponding relation in $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. By Theorem 3.9, the equation

$$f(q,p_1^{-1}q^{Q_1\partial_{Q_1}},\ldots,p_r^{-1}q^{Q_r\partial_{Q_r}},Q)J(Q,q)=0$$

implies

$$\sum_{\beta\in Q_+^\vee,\vec{n}\in\mathbb{Z}_{\geq 0}^r} f_{\beta,\vec{n}}(q)q^{-\langle\beta,\lambda[\vec{n}]\rangle}\chi(\mathcal{Q}(\gamma-\beta),\mathcal{O}_{\mathcal{Q}(\gamma-\beta)}(\lambda[\vec{n}]))=0\quad\lambda\in P_+,\gamma\in Q_+^\vee.$$

By Proposition 3.10 and [28, Proposition D.1], this further implies

$$\sum_{\beta\in Q^\vee_+,\vec{n}\in\mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(q)\mathrm{gch}\, H^0(\mathbf{Q}(t_\beta),\mathcal{O}_{\mathbf{Q}(t_\beta)}(\lambda[\vec{n}]))=0 \quad \lambda\in P_+.$$

Taking [28, Corollary 5.9] into account (and the fact that our K-group intersects with the dense subset of the K-group in [28, §6]), we derive

$$\sum_{\beta\in Q^\vee_+,\vec{n}\in\mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(q)[\mathcal{O}_{\mathbf{Q}(t_\beta)}(-\sum_{i=1}^r n_i\varpi_i)]=0\in K_{\mathbf{I}\ltimes\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}}),$$

whose $q=1$ specialization (possible by our choice of f) yields

$$\sum_{\beta\in Q^\vee_+,\vec{n}\in\mathbb{Z}^r_{\geq 0}} f_{\beta,\vec{n}}(1)[\mathcal{O}_{\mathbf{Q}(t_\beta)}(-\sum_{i=1}^r n_i\varpi_i)]=0\in K_H(\mathbf{Q}_G^{\mathrm{rat}}).$$

This induces a $\mathbb{C}P$-linear map $\Psi: qK_H(\mathcal{B})_{\mathrm{loc}}\longrightarrow K_H(\mathbf{Q}_G^{\mathrm{rat}})$ that sends $[\mathcal{O}_\mathcal{B}]$ to $[\mathcal{O}_{\mathbf{Q}_G(e)}]$, and the multiplication by Q^β to the right multiplication by β for each $\beta\in Q^\vee$. In Theorem 3.8 2), the multiplication by $p_i^{-1}q^{Q_i\partial_{Q_i}}$ on the first factor $J'(Qq^\lambda,q)$ results in the line bundle twist by $\mathcal{O}_{\mathfrak{Q}(\beta)}(-\varpi_i)$. Hence, it corresponds to the line bundle twist by $\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)$ in $K_H(\mathbf{Q}_G^{\mathrm{rat}})$. In view of Theorem 1.17 (and the definition of the shift operators), the quantum multiplication by a_i becomes the endomorphism $\Xi(-\varpi_i)$ $(i\in\mathtt{I})$ via Ψ.

By Proposition 1.16, the map Ψ is surjective. It must be injective as the both sides are free modules of rank $|W|$ over the ring $\mathbb{C}P\otimes(\mathbb{C}Q^\vee\otimes_{\mathbb{C}Q^\vee_+}\mathbb{C}[\![Q^\vee_+]\!])$ (see Lemma 1.11). □

4 Identification of the bases

Keep the setting of the previous sections. We prove the following result in order to complete the proof of Corollary 3.2.

Theorem 4.1. *The map Ψ constructed in Theorem 3.1 restricts to an isomorphism $qK_H(\mathcal{B})\cong K_H(\mathbf{Q}_G(e))$ of $\mathbb{C}P\otimes\mathbb{C}Q^\vee_+$-modules, and we have*

$$\Psi([\mathcal{O}_{\mathcal{B}(w)}])=[\mathcal{O}_{\mathbf{Q}(w)}]\qquad w\in W.$$

Since Theorem 1.20 is used only when we reformulate Theorem 3.11 into Theorem 3.1, we obtain an alternative proof of the following:

Theorem 4.2 (= Theorem 1.20 due to Anderson-Chen-Tseng)**.** *For each $i\in\mathtt{I}$, we have $A_i(q)([\mathcal{O}_\mathcal{B}])=[\mathcal{O}_\mathcal{B}(-\varpi_i)]$.*

Proof. By Theorem 3.11 and Remark 3.12, we know that $\Psi_q(A_i(q)([\mathcal{O}_\mathcal{B}]))=[\mathcal{O}_{\mathbf{Q}_G}(e)(-\varpi_i)]$. Now we argue as:

$$\begin{aligned}A_i(q)([\mathcal{O}_\mathcal{B}])&=\Psi_q^{-1}([\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)])\\&=e^{-\varpi_i}\Psi_q^{-1}([\mathcal{O}_{\mathbf{Q}_G(e)}]-[\mathcal{O}_{\mathbf{Q}_G(s_i)}]) && \text{by Lemma 1.14}\\&=e^{-\varpi_i}([\mathcal{O}_\mathcal{B}(e)]-[\mathcal{O}_{\mathcal{B}(s_i)}]) && \text{by Theorem 4.1}\\&=[\mathcal{O}_\mathcal{B}(-\varpi_i)] && \text{by (1.3).}\end{aligned}$$

Here we remark that the short exact sequence (1.7) yields an equality also in $K_{\mathbf{I}\ltimes\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$. These imply the result. □

Corollary 4.3. *For each $i \in \mathtt{I}$ and $w \in W$, the element $A_i(q)([\mathcal{O}_{\mathcal{B}(w)}])$ is a finite $\mathbb{C}[q^{\pm 1}]P$-linear combination of $\{[\mathcal{O}_{\mathcal{B}(w)}]Q^{\beta}\}_{w \in W, \beta \in Q_+^\vee}$.*

Proof. By Remark 3.12 and Theorem 4.1, the problem reduces to the corresponding problem in $K_{\mathbf{I} \ltimes \mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$. The latter is explained in either [27, Theorem 3.7] (as in Corollary 3.3) or [41, Theorem 1] (as an explicit formula), though author's original reasoning is by the finiteness of the (global version of the) decomposition procedure in [15] (as their global generalized Weyl modules are exactly $\Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda))^*$; see e.g. [25, §5]). □

4.1 Graph and map spaces and their line bundles

We refer [31, 17, 20] for the precise definitions of the notions appearing in this subsection.

For each non-negative integer n and $\beta \in Q_+^\vee$, we set $\mathcal{GB}_{n,\beta}$ to be the space of stable maps of genus zero curves with n-marked points to $(\mathbb{P}^1 \times \mathcal{B})$ of bidegree $(1, \beta)$, that is also called the graph space of $\mathcal{B}$. A point of $\mathcal{GB}_{n,\beta}$ is a(n arithmetic) genus zero curve C with n-marked points $\{x_1, \ldots, x_n\}$, together with a map to $\mathbb{P}^1$ of degree one. Hence, we have a unique $\mathbb{P}^1$-component of C that maps isomorphically onto $\mathbb{P}^1$. We call this component the main component of C and denote it by C_0. For a genus zero curve C, let $|C|$ denote the number of its irreducible components. The space $\mathcal{GB}_{n,\beta}$ is a normal projective variety by [17, Theorem 2] that have at worst quotient singularities arising from the automorphism of stable maps. The natural $(H \times \mathbb{G}_m)$-action on $(\mathbb{P}^1 \times \mathcal{B})$ induces a natural $(H \times \mathbb{G}_m)$-action on $\mathcal{GB}_{n,\beta}$. Moreover, $\mathcal{GB}_{0,\beta}$ has only finitely many isolated $(H \times \mathbb{G}_m)$-fixed points, and thus we can apply the formalism of Atiyah-Bott-Lefschetz localization (cf. [20, p200L26] and [7, Proof of Lemma 5]).

We have a morphism $\pi_{n,\beta} : \mathcal{GB}_{n,\beta} \to \mathcal{Q}(\beta)$ that factors through $\mathcal{GB}_{0,\beta}$ (Givental's main lemma [21]; see [12, §8] and [17, §1.3]). Let $\widetilde{\mathtt{ev}}_j : \mathcal{GB}_{n,\beta} \to \mathbb{P}^1 \times \mathcal{B}$ $(1 \le j \le n)$ be the evaluation at the j-th marked point, and let $\mathtt{ev}_j : \mathcal{GB}_{n,\beta} \to \mathcal{B}$ be its composition with the second projection.

A variety $\mathfrak{Y}$ is said to have rational singularities if there exists a resolution of singularities $g : \mathfrak{Z} \to \mathfrak{Y}$ that satisfies $g_* \mathcal{O}_{\mathfrak{Z}} \cong \mathcal{O}_{\mathfrak{Y}}$ and $\mathbb{R}^{>0}(g)_* \mathcal{O}_{\mathfrak{Z}} \cong \{0\}$ ([30, Theorem 5.10]). This is equivalent to assume that all the resolutions of singularities of $\mathfrak{Y}$ have the same property ([*loc. cit.*]).

By the theorem on formal functions ([22, II §11]), the rationality of singularities on a normal variety (i.e. the vanishing of the higher direct images of the structure sheaf from a resolution) is detected by its completion. Therefore, in case $\mathfrak{Y}$ is a normal variety with its (not necessarily closed) point y, we say the completion of $\mathfrak{Y}$ along y has a rational singularity if $\mathfrak{Y}$ has a rational singularity along y (cf. [43, Lemma 15.51.6]).

Remark 4.4. In the below, Theorem 4.5 and other assertions about the rational resolutions of singularities $f : \mathfrak{X} \to \mathfrak{Y}$ are used only to ensure $f_* \mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_{\mathfrak{Y}}$ and $\mathbb{R}^{>0}(f)_* \mathcal{O}_{\mathfrak{X}} \cong \{0\}$. To this end, it suffices to assume that f is a birational projective morphism and $\mathfrak{X}$ has only rational singularities (or quotient singularities by [30, Proposition 5.15]) as we can replace $\mathfrak{X}$ with its resolution of singularities.

Theorem 4.5 (Braverman-Finkelberg [6, 7, 8])**.** *The morphism $\pi_{0,\beta}$ is a rational resolution of singularities* (*in an orbifold sense*). □

Since $\mathcal{Q}(\beta)$ is irreducible (Remark 3.7), Theorem 4.5 asserts that $\mathcal{GB}_{n,\beta}$ is irreducible ([44, 29]).

For each $\lambda \in P$, we have a line bundle $\mathcal{O}_{\mathcal{GB}_{n,\beta}}(\lambda) := \pi_{n,\beta}^* \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)$. For a $(H \times \mathbb{G}_m)$-equivariant coherent sheaf on a projective $(H \times \mathbb{G}_m)$-variety $\mathcal{X}$, let $\chi(\mathcal{X}, \mathcal{F}) \in \mathbb{C}P[q^{\pm 1}]$ denote its $(H \times \mathbb{G}_m)$-equivariant Euler-Poincaré characteristic (that enhances the element $\chi(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda))$ defined in §3.1).

Let $\mathcal{X}(\beta)$ denote the subvariety of $\mathcal{GB}_{2,\beta}$ consisting of the stable maps whose first marked point projects to $0 \in \mathbb{P}^1$, and whose second marked point projects to $\infty \in \mathbb{P}^1$ through the projection of a genus zero domain curve C to the main component $C_0 \cong \mathbb{P}^1$. Let us denote the restriction of $\mathtt{ev}_i$ ($i = 1, 2$) to $\mathcal{X}(\beta)$ by the same letter. By Theorem 4.5, $\mathcal{X}(\beta)$ also gives a resolution of singularities $\pi_\beta : \mathcal{X}(\beta) \to \mathcal{Q}(\beta)$. The following is a result of Buch-Chaput-Mihalcea-Perrin [10]:

Theorem 4.6 ([10] Corollary 3.8, cf. [26] Theorem 4.1)**.** *The variety*

$$\mathtt{ev}_1^{-1}(\mathcal{B}(w)) \cap \mathtt{ev}_2^{-1}(\mathcal{B}^{\mathrm{op}}(v)) \subset \mathcal{X}(\beta)$$

is irreducible, normal, and has rational singularities (that we denote by $\mathcal{X}(\beta, w, v)$) for each $w, v \in W$. □

We remark that $\mathcal{X}(\beta) = \mathcal{X}(\beta, e, w_0)$. We set $\mathcal{X}(\beta, w) := \mathcal{X}(\beta, w, w_0)$ and $\mathcal{Q}(\beta, w, v) := \pi_\beta(\mathcal{X}(\beta, w, v)) \subset \mathcal{Q}(\beta, w)$ (this latter space is the Richardson variety of a semi-infinite flag manifolds studied in [26, §4]). Then, the map π_β restricts to a $(H \times \mathbb{G}_m)$-equivariant birational proper map

$$\pi_{\beta,w,v} : \mathcal{X}(\beta, w, v) \to \mathcal{Q}(\beta, w, v)$$

by [26, §4.3]. We denote π_{β,w,w_0} by $\pi_{\beta,w}$ for simplicity. Let $\mathcal{O}_{\mathcal{X}(\beta,w,v)}(\lambda)$ denote the restriction of $\mathcal{O}_{\mathcal{GB}_{2,\beta}}(\lambda)$ to $\mathcal{X}(\beta, w, v)$ for each $\lambda \in P$.

4.2 Cohomology calculation for $\mathcal{X}(\beta, w)$

Theorem 4.7 ([26])**.** *For each $w, v \in W$ and $\beta \in Q_+^\vee$, we have:*

1. *The variety $\mathcal{Q}(\beta, w, v)$ is normal;*
2. *We have $H^{>0}(\mathcal{Q}(\beta, w, v), \mathcal{O}_{\mathcal{Q}(\beta,w,v)}(\lambda)) = \{0\}$ for each $\lambda \in P_+$;*
3. *For $\beta' \in Q_+^\vee$ such that $\beta < \beta'$ and $\lambda \in P_+$, the natural restriction map*
$$H^0(\mathcal{Q}(\beta', w), \mathcal{O}_{\mathcal{Q}(\beta',w)}(\lambda)) \longrightarrow H^0(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda))$$
is surjective;
4. *Let $i \in \mathtt{I}$ be such that $s_i w < w$ and $s_i v < v$. Then, the variety $\mathcal{Q}(\beta, w, v)$ is B_i-stable, and we have an inflation map $\pi_i : SL(2, i) \times^{B_i} \mathcal{Q}(\beta, w, v) \to \mathcal{Q}(\beta, s_i w, v)$. We have $\mathbb{R}^\bullet(\pi_i)_* \mathcal{O}_{SL(2,i) \times^{B_i} \mathcal{Q}(\beta,w,v)} \cong \mathcal{O}_{\mathcal{Q}(\beta,s_i w,v)}$;*
5. *Assume that $s_\vartheta w > w$ and $s_\vartheta v > v$. Then, the variety $\mathcal{Q}(\beta, w, v)$ admits a B_0-action and we have an inflation map*
$$\pi_0 : SL(2, 0) \times^{B_0} \mathcal{Q}(\beta, w, v) \to \mathcal{Q}(\beta - w^{-1}\vartheta^\vee, s_\vartheta w, v).$$
We have $\mathbb{R}^\bullet(\pi_0)_* \mathcal{O}_{SL(2,0) \times^{B_0} \mathcal{Q}(\beta,w,v)} \cong \mathcal{O}_{\mathcal{Q}(\beta - w^{-1}\vartheta^\vee, s_\vartheta w, v)}$.

Proof. The first item is [26, Theorem 4.19]. The second and the third items are [26, Theorem 3.33]. The fourth item follows from [26, Proposition 3.39] and the discussion in [26, §4.2]. The same is true for the fifth item if we make a suitable renormalization of the translation part by [26, Lemma 3.7]. □

Theorem 4.8. *For each $w \in W$ and $\beta \in Q^\vee_+$, the variety $\mathcal{Q}(\beta, w)$ has rational singularities. In particular, $\mathcal{Q}(\beta, w)$ is Cohen-Macauley.*

Proof. We examine more general assertion that when $\mathcal{Q}(\beta, w, v)$ ($w, v \in W, \beta \in Q^\vee_+$) has rational singularities until the last two paragraphs of this proof. By the normality of $\mathcal{Q}(\beta, w, v)$ and the fact that all the fibers of $\pi_{\beta,w,v}$ are connected ([26, Corollary 4.18]), we have

$$(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)} \cong \mathcal{O}_{\mathcal{Q}(\beta,w,v)}.$$

Let us consider the open subset $\mathring{\mathcal{Q}}(\beta, w, v) \subset \mathcal{Q}(\beta, w, v)$ consisting of quasi-maps defined at $0 \in \mathbb{P}^1$ (i.e. have no defect at 0) and their values at 0 belong to $\mathbb{O}_{\mathcal{B}}(w)$. The variety $\mathcal{Q}(\beta, w, v)$ is decomposed into the disjoint union of $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$'s, where $0 \le \beta' \le \beta$ and $w' \in W$, and the inclusion map is given by adding the defect β' at 0 (cf. (3.3)). We put $\mathring{\mathcal{Q}}(\beta, w) := \mathring{\mathcal{Q}}(\beta, w, w_0)$. Similarly, we consider the open subset $\mathring{\mathcal{Q}}(\beta) \subset \mathcal{Q}(\beta)$ consisting of quasi-maps defined at zero (we have $\mathring{\mathcal{Q}}(\beta) = G\mathring{\mathcal{Q}}(\beta, w)$ for every $w \in W$).

We also consider the open subsets $\dot{\mathcal{Q}}(\beta, w, v) \subset \mathring{\mathcal{Q}}(\beta, w, v)$ consisting of quasi-maps without defect at $\infty \in \mathbb{P}^1$ (and they have no defects at $0, \infty$).

Claim A. *The inclusion $\mathring{\mathcal{Q}}(\beta - \beta', w', v) \subset \mathcal{Q}(\beta, w, v)$ induces a locally trivial fiber bundle over $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$ whose fiber is the complete local ring of its transversal slice. In addition, its fiber does not depend on β and v.*

Proof. In case $w = w' = v = e$, we can identify $\dot{\mathcal{Q}}(\beta - \beta', e, e)$ with an open subset of $\mathcal{Z}(\beta - \beta')$ by swapping $0, \infty \in \mathbb{P}^1$ through the involutions $z \mapsto z^{-1}$ and $B \to B^-$. In the same vein, we can take $\mathcal{Z}(\beta) \subset \mathcal{Q}(\beta, e, e)$ as an open neighbourhood of $\dot{\mathcal{Q}}(\beta - \beta', e, e)$. Let $p_0 \in \mathcal{Z}(\beta')$ be the unique $(H \times \mathbb{G}_m)$-fixed point. Let $\mathcal{U} \subset \mathbb{A}^{(\beta)}$ be a (complex analytic) open subset of the form $(U_1)^{(\beta')} \times (U_2)^{(\beta - \beta')}$, where two open subsets $0 \in U_1, U_2 \subset \mathbb{A}^1$ satisfy $U_1 \cap U_2 = \emptyset$. On the π^β-preimage of $\mathcal{U}$, the transversal slice of $\dot{\mathcal{Q}}(\beta - \beta', e, e) \subset \mathcal{Z}(\beta)$ is isomorphic to an open neighbourhood of $p_0 \in \mathcal{Z}(\beta')$ given as the factor $(\pi^{\beta'})^{-1}((U_1)^{(\beta')})$ of $(\pi^\beta)^{-1}(\mathcal{U})$ by Theorem 3.6 (here we warn that 0 and ∞ in $\mathbb{P}^1$ are swapped).

By the definition of π^β, we can apply the N^--action to move the configuration points in $\mathbb{A}^{(\beta)}$ that does not come from defects. More precisely, if a point of $\dot{\mathcal{Q}}(\beta - \beta', e, e) \subset \mathcal{Z}(\beta)$ have additional multiplicity $[0]$ (in its image of π^β) as nondefect, then some of the coordinate $u_i \in L(\varpi_i)^* \otimes \mathbb{C}[z]$ ($i \in \mathtt{I}$) have zero of order $\langle \beta', \varpi_i \rangle$ at $z = 0$ and it has zero of order $> \langle \beta', \varpi_i \rangle$ at $z = 0$ after paired with the highest weight part of $L(\varpi_i)$. Thus, we can utilize the N^--action to through in the lower degree part to remove the additional multiplicity at $[0]$. This can be understood as the rearrangement of the transversal slices. In particular, we have a locally trivial family of transversal slices along an open neighbourhood of each point of $\dot{\mathcal{Q}}(\beta - \beta', e, e)$.

The local rings of transversal slices (at meeting points) do not depend on the choice of a local trivialization of the family of transversal slices up to an isomorphism. By replacing local rings with the completed local rings, we can further identify those arising from either of the complex analytic/algebraic versions of the transversal slices.

Since the inclusion $\dot{\mathcal{Q}}(\beta - \beta', e, e) \subset \mathcal{Z}(\beta)$ is $(B^- \times \mathbb{G}_m)$-stable, the above description of transversal slices prolongs to the whole $\dot{\mathcal{Q}}(\beta - \beta', e, e)$ as a $(B^- \times \mathbb{G}_m)$-equivariant locally trivial family of complete local rings (through the presentation by the normal sheaf) whose fiber is isomorphic to $\mathcal{O}^{\wedge}_{\mathcal{Z}(\beta'),p_0}$.

Taking into account the G-action, the transversal slices of $\dot{\mathcal{Q}}(\beta - \beta', e, v) \subset \mathcal{Q}(\beta, e, v)$ give rise to a locally trivial family of complete local rings for each v. It gives rise to a family of complete local rings over $\dot{\mathcal{Q}}(\beta - \beta')$ equipped with the $(G \times \mathbb{G}_m)$-action. In this case, $\infty \in \mathbb{P}^1$ and other points in $\mathbb{P}^1 \setminus \{0\}$ can be swapped by using an automorphism of $\mathbb{P}^1$ that fixes 0. Therefore, we can extend our family to the whole of $\mathring{\mathcal{Q}}(\beta - \beta') \subset \mathcal{Q}(\beta)$. This family inherits the $(G \times \mathbb{G}_m)$-action. Moreover, this construction is also compatible with varying β by adding defects at $\infty \in \mathbb{P}^1$ (by using an automorphism of $\mathbb{P}^1$). This also implies that a local trivialization of this family always extend by adding defects at $\infty \in \mathbb{P}^1$. It results in an inductive system of the locally trivial family of complete local rings on an ind-scheme $\bigcup_{\beta \in Q^\vee_+} \mathring{\mathcal{Q}}(\beta - \beta')$. This inductive system admits an additional ind-action of $G[z] \ltimes \mathbb{G}_m \subset G[\![z]\!] \ltimes \mathbb{G}_m$, where $\mathbb{G}_m$ is the loop rotation.

The projective limit of the coordinate ring of our family has truncation morphisms that are $\mathbb{G}_m$-equivariant. Its local trivializations can be also (taken to be) $\mathbb{G}_m$-stable and compatible with these truncation morphisms by construction. Taking the $\mathbb{G}_m$-finite part along the zero section, we obtain a $G[\![z]\!] \ltimes \mathbb{G}_m$-equivariant fiber bundle $\mathcal{F}'$ on $G\mathbb{O}(t_{\beta'})$ whose fiber is isomorphic to $\mathcal{O}^{\wedge}_{\mathcal{Z}(\beta'),p_0}$ (this family must be isomorphic to the locally trivial family of complete local rings of transversal slices from $G\mathbb{O}(t_{\beta'})$ to $G\mathbb{O}(e)$, that is $G[\![z]\!]$-equivariant). The restriction of $\mathcal{F}'$ to each $\mathring{\mathcal{Q}}(\beta - \beta')$ recovers the family of complete local rings of transversal direction.

Now we restrict $\mathcal{F}'$ to $\mathbb{O}(w' t_{\beta'})$, and add (i.e. take a completed tensor product with) the complete local ring of the transversal direction to $\mathbf{Q}_G(t_{\beta'})$, that is the same as the complete local ring of a transversal slice from $\mathbb{O}_{\mathcal{B}}(w')$ to $\mathcal{B}$ (it is a regular local ring). This yields an $\mathbf{I}$-equivariant family $\mathcal{F}''$ of complete local rings on $\mathbb{O}(w' t_{\beta'})$. Now we intersect $\mathcal{F}''$ with $\mathbf{Q}(w)$ to obtain an $\mathbf{I}$-equivariant family $\mathcal{F}$ of local rings. The fibers of $\mathcal{F}''$ are the local counterparts of transversal slices from $\mathbb{O}(w' t_{\beta'})$ to $\mathbf{Q}_G(e)$, and hence the fibers of $\mathcal{F}$ are the local counterparts of transversal slices from $\mathbb{O}(w' t_{\beta'})$ to $\mathbf{Q}_G(w)$. The fiber space $\mathcal{F}$ is locally trivial as $\mathcal{F}'$ and $\mathcal{F}''$ are so and construction is homogeneous under the action of $\mathbf{I}$. Since the family $\mathcal{F}'$ restricts to the family of complete local rings of the transversal slices from $\mathring{\mathcal{Q}}(\beta - \beta')$ to $\mathcal{Q}(\beta)$, we further conclude that $\mathcal{F}$ restricts to the family of complete local rings of transversal slices from $\mathring{\mathcal{Q}}(\beta - \beta', w')$ to $\mathcal{Q}(\beta, w)$. This restriction is the desired locally trivial fiber bundle.

Imposing different β or $v \neq w_0$ is obtained by restricting the above fiber bundle, and hence we conclude that the fiber does not depend on v. □

Claim B. *The transversal slice of $\mathring{\mathcal{Q}}(\beta - \beta', w', v) \subset \mathcal{Q}(\beta, w, v)$ in Claim A have rational singularities along $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$.*

Proof. We assume to the contrary to deduce contradiction. A product of local rings have rational singularities if and only if each of them admits rational singularities. By Claim A, if a transversal slice of $\mathring{\mathcal{Q}}(\beta - \beta', w', v) \subset \mathcal{Q}(\beta, w, v)$ have singularities worse than rational singularities along $\mathring{\mathcal{Q}}(\beta - \beta', w', v)$, then we have

$$\mathcal{Q}(\beta - \beta', w', v) \subset \operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}. \tag{4.1}$$

This containment is independent of the choice of v, though the LHS maybe an empty set. We enlarge β if necessary to guarantee $\mathcal{Q}(\beta - \beta', w', v) \neq \emptyset$ for every $v \in W$, and some point (and hence general points) of $\mathcal{Q}(\beta - \beta', w', v)$ has no defect at ∞ and its value belongs to $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v)$ by [16, Lemma 8.5.1]. Let $Z(v)$ be an irreducible component of $\operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}$ that contains $\mathcal{Q}(\beta - \beta', w', v)$. General points of $Z(v)$ have no defect at ∞, and their values at ∞ belong to $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v)$. We set $U := N \cap \dot{v} N \dot{v}^{-1}$. Then, the multiplication map $U \times \mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v) \subset \mathcal{B}$ defines an embedding of an open dense subset. Thus, $\overline{U \times Z(v)}$ is an irreducible component of the support of (4.1) for $v = w_0$ that contains $\mathcal{Q}(\beta - \beta', w', w_0)$. Therefore, we obtain a family $\{Z(v)\}_{v \in W}$ of irreducible components of $\operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}$ such that $\mathcal{Q}(\beta - \beta', w', v) \subset Z(v)$, $Z(v) \subset Z(u)$ if $v \le u$, and $\overline{BZ(v)}$ is independent of v.

Let $i \in \mathtt{I}_{\mathrm{af}}$ be such that $SL(2,i)Z(v) \not\subset Z(v)$ for $i \neq 0$ and $s_i v < v$, or $i = 0$ and $s_\vartheta v > v$, inside $\mathcal{Q}(\beta)$ $(i \neq 0)$ or $\mathcal{Q}(\beta - (w)^{-1}\vartheta)$ ($i = 0$, see Theorem 4.7 4)). Since B_i acts on $\mathcal{Q}(\beta, w, v)$ (by Theorem 4.7), it follows that $Z(v)$ is B_i-stable. We have $s_i w < w$ $(i \neq 0)$ or $s_\vartheta w > w$ $(i = 0)$ as otherwise $SL(2,i)$ acts on $\mathcal{Q}(\beta, w, v)$ and hence on $Z(v)$, that is a contradiction.

The map π_i restricted to $SL(2,i) \times^{B_i} Z(v)$ is birational onto its image. In particular, there exists a Zariski open subset $V \subset SL(2,i)Z(v)$ such that $\pi_i^{-1}(V) \cap SL(2,i) \times^{B_i} Z(v)$ forms an irreducible component of

$$\pi_i^{-1}(V) \cap \left(SL(2,i) \times^{B_i} \operatorname{Supp} \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}\right) \subset SL(2,i) \times^{B_i} \mathcal{Q}(\beta, w, v).$$

Hence, $(\pi_i)_*$ sends the inflation of $\mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)}$ to a non-zero sheaf whose support contains $SL(2,i)Z(v)$.

If the variety $\mathcal{Q}(\beta, w, v)$ is B_i-stable for $i \in \mathtt{I}$, then the G-action on $\mathcal{X}(\beta)$ restricts to the B_i-action on $\mathcal{X}(\beta, w, v)$. If the variety $\mathcal{Q}(\beta, w, v)$ is B_0-stable, then there exists a smooth projective variety $\mathcal{X}'(\beta, w, v)$ with the B_0-action that yields a B_0-equivariant resolution of singularities of $\mathcal{Q}(\beta, w, v)$ (see e.g. [45, Corollary 7.6.3]). We can replace $\pi_{\beta,w,v} : \mathcal{X}(\beta, w, v) \to \mathcal{Q}(\beta, w, v)$ with $\mathcal{X}'(\beta, w, v) \to \mathcal{Q}(\beta, w, v)$ in this case since both of $\mathcal{X}(\beta, w, v)$ and $\mathcal{X}'(\beta, w, v)$ have rational singularities and there exists yet another resolution of singularities of $\mathcal{Q}(\beta, w, v)$ that dominates the both. Let us consider the map

$$SL(2,i) \times^{B_i} \mathcal{X}(\beta, w, v) \to SL(2,i) \times^{B_i} \mathcal{Q}(\beta, w, v) \xrightarrow{\pi_i} \mathcal{Q}(\gamma, u, v), \tag{4.2}$$

where the first map is the inflation of $\pi_{\beta,w,v}$, $\gamma = \beta$ and $u = s_i w$ $(i \neq 0)$, or $\gamma = \beta - w^{-1}\vartheta^\vee$ and $u = s_\vartheta w$ $(i = 0)$. Since $\mathcal{X}(\beta, w, v)$ has rational singularities, so is $SL(2,i) \times^{B_i} \mathcal{X}(\beta, w, v)$. In addition, the composition map (4.2) is birational and projective. Thus, it is another resolution of $\mathcal{Q}(\gamma, u, v)$ by a variety that has rational singularities. Therefore, we can replace $\mathcal{X}(\gamma, u, v)$ with $SL(2,i) \times^{B_i} \mathcal{X}(\beta, w, v)$ to compute $\mathbb{R}^\bullet(\pi_{\gamma,u,v})_* \mathcal{O}_{\mathcal{X}(\gamma,u,v)}$. Applying the Leray spectral sequence to (4.2) using Theorem 4.7 4), 5), we have

$$\mathbb{R}^0(\pi_{\gamma,u,v})_* \mathcal{O}_{\mathcal{X}(\gamma,u,v)} \cong \mathcal{O}_{\mathcal{Q}(\gamma,u,v)} \quad \text{and}$$
$$\mathbb{R}^{>0}(\pi_{\gamma,u,v})_* \mathcal{O}_{\mathcal{X}(\gamma,u,v)} \neq \{0\}. \tag{4.3}$$

Moreover, the support of (4.3) contains $SL(2,i)Z(v)$. By construction, general points of $SL(2,i)Z(v)$ have no defect at ∞, and their values belong to $\mathbb{O}^{\mathrm{op}}_{\mathcal{B}}(v)$. Therefore, an irreducible component $Z'(v)$ of the support of (4.3) that contains $SL(2,i)Z(v)$ again comes as a family $\{Z'(v)\}_{v \in W}$ such that $Z'(v) \subset Z'(u)$ if $v \le u$, and $\overline{BZ'(v)}$ is independent of v (in particular, we have $Z'(v)$ even if $i \neq 0$ and $s_i v > v$, or $i = 0$ and $s_\vartheta v < v$). Thus, we can repeat the above procedure by replacing $\mathcal{Q}(\beta, w, v)$ with $\mathcal{Q}(\gamma, u, v)$ and $\{Z(v)\}_{v \in W}$ with $\{Z'(v)\}_{v \in W}$. Note that we eventually attain $Z'(v) = SL(2,i)Z(v)$ for any application of the above procedures as the strict inclusion forces

$$(0 <) \quad \mathrm{codim}_{\mathcal{Q}(\gamma,u,v)} Z'(v) < \mathrm{codim}_{\mathcal{Q}(\gamma,u,v)} SL(2,i)Z(v) = \mathrm{codim}_{\mathcal{Q}(\beta,w,v)} Z(v),$$

that cannot be repeated infinitely many times.

Consider the minimal $\mathcal{Q}(\theta, t, v)$ ($\theta \in Q^\vee_+$ and $t \in W$) that contains $Z(v)$. Here $\mathring{\mathcal{Q}}(\theta, t, v)$ contains a point of $Z(v)$ as $Z(v)$ is irreducible and the inclusion relations among $\mathcal{Q}(\beta - \bullet, \bullet, v)$ obey the closure relation of $\mathbf{I}$-orbits of $\mathbf{Q}^{\mathrm{rat}}_G$ described in Theorem 1.10. Hence, the condition $SL(2,i)Z(v) \not\subset Z(v)$ is achieved if it holds for $\mathcal{Q}(\theta, t, v)$. Since $\overline{BZ(v)}$ is common for every $v \in W$, we deduce that θ and u are independent of v. In addition, $\mathcal{Q}(\theta, t, v)$ is transformed to $\mathcal{Q}(\theta, s_i t, v)$ ($i \neq 0$) or $\mathcal{Q}(\theta - t^{-1}\vartheta, s_\vartheta t, v)$ ($i \neq 0$) by an application of the above procedure.

In view of [25, Theorem 4.6] (cf. arguments around there), we repeat these procedures if necessary to assume $t = w_0$. The condition $SL(2,i)Z(v) \not\subset Z(v)$ implies $SL(2,i)\mathcal{Q}(\beta, w, v) \not\subset \mathcal{Q}(\beta, w, v)$ (otherwise $Z(v)$ is $SL(2,i)$-stable) asserts that $s_i t < t$ implies $s_i w < w$ ($i \in \mathtt{I}$). In case $t = w_0$, this implies $w = w_0$. Again by repeating the above procedures, we can rearrange the situation to assume $w = t = e$ and $v = w_0$. In this case, we have $\mathcal{Q}(\beta) = \mathcal{Q}(\beta, e, w_0)$, that has rational singularities by Theorem 4.5. From this, we find a contradiction on the existence of $Z(v)$. This in turn implies a contradiction to the existence of (β', w'). Therefore, we conclude that our transversal slices have rational singularities along the origin. $\Box$

We return to the proof of Theorem 4.8. We assume to the contrary to deduce contradiction. Namely, we assume that $\mathcal{Q}(\beta, w, v)$ has singularities worse than rational singularities. In view of Claim A and Claim B, if the worse than rational singularity locus of $\mathcal{Q}(\beta, w, v)$ is contained in $\mathcal{Q}(\gamma, u, v)$, then the variety $\mathcal{Q}(\gamma, u, v)$ itself have singularities worse than rational singularities. Therefore, by rearranging (β, w) if necessary, we can assume

$$\mathring{\mathcal{Q}}(\beta, w, v) \cap \mathrm{Supp}\, \mathbb{R}^{>0}(\pi_{\beta,w,v})_* \mathcal{O}_{\mathcal{X}(\beta,w,v)} \neq \emptyset. \tag{4.4}$$

Now we assume $v = w_0$ in addition to (4.4). Let $Y \subset \mathring{\mathcal{Q}}(\beta, w)$ be the locus on which the singularity of $\mathring{\mathcal{Q}}(\beta, w)$ is worse than rational singularities. We set $U := N^- \cap \dot{w} N \dot{w}^{-1}$. Then, the U-action on $\mathbb{O}(w) \subset \mathcal{B}$ is free, and the image of the multiplication map $U \times \mathbb{O}(w) \to \mathcal{B}$ is open dense. The U-action on $\mathring{\mathcal{Q}}(\beta, w)$ is also free. It follows that $UY \subset U\mathring{\mathcal{Q}}(\beta, w)$ is a locus on which the singularity is worse than rational singularities, and $(U \times \mathring{\mathcal{Q}}(\beta, w)) \cong U\mathring{\mathcal{Q}}(\beta, w) \subset \mathring{\mathcal{Q}}(\beta)$ is open dense.

However, the variety $\mathring{\mathcal{Q}}(\beta)$ has only rational singularities (Theorem 4.5). This is a contradiction to our assumption. Hence $\mathcal{Q}(\beta, w)$ must have rational singularities. The latter assertion follows from [30, Theorem 5.10]. $\Box$

Corollary 4.9. *Let $\lambda \in P_+$. For each $w \in W$ and $\beta \in Q^\vee$, we have*

$$H^{>0}(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \{0\}.$$

Proof. In view of Theorem 4.8, we apply [30, Theorem 5.10] and the Leray spectral sequence to reduce the assertion to $H^{>0}(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) = \{0\}$. This is Theorem 4.7 2). □

Proposition 4.10. *Let $w \in W$ and $\lambda \in P_+$. We have*

$$\lim_{\beta \to \infty} \chi(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)).$$

Proof. By Corollary 4.9, we have

$$\chi(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda))$$

for every $\beta \in Q^\vee_+$.

By Theorem 4.8, we deduce

$$H^0(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = H^0(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda))$$

for every $\lambda \in P_+$ and $\beta \in Q^\vee_+$.

By Theorem 4.7 2), we have

$$\lim_{\beta \to \infty} \chi(\mathcal{X}(\beta, w), \mathcal{O}_{\mathcal{X}(\beta,w)}(\lambda)) = \lim_{\beta \to \infty} \chi(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) \qquad \lambda \in P_+$$

and it is uniquely determined by Theorem 4.7 3). In addition, the comparison of Theorem 4.7 3) with [25, Theorem 4.12] implies

$$\lim_{\beta \to \infty} \chi(\mathcal{Q}(\beta, w), \mathcal{O}_{\mathcal{Q}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \qquad \lambda \in P_+.$$

Combining these implies the desired equality. □

4.3 Proof of Theorem 4.1

The whole of this subsection is devoted to the proof of Theorem 4.1. In this subsection, $\otimes$ is understood to be $\otimes_{\mathcal{O}_Z}$, where Z is the variety we are considering.

By the proof of Theorem 3.11 and the properties of $\star$-products of H-equivariant quantum K-groups (see §1.5), $qK_H(\mathcal{B})$ is the subspace of $qK_H(\mathcal{B})_{\mathrm{loc}}$ (topologically) generated by $\mathbb{C}P$, $Q^\vee_+$, and $[\mathcal{O}_{\mathcal{B}}(\pm\varpi_i)] \star$ ($i \in \mathtt{I}$). As each of them (transferred by Ψ) preserves $K_H(\mathbf{Q}_G(e))$ and $\Psi([\mathcal{O}_{\mathcal{B}}]) = [\mathcal{O}_{\mathbf{Q}_G(e)}]$, we deduce that Ψ embeds $qK_H(\mathcal{B})$ into $K_H(\mathbf{Q}_G(e))$.

We consider the $\mathbb{C}P[q^{\pm 1}]$-valued functional $F^\lambda_\beta(\bullet)$ on $K_H(\mathcal{B}) \otimes \mathbb{C}[q^{\pm 1}]Q^\vee_+$ with parameters $\beta \in Q^\vee_+$ and $\lambda \in P_+$:

$$\begin{aligned}\sum_{\beta \in Q^\vee_+} F^\lambda_\beta(\bullet) Q^\beta &:= \sum_{\gamma \in Q^\vee_+} \chi(\mathcal{X}(\gamma), \mathcal{O}_{\mathcal{X}(\gamma)}(\lambda) \otimes \mathtt{ev}_1^*(\bullet) \otimes \mathtt{ev}_2^*(\mathcal{O}_{\mathcal{B}})) Q^\gamma \\ &= \chi(T(\prod_{i \in \mathtt{I}} A_i(q)^{-\langle \alpha_i^\vee, \lambda \rangle}(\bullet)) \cdot \overline{T}([\mathcal{O}_{\mathcal{B}}]))\end{aligned}$$

(where the second equality is a reformulation of [24, Proposition 2.13]). Note that this collection of functionals $\{F^{\lambda}_{\beta}(\bullet)\}_{\beta,\lambda}$ is uniquely determined by the calculations from §4.2 and Theorem 3.8

$$\begin{aligned}\sum_{\beta \in Q^{\vee}_{+}} F^{\lambda}_{\beta}([\mathcal{O}_{\mathcal{B}}])Q^{\beta} &= \sum_{\beta \in Q^{\vee}_{+}} \chi(\mathfrak{X}(\beta), \mathcal{O}_{\mathfrak{X}(\beta)}(\lambda) \otimes \mathtt{ev}_1^*(\mathcal{O}_{\mathcal{B}}) \otimes \mathtt{ev}_2^*(\mathcal{O}_{\mathcal{B}}))Q^{\beta} \\ &= \sum_{\beta \in Q^{\vee}_{+}} \chi(\mathfrak{Q}(\beta), \mathcal{O}_{\mathfrak{Q}(\beta)}(\lambda))Q^{\beta} = D_{w_0}(J'(Qq^{\lambda}, q)e^{w_0\lambda}J'(Q, q^{-1})),\end{aligned}$$

as $\sum_{\beta} F^{\lambda}_{\beta}(\bullet)Q^{\beta}$ commutes with the $\mathbb{C}P[q^{\pm 1}]$-action and the $\mathbb{C}Q^{\vee}$-action, and intertwines the shift operator $A_i(q)$ with the line bundle twist by $\mathcal{O}_{\mathfrak{X}(\beta)}(-\varpi_i)$ for each $i \in \mathtt{I}$.

For each $a \in K_H(\mathcal{B}) \otimes \mathbb{C}[q^{\pm 1}][\![Q^{\vee}_{+}]\!]$, we have the class $\Psi(a) \in K_H(\mathbf{Q}_G(e))$ written as

$$\Psi(a) = \sum_{w \in W_{\mathrm{af}}} c^w(a)[\mathcal{O}_{\mathbf{Q}_G(w)}] \qquad w \in W_{\mathrm{af}}, c^w(a) \in \mathbb{C}[q^{\pm 1}]P.$$

In view of the definition of our shift operators (1.9) and the proof of Theorem 3.11, the coefficients $c^w(a) \in \mathbb{C}P$ is characterized as

$$c^w(a) = c^w_q(a)|_{q=1}$$

if we have elements $c^w_q(a) \in \mathbb{C}[q^{\pm 1}]P$ $(w \in W_{\mathrm{af}})$ determined by

$$\lim_{\beta \to \infty} F^{\lambda}_{\beta}(a) = \sum_{w \in W_{\mathrm{af}}} c^w_q(a) \operatorname{gch} \Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)) \qquad \lambda \in P_+.$$

We have

$$\lim_{\beta \to \infty} \chi(\mathfrak{X}(\beta, w), \mathcal{O}_{\mathfrak{X}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(\beta,w)}(\lambda))$$

for each $\lambda \in P_+$ by Proposition 4.10.

Thus, we have

$$\begin{aligned}\lim_{\beta \to \infty} F^{\lambda}_{\beta}([\mathcal{O}_{\mathcal{B}(w)}]) &= \lim_{\beta \to \infty} \chi(\mathfrak{X}(\beta), \mathcal{O}_{\mathfrak{X}(\beta)}(\lambda) \otimes \mathtt{ev}_1^*(\mathcal{O}_{\mathcal{B}(w)}) \otimes \mathtt{ev}_2^*(\mathcal{O}_{\mathcal{B}})) \\ &= \lim_{\beta \to \infty} \chi(\mathfrak{X}(\beta), \mathcal{O}_{\mathfrak{X}(\beta)}(\lambda) \otimes \mathtt{ev}_1^*(\mathcal{O}_{\mathcal{B}(w)})) \qquad (4.5) \\ &= \lim_{\beta \to \infty} \chi(\mathfrak{X}(\beta, w), \mathcal{O}_{\mathfrak{X}(\beta,w)}(\lambda)) = \operatorname{gch} H^0(\mathbf{Q}(w), \mathcal{O}_{\mathbf{Q}(w)}(\lambda))\end{aligned}$$

for each $\lambda \in P_+$ and $w \in W$.

Therefore, we conclude

$$\Psi([\mathcal{O}_{\mathcal{B}(w)}]) = [\mathcal{O}_{\mathbf{Q}_G(w)}] \qquad w \in W.$$

This proves the second assertion. By examining the $\mathbb{C}P$-bases between $qK_H(\mathcal{B})$ and $K_H(\mathbf{Q}_G(e))$, we also deduce $\operatorname{Im} \Psi = K_H(\mathbf{Q}_G(e))$. This is the first assertion. These complete the proofs of all the assertions.

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