On the monoidality of Saito reflection functors

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Abstract

We extend the definition of the Saito reflection functor of the Khovanov-Lauda-Rouquier algebras to the case of symmetric Kac-Moody algebras and prove that it defines a monoidal functor.

Introduction

In [6], the Saito reflection functors for the Khovanov-Lauda-Rouquier algebras of type ADE are introduced. It categorifies Lusztig's braid group action [11, §39] on (a subalgebra of) of the positive half of the quantum groups in the sense of Khovanov-Lauda-Rouquier [8, 16]. They are main ingredients to construct PBW bases in the spirit of Lusztig [11], and provided a certain role in the representation theory of the Khovanov-Lauda-Rouquier algebras.

The goal of this paper is to develop it little bit further, and provide some basic properties in more general setting than that of [6]. Let $\mathcal{A} := \mathbb{Z}[t^{\pm 1}]$. Let $\mathfrak{g}$ be a symmetric Kac-Moody Lie algebra, and let U^+ be the positive half of the $\mathcal{A}$-integral version of the quantum group of $\mathfrak{g}$ (see e.g. Lusztig [11] §1). Let $Q^+ := \mathbb{Z}_{\geq 0} I$, where I is the set of positive simple roots. We have a weight space decomposition $U^+ = \bigoplus_{\beta \in Q^+} U^+_\beta$. We have the Weyl group W of $\mathfrak{g}$ with its set of simple reflections $\{s_i\}_{i \in I}$. For each $\beta \in Q^+$, we have a finite set $B(\infty)_\beta$ which parameterizes a pair of distinguished bases $\{G^{up}(b)\}_{b \in B(\infty)_\beta}$ and $\{G^{low}(b)\}_{b \in B(\infty)_\beta}$ of $\mathbb{Q}(t) \otimes_{\mathcal{A}} U^+_\beta$. The Khovanov-Lauda-Rouquier algebra R_β is a certain graded algebra whose grading is bounded from below with the following properties:

- The set of isomorphism classes of simple graded R_β-modules (up to grading shifts) is also parameterized by $B(\infty)_\beta$;
- For each $b \in B(\infty)_\beta$, we have a simple graded R_β-module L_b and its projective cover P_b. Let $L_{b'} \langle k \rangle$ be the grade k shift of $L_{b'}$, and let $[P_b : L_{b'} \langle k \rangle]_0$ be the multiplicity of $L_{b'} \langle k \rangle$ in P_b (that is finite). Then, we have

$$G^{low}(b) = \sum_{b' \in B(\infty)_\beta, k \in \mathbb{Z}} t^k [P_b : L_{b'} \langle k \rangle]_0 G^{up}(b');$$

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- For each $\beta, \beta' \in Q^+$, there exists an induction functor

$$\star : R_\beta\text{-}\mathsf{gmod} \times R_{\beta'}\text{-}\mathsf{gmod} \ni (M, N) \mapsto M \star N \in R_{\beta+\beta'}\text{-}\mathsf{gmod};$$

- $\mathbf{K} := \bigoplus_{\beta \in Q^+} \mathbb{Q}(t) \otimes_{\mathcal{A}} K(R_\beta\text{-}\mathsf{gmod})$ is an associative algebra isomorphic to $\mathbb{Q}(t) \otimes_{\mathcal{A}} U^+$ with its product inherited from $\star$ (and the t-action is a grading shift).

For each $i \in I$ and $\beta \in Q^+$, we have certain quotients ${}_iR_\beta$ and ${}^iR_\beta$ of R_β. In case $s_i\beta \in Q^+$, an interpretation of Lusztig's geometric construction yields that ${}_iR_\beta$ and ${}^iR_{s_i\beta}$ must be Morita equivalent. This naturally enables us to define a right exact functor

$$\mathbb{T}_i : R_\beta\text{-}\mathsf{gmod} \twoheadrightarrow {}^iR_\beta\text{-}\mathsf{gmod} \xrightarrow{\cong} {}_iR_{s_i\beta}\text{-}\mathsf{gmod} \hookrightarrow R_{s_i\beta}\text{-}\mathsf{gmod}$$

that we call the Saito reflection functor. Under this setting, our main results read:

Theorem A (Theorems 3.8 + 3.9 + 4.1)**.** *The functors $\{\mathbb{T}_i\}_{i \in I}$ satisfies the following:*

1. *There exist a right adjoint functor $\mathbb{T}_i^*$ of $\mathbb{T}_i$;*
2. *For each $M \in {}^iR_\beta\text{-}\mathsf{gmod}$ and $N \in {}_iR_{s_i\beta}\text{-}\mathsf{gmod}$, we have*

$$\mathrm{ext}^*_{R_{s_i\beta}}(\mathbb{T}_i M, N) \cong \mathrm{ext}^*_{R_\beta}(M, \mathbb{T}_i^* N);$$

3. *They satisfy the braid relations;*
4. *For each $\beta_1, \beta_2 \in Q^+ \cap s_iQ^+$ and $M_1 \in {}^iR_{\beta_1}\text{-}\mathsf{gmod}$, $M_2 \in {}^iR_{\beta_2}\text{-}\mathsf{gmod}$, we have a natural isomorphism*

$$\mathbb{T}_i(M_1 \star M_2) \cong (\mathbb{T}_i M_1) \star (\mathbb{T}_i M_2).$$

Here we understand M_1, M_2 as modules of R_{β_1} and R_{β_2} through the pullbacks.

We remark that Theorem A confirms a conjecture in [5] and provides one way to correct an error in [6] (see Remark 4.2 or the arXiv version of [6]). Also, the above result should extend to the positive characteristic case at least when $\mathfrak{g}$ is of type ADE by using [15].

We note that Peter McNamara sent me a version of [14] during the preparation of this paper that partly overlaps with the content of this paper.

1 Conventions and recollections

An algebra R is a (not necessarily commutative) unital $\mathbb{C}$-algebra. A variety $\mathfrak{X}$ is a separated reduced scheme $\mathfrak{X}_0$ of finite type over some localization $\mathbb{Z}_S$ of $\mathbb{Z}$ specialized to $\mathbb{C}$. It is called a G-variety if we have an action of a connected affine algebraic group scheme G flat over $\mathbb{Z}_S$ on $\mathfrak{X}_0$ (specialized to $\mathbb{C}$). As in [1, §6] and [2] (see alto [7]), we transplant the notion of weights to the derived category of (G-equivariant) constructible sheaves with finite monodromy on $\mathfrak{X}$. Let us denote by $D^b(\mathfrak{X})$ (resp. $D^+(\mathfrak{X})$) the bounded (resp. bounded from the below) derived category of the category of constructible sheaves on $\mathfrak{X}$, and denote by $D^+_G(\mathfrak{X})$ the G-equivariant derived category of $\mathfrak{X}$. We have a natural forgetful functor $D^+_G(\mathfrak{X}) \to D^+(\mathfrak{X})$, whose preimage of $D^b(\mathfrak{X})$ is denoted by $D^b_G(\mathfrak{X})$. For an object of $D^b_G(\mathfrak{X})$, we may denote its image in $D^b(\mathfrak{X})$ by the same letter.

2 Quivers and the KLR algebras

Let $\Gamma = (I, \Omega)$ be an oriented graph with the set of its vertex I and the set of its oriented edges Ω. Here I is fixed, and Ω might change so that the underlying graph Γ_0 of Γ is fixed. We refer Ω as the orientation of Γ. We form a path algebra $\mathbb{C}[\Gamma]$ of Γ.

For $h \in \Omega$, we define $h' \in I$ to be the source of h and $h'' \in I$ to be the sink of h. We denote $i \leftrightarrow j$ for $i, j \in I$ if and only if there exists $h \in \Omega$ such that $\{h', h''\} = \{i, j\}$. A vertex $i \in I$ is called a sink of Γ (or Ω) if $h' \neq i$ for every $h \in \Omega$. A vertex $i \in I$ is called a source of Γ (or Ω) if $h'' \neq i$ for every $h \in \Omega$. We assume that $h' \neq h''$ for every $h \in \Omega$ in the below (i.e. Γ has no edge loop). We have a symmetric Kac-Moody algebra $\mathfrak{g}$ with its Dynkin diagram Γ_0.

Let Q^+ be the free abelian semi-group generated by $\{\alpha_i\}_{i\in I}$, and let $Q^+ \subset Q$ be the free abelian group generated by $\{\alpha_i\}_{i\in I}$. We sometimes identify Q with the root lattice of $\mathfrak{g}$ with a set of its simple roots $\{\alpha_i\}_{i\in I}$. Let $W = W(\Gamma_0)$ denote the Weyl group of type Γ_0 with a set of its simple reflections $\{s_i\}_{i\in I}$. The group W acts on Q via the above identification.

An I-graded vector space V is a vector space over $\mathbb{C}$ equipped with a direct sum decomposition $V = \bigoplus_{i\in I} V_i$.

Let V be an I-graded vector space. For $\beta \in Q^+$, we declare $\underline{\dim} V = \beta$ if and only if $\beta = \sum_{i\in I}(\dim V_i)\alpha_i$. We call $\underline{\dim} V$ the dimension vector of V. Form a vector space

$$E^\Omega_V := \bigoplus_{h\in\Omega} \mathrm{Hom}_{\mathbb{C}}(V_{h'}, V_{h''}).$$

We set $G_V := \prod_{i\in I} GL(V_i)$. The group G_V acts on E^Ω_V through its natural action on V. The space E^Ω_V can be identified with the based space of $\mathbb{C}[\Gamma]$-modules with its dimension vector β.

For each $k \geq 0$, we consider a sequence $\mathbf{m} = (m_1, m_2, \ldots, m_k) \in I^k$. We abbreviate this as $\mathsf{ht}(\mathbf{m}) = k$. We set $\mathsf{wt}(\mathbf{m}) := \sum_{j=1}^k \alpha_{m_j} \in Q^+$. For $\beta = \mathsf{wt}(\mathbf{m}) \in Q^+$, we set $\mathsf{ht}\,\beta = k$. For a sequence $\mathbf{m}' := (m'_1, \ldots, m'_{k'}) \in I^{k'}$, we set

$$\mathbf{m} + \mathbf{m}' := (m_1, \ldots, m_k, m'_1, \ldots, m'_{k'}) \in I^{k+k'}.$$

For $i \in I$ and $k \geq 0$, we understand that $ki = (i, \ldots, i) \in I^k$.

For each $\beta \in Q^+$, we set Y^β to be the set of all sequences $\mathbf{m}$ such that $\mathsf{wt}(\mathbf{m}) = \beta$. For each $\beta \in Q^+$ with $\mathsf{ht}\,\beta = n$ and $1 \leq i < n$, we define an action of $\{\sigma_i\}_{i=1}^{n-1}$ on Y^β as follows: For each $1 \leq i < n$ and $\mathbf{m} = (m_1, \ldots, m_n) \in Y^\beta$, we set

$$\sigma_i \mathbf{m} := (m_1, \ldots, m_{i-1}, m_{i+1}, m_i, m_{i+2}, \ldots, m_n).$$

It is clear that $\{\sigma_i\}_{i=1}^{n-1}$ generates a $\mathfrak{S}_n$-action on Y^β. In addition, $\mathfrak{S}_n$ naturally acts on a set of integers $\{1, 2, \ldots, n\}$. For $1 \leq i < n$, we set $h_{\mathbf{m},i} := \#\{h \in \Omega \mid h' = m_i, h'' = m_{i+1}\}$ and $a_{\mathbf{m},i} := h_{\mathbf{m},i} + h_{\sigma_i\mathbf{m},i}$.

Definition 2.1 (Khovanov-Lauda [8], Rouquier [16]). *Let $\beta \in Q^+$ so that $n = \mathsf{ht}\,\beta$. We define the KLR algebra R_β as the unital algebra generated by the elements $z_1, \ldots, z_n$, $\tau_1, \ldots, \tau_{n-1}$, and $e(\mathbf{m})$ ($\mathbf{m} \in Y^\beta$) subject to the following relations:*

1. $\deg z_i e(\mathbf{m}) = 2$ *for every i, and*
$$\deg \tau_i e(\mathbf{m}) = \begin{cases} -2 & (m_i = m_{i+1}) \\ a_{\mathbf{m},i} & (m_i \leftrightarrow m_{i+1}) \\ 0 & (\textit{otherwise}) \end{cases};$$
2. $[z_i, z_j] = 0$, $e(\mathbf{m})e(\mathbf{m}') = \delta_{\mathbf{m},\mathbf{m}'} e(\mathbf{m})$, *and* $\sum_{\mathbf{m} \in Y^\beta} e(\mathbf{m}) = 1$*;*
3. $\tau_i e(\mathbf{m}) = e(\sigma_i \mathbf{m}) \tau_i e(\mathbf{m})$, *and* $\tau_i \tau_j e(\mathbf{m}) = \tau_j \tau_i e(\mathbf{m})$ *for* $|i - j| > 1$*;*
4. $\tau_i^2 e(\mathbf{m}) = Q_{\mathbf{m},i}(z_i, z_{i+1}) e(\mathbf{m})$*;*
5. *For each $1 \le i < n$, we have*
$$\tau_{i+1}\tau_i\tau_{i+1}e(\mathbf{m}) - \tau_i\tau_{i+1}\tau_i e(\mathbf{m}) = \begin{cases} \frac{Q_{\mathbf{m},i}(z_{i+2}, z_{i+1}) - Q_{\mathbf{m},i}(z_i, z_{i+1})}{z_{i+2} - z_i} e(\mathbf{m}) & (m_{i+2} = m_i) \\ 0 & (\textit{otherwise}) \end{cases};$$
6. $$\tau_i z_k e(\mathbf{m}) - z_{\sigma_i(k)} \tau_i e(\mathbf{m}) = \begin{cases} -e(\mathbf{m}) & (i = k, m_i = m_{i+1}) \\ e(\mathbf{m}) & (i = k - 1, m_i = m_{i+1}) \\ 0 & (\textit{otherwise}) \end{cases}.$$

Here we set
$$Q_{\mathbf{m},i}(u, v) = \begin{cases} 1 & (m_i \neq m_{i+1}, m_i \not\leftrightarrow m_{i+1}) \\ (-1)^{h_{\mathbf{m},i}} (u - v)^{a_{\mathbf{m},i}} & (m_i \leftrightarrow m_{i+1}) \\ 0 & (\textit{otherwise}) \end{cases},$$
where u, v are indeterminants. □

Remark 2.2. *Note that the algebra R_β a priori depends on the orientation Ω through $Q_{\mathbf{m},i}(u, v)$. Since the graded algebras R_β are known to be mutually isomorphic for any two choices of Ω (cf. [16, §3.2.4] and Theorem 2.3), we suppress this dependence in the below.*

For an I-graded vector space V with $\underline{\dim} V = \beta$, we define
$$F_\beta^\Omega := \left\{ (\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \,\middle|\, \begin{array}{c} x \in E_V^\Omega. \text{ For each } 0 < j \le \mathsf{ht}\beta, \\ F_j \subset V \text{ is an } I\text{-graded vector subspace,} \\ F_{j+1} \subsetneq F_j, \text{ and satisfies } xF_j \subset F_{j+1}. \end{array} \right\} \quad \text{and}$$
$$\mathcal{B}_\beta^\Omega := \left\{ \{F_j\}_{j=0}^{\mathsf{ht}\beta} \,\middle|\, F_j \subset V \text{ is an } I\text{-graded vector subspace s.t. } F_{j+1} \subsetneq F_j. \right\}.$$

We have a projection
$$\varpi_\beta^\Omega : F_\beta^\Omega \ni (\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \mapsto \{F_j\}_{j=0}^{\mathsf{ht}\beta} \in \mathcal{B}_\beta^\Omega,$$

which is G_V-equivariant. For each $\mathbf{m} \in Y^\beta$, we have a connected component

$$F^\Omega_{\mathbf{m}} := \{(\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \in F^\Omega_\beta \mid \underline{\dim} F_j / F_{j+1} = \alpha_{m_{j+1}} \ \ \forall j\} \subset F^\Omega_\beta,$$

that is smooth of dimension $d^\Omega_{\mathbf{m}}$. We set $\mathcal{B}^\Omega_{\mathbf{m}} := \varpi^\Omega_\beta(F^\Omega_{\mathbf{m}})$, that is an irreducible component of $\mathcal{B}^\Omega_\beta$. Let

$$\pi^\Omega_{\mathbf{m}} : F^\Omega_{\mathbf{m}} \ni (\{F_j\}_{j=0}^{\mathsf{ht}\beta}, x) \mapsto x \in E^\Omega_V$$

be the second projection that is also G_V-equivariant. The map $\pi^\Omega_{\mathbf{m}}$ is projective, and hence

$$\mathcal{L}^\Omega_{\mathbf{m}} := (\pi^\Omega_{\mathbf{m}})_! \underline{\mathbb{C}} [d^\Omega_{\mathbf{m}}]$$

decomposes into a direct sum of (shifted) irreducible perverse sheaves with their coefficients in $D^b(\mathrm{pt})$ (Gabber's decomposition theorem, [1, Théorème 6.2.5]). Let us denote by $\mathcal{Q}^\Omega_{\mathbf{m}}$ be the set of isomorphism classes of simple irreducible perverse sheaves that appear as a direct summand of $\mathcal{L}^\Omega_{\mathbf{m}}$ (with some shifts). We set $\mathcal{L}^\Omega_\beta := \bigoplus_{\mathbf{m} \in Y^\beta} \mathcal{L}^\Omega_{\mathbf{m}}$ and $\mathcal{Q}^\Omega_\beta := \bigcup_{\mathbf{m} \in Y^\beta} \mathcal{Q}^\Omega_{\mathbf{m}}$. We might also denote $\mathcal{L}^\Omega_\beta$ by $\mathcal{L}^\Omega_V$ in order to clarify the dependence on V. Let $e(\mathbf{m})$ be the idempotent in $\mathrm{End}(\mathcal{L}^\Omega_\beta)$ so that $e(\mathbf{m})\mathcal{L}^\Omega_\beta = \mathcal{L}^\Omega_{\mathbf{m}}$. Since $\pi^\Omega_{\mathbf{m}}$ is projective, we conclude that $\mathbb{D}\mathcal{L}^\Omega_{\mathbf{m}} \cong \mathcal{L}^\Omega_{\mathbf{m}}$ for each $\mathbf{m} \in Y^\beta$, and hence

$$\mathbb{D}\mathcal{L}^\Omega_\beta \cong \mathcal{L}^\Omega_\beta. \tag{2.1}$$

Theorem 2.3 (Varagnolo-Vasserot [17])**.** *Under the above settings, we have an isomorphism of graded algebras:*

$$R_\beta \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_{G_V}(\mathcal{L}^\Omega_\beta, \mathcal{L}^\Omega_\beta).$$

In particular, the RHS does not depend on the choice of an orientation Ω of Γ_0.

For each $\mathbf{m}, \mathbf{m}' \in Y^\beta$, we set

$$R_{\mathbf{m},\mathbf{m}'} := e(\mathbf{m}) R_\beta e(\mathbf{m}') = \bigoplus_{i \in \mathbb{Z}} \mathrm{Ext}^i_{G_V}(\mathcal{L}^\Omega_{\mathbf{m}'}, \mathcal{L}^\Omega_{\mathbf{m}}).$$

We set $S_\beta \subset R_\beta$ to be a subalgebra which is generated by $e(\mathbf{m})$ ($\mathbf{m} \in Y^\beta$) and $z_1, \ldots, z_n$.

For each $\beta_1, \beta_2 \in Q^+$ with $\mathsf{ht}\,\beta_1 = n_1$ and $\mathsf{ht}\,\beta_2 = n_2$, we have a natural inclusion:

$$\begin{array}{ccccccc} R_{\beta_1} \boxtimes R_{\beta_2} & \ni & e(\mathbf{m}) \boxtimes e(\mathbf{m}') & \mapsto & e(\mathbf{m} + \mathbf{m}') & \in & R_{\beta_1+\beta_2}\ . \\ R_{\beta_1} \boxtimes 1 & \ni & z_i \boxtimes 1, \tau_i \boxtimes 1 & \mapsto & z_i, \tau_i & \in & R_{\beta_1+\beta_2} \\ 1 \boxtimes R_{\beta_2} & \ni & 1 \boxtimes z_i, 1 \boxtimes \tau_i & \mapsto & z_{i+n_1}, \tau_{i+n_1} & \in & R_{\beta_1+\beta_2} \end{array}$$

This defines an exact functor

$$\star : R_{\beta_1} \boxtimes R_{\beta_2}\text{-}\mathsf{gmod} \ni M_1 \boxtimes M_2 \mapsto R_{\beta_1+\beta_2} \otimes_{R_{\beta_1} \boxtimes R_{\beta_2}} (M_1 \boxtimes M_2) \in R_{\beta_1+\beta_2}\text{-}\mathsf{gmod}.$$

The functor $\star$ restricts to an exact functor in the category of graded projective modules (see e.g. [8, Proposition 2.16]):

$$\star : R_{\beta_1} \boxtimes R_{\beta_2}\text{-}\mathsf{proj} \ni M_1 \boxtimes M_2 \mapsto R_{\beta_1+\beta_2} \otimes_{R_{\beta_1} \boxtimes R_{\beta_2}} (M_1 \boxtimes M_2) \in R_{\beta_1+\beta_2}\text{-}\mathsf{proj}.$$

If $i \in I$ is a source of Γ and $f = (f_h)_{h \in \Omega} \in E_V^{\Omega}$, then we define

$$\epsilon_i^*(f) := \dim \ker \bigoplus_{h \in \Omega, h' = i} f_h \le \dim V_i.$$

If $i \in I$ is a sink of Γ and $f = (f_h)_{h \in \Omega} \in E_V^{\Omega}$, then we define

$$\epsilon_i(f) := \dim \operatorname{coker} \bigoplus_{h \in \Omega, h'' = i} f_h \le \dim V_i.$$

Each of $\epsilon_i^*(f)$ or $\epsilon_i(f)$ does not depend on the choice of a point in a G_V-orbit, and is a constructible function on E_V^{Ω}. Hence, ϵ_i or ϵ_i^* induces a function on E_V^{Ω} that is constant on each G_V-orbit, and a function on $\mathcal{Q}_\beta^{\Omega}$ through its value on an open dense subset of the support of its element whenever i is a source or a sink.

Proposition 2.4 (Lusztig [13]). *For each $i \in I$, the functions ϵ_i and ϵ_i^* descend to functions on $\mathcal{Q}_\beta^{\Omega}$ for each $\beta \in Q^+$. In particular, it gives rise to functions on the set of isomorphism classes of simple graded R_β-modules (up to degree shifts).*

Proof. Note that [13, Proposition 6.6] considers only ϵ_i, but ϵ_i^* is obtained by swapping the order of the convolution operation. $\square$

Theorem 2.5 (Khovanov-Lauda [8], Rouquier [16], Varagnolo-Vasserot [17]). *In the above setting, we have:*

1. *For each $i \in I$ and $n \ge 0$, $R_{n\alpha_i}$ has a unique indecomposable projective module P_{ni} up to grading shifts;*

2. *The functor $\star$ induces a $\mathbb{Z}[t^{\pm 1}]$-algebra structure on*

$$\mathbf{K} := \bigoplus_{\beta \in Q^+} K(R_\beta\text{-}\mathsf{proj});$$

3. *The algebra $\mathbf{K}$ is isomorphic to the integral form U^+ of the positive part of the quantized enveloping algebra of type Γ_0 by identifying $[P_{ni}]$ with the n-th divided power of a Chevalley generator of U^+;*

4. *The above isomorphism identifies a class of indecomposable graded projective R_β-module ($\beta \in Q^+$) up to grading shifts with an element of the lower global basis of U^+ in the sense of [4];*

5. *There exists a set $B(\infty) = \bigsqcup_{\beta \in Q^+} B(\infty)_\beta$ that parameterizes the set of lower global basis of U^+ ([4]). The above bijection identifies the functions ϵ_i, ϵ_i^* ($i \in I$) on the indecomposable graded projective (or graded simple) modules of $\bigsqcup_\beta R_\beta$-gmod with the corresponding functions on $B(\infty)$.*

Proof. See [6, Theorem 2.5]. $\square$

Proposition 2.6 ([6]). *The sheaf $\mathcal{L}_\beta^{\Omega}$ can be equipped with the structure of pure weight 0. In particular, the graded algebra R_β itself is pure of weight 0.*

Proof. The statement of [6, Proposition 2.7] is only when Γ_0 is a Dynkin quiver, but the argument works in general. $\square$

Thanks to Theorem 2.3 and Theorem 2.5 5), we have an identification $B(\infty)_\beta \cong \mathcal{Q}^\Omega_\beta$. Via this idenfication, each $b \in B(\infty)_\beta$ defines a G_V-equivariant simple perverse sheaf $\mathsf{IC}^\Omega(b)$ on E^Ω_V, where $\underline{\dim}\, V = \beta$. Each $b \in B(\infty)_\beta$ defines an indecomposable graded projective module P_b of R_β with simple head L_b that is isomorphic to its graded dual L_b^*.

Let $\beta \in Q^+$ so that $\mathsf{ht}\,\beta = n$. For each $i \in I$ and $k \geq 0$, we set

$$Y^\beta_{k,i} := \{\mathbf{m} = (m_j) \in Y^\beta \mid m_1 = \cdots = m_k = i\} \text{ and}$$
$$Y^{\beta;*}_{k,i} := \{\mathbf{m} = (m_j) \in Y^\beta \mid m_n = \cdots = m_{n-k+1} = i\}.$$

In addition, we define two idempotents of R_β as:

$$e_i(k) := \sum_{\mathbf{m} \in Y^\beta_{k,i}} e(\mathbf{m}), \quad \text{and} \quad e^*_i(k) := \sum_{\mathbf{m} \in Y^{\beta;*}_{k,i}} e(\mathbf{m}).$$

Theorem 2.7 (Lusztig [10] §6, Lauda-Vazirani [9] §2.5.1)**.** *Let $\beta \in Q^+$. For each $b \in B(\infty)_\beta$ and $i \in I$, we have*

$$\epsilon_i(b) = \max\{k \mid e_i(k)L_b \neq \{0\}\} \text{ and}$$
$$\epsilon^*_i(b) = \max\{k \mid e^*_i(k)L_b \neq \{0\}\}.$$

Moreover, $e_i(\epsilon_i(b))L_b$ and $e^*_i(\epsilon^*_i(b))L_b$ are irreducible $R_{\epsilon_i(b)\alpha_i} \boxtimes R_{\beta-\epsilon_i(b)\alpha_i}$-module and $R_{\beta-\epsilon^*_i(b)\alpha_1} \boxtimes R_{\epsilon^*_i(b)\alpha_i}$-module, respectively. In addition, if we have distinct $b, b' \in B(\infty)_\beta$ so that $\epsilon_i(b) = k = \epsilon_i(b')$ with $k \geq 0$, then $e_i(k)L_b$ and $e_i(k)L_{b'}$ are not isomorphic as an $R_{k\alpha_i} \boxtimes R_{\beta-k\alpha_i}$-module. $\square$

3 Saito reflection functors

Let Ω_i be the set of edges $h \in \Omega$ with $h'' = i$ or $h' = i$. Let $s_i\Omega_i$ be a collection of edges obtained from $h \in \Omega_i$ by setting $(s_ih)' = h''$ and $(s_ih)'' = h'$. We define $s_i\Omega := (\Omega \backslash \Omega_i) \cup s_i\Omega_i$ and set $s_i\Gamma := (I, s_i\Omega)$. Note that $\Gamma_0 = (s_i\Gamma)_0$.

Let V be an I-graded vector space with $\underline{\dim}\, V = \beta$. For a sink i of Γ, we define

$${}_iE^\Omega_V := \big\{(f_h)_{h\in\Omega} \in E^\Omega_V \mid \operatorname{coker}(\bigoplus_{h\in\Omega, h''=i} f_h : \bigoplus_{h'} V_{h'} \to V_i) = \{0\}\big\}.$$

For a source i of Γ, we define

$${}^iE^\Omega_V := \big\{(f_h)_{h\in\Omega} \in E^\Omega_V \mid \ker(\bigoplus_{h\in\Omega, h'=i} f_h : V_i \to \bigoplus_{h''} V_{h''}) = \{0\}\big\}.$$

Let Ω be an orientation of Γ so that $i \in I$ is a sink. Let $\beta \in Q^+ \cap s_iQ^+$. Let V and V' be I-graded vector spaces with $\underline{\dim}\, V = \beta$ and $\underline{\dim}\, V' = s_i\beta$, respectively. We fix an isomorphism $\phi : \oplus_{j\neq i} V_j \xrightarrow{\cong} \oplus_{j\neq i} V'_j$ as I-graded vector spaces. We define:

$$Z^\Omega_{V,V'} := \left\{ \{(f_h)_{h\in\Omega}, (f'_h)_{h\in s_i\Omega}, \psi\} \;\middle|\; \begin{array}{c} (f_h) \in {}_iE^\Omega_V, (f'_h) \in {}^iE^{s_i\Omega}_{V'}, \\ \phi f_h = f'_h \phi \text{ for } h \notin \Omega_i \\ \psi : V'_i \xrightarrow{\cong} \ker(\bigoplus_{h\in\Omega_i} f_h : \bigoplus_h V_{h'} \to V_i) \end{array} \right\}.$$

We have a diagram:

$$E_V^\Omega \xleftarrow{j_V} {}_iE_V^\Omega \xleftarrow{q_V^i} Z_{V,V'}^\Omega \xrightarrow{p_{V'}^i} {}^iE_{V'}^{s_i\Omega} \xrightarrow{\jmath_{V'}} E_{V'}^{s_i\Omega} \quad . \tag{3.1}$$

If we set

$$G_{V,V'} := GL(V_i) \times GL(V_i') \times \prod_{j \neq i} GL(V_j) \cong GL(V_i) \times GL(V_i') \times \prod_{j \neq i} GL(V_j'),$$

then the maps $p_{V'}^i$ and q_V^i are $G_{V,V'}$-equivariant.

Proposition 3.1 (Lusztig [12])**.** *The morphisms p_V^i and q_V^i in (3.1) are $\mathrm{Aut}(V_i)$-torsor and $\mathrm{Aut}(V_i')$-torsor, respectively.* □

When $\beta = \underline{\dim}\, V$, we set

$${}_iR_\beta^\Omega := \mathrm{Ext}^\bullet_{G_V}(j_V^*\mathcal{L}_V^\Omega, j_V^*\mathcal{L}_V^\Omega) \quad \text{and} \quad {}^iR_{s_i\beta}^{s_i\Omega} := \mathrm{Ext}^\bullet_{G_{V'}}(\jmath_{V'}^*\mathcal{L}_{V'}^{s_i\Omega}, \jmath_{V'}^*\mathcal{L}_{V'}^{s_i\Omega}).$$

For each $k > 0$, we fix an I-graded vector subspace $U_k \subset V$ so that $\underline{\dim}\, U_k = \beta - k\alpha_i$ and an I-graded vector subspace $U_k' \subset V'$ so that $\underline{\dim}\, U_k' = s_i\beta - k\alpha_i$. We have natural embeddings $\kappa_k : E_{U_k}^\Omega \subset E_V^\Omega$ and $\eta_k : E_{U_k'}^{s_i\Omega} \subset E_{V'}^{s_i\Omega}$ by adding direct sums of k copies of one-dimensional $\mathbb{C}[\Gamma]$-modules and $\mathbb{C}[s_i\Gamma]$-modules of their dimension vectors α_i, respectively.

Theorem 3.2 (Lusztig [13])**.** *Let $k > 0$. The restriction $\kappa_k^*\mathcal{L}_\beta^\Omega$ is a direct sum of shifted perverse sheaves in $\mathcal{Q}_{\beta-k\alpha_i}^\Omega$. Similarly, the restriction $\eta_k^*\mathcal{L}_{s_i\beta}^{s_i\Omega}$ is a direct sum of shifted perverse sheaves in $\mathcal{Q}_{s_i\beta-k\alpha_i}^{s_i\Omega}$.*

Proof. The assertion is exactly [13, Proposition 4.2] since the projection map p (in the notation of [13]) is an isomorphism (onto the image) if we appropriately arrange $\mathbf{W}$ and $\mathbf{T}$ in [13, §4.1]. □

We set ${}_iE_{V,k}^\Omega := G_V E_{U_k}^\Omega$, ${}_iE_{V,(k)}^\Omega := {}_iE_{V,k}^\Omega \backslash {}_iE_{V,k+1}^\Omega$, $i_k : {}_iE_{V,k}^\Omega \hookrightarrow E_V^\Omega$ and $j_k : {}_iE_{V,(k)}^\Omega \hookrightarrow {}_iE_{V,k}^\Omega$ for each $k > 0$. We have ${}_iE_{V,k}^\Omega = \bigsqcup_{k' \geq k} {}_iE_{V,(k')}^\Omega$, and we have $\epsilon_i(x) = k$ for $x \in {}_iE_{V,(k)}^\Omega$. The map i_k is closed immersion, and the map j_k is an open embedding. We set ${}^iE_{V',k}^{s_i\Omega} := G_{V'} E_{U_k'}^{s_i\Omega}$, and we define similar maps $\imath_k, \jmath_k$ for them that we use only as "an analogous" situation.

Proposition 3.3. *Let $k > 0$. The sheaf $i_k^*\mathcal{L}_\beta^\Omega$ is the direct sum of shifted perverse sheaves in $\mathcal{Q}_\beta^\Omega$ supported on ${}_iE_{V,k}^\Omega$ if we restrict them to ${}_iE_{V,(k)}^\Omega$. Similarly, the restriction $\imath_k^*\mathcal{L}_{s_i\beta}^{s_i\Omega}$ is a direct sum of shifted perverse sheaves in $\mathcal{Q}_{s_i\beta}^{s_i\Omega}$ supported on ${}^iE_{V',k}^{s_i\Omega}$ along the loci with $\epsilon_i^* = k$.*

Proof. As the proofs of the both cases are completely parallel, we concentrate into the case of $i_k^*\mathcal{L}_\beta^\Omega$.

The map κ_k factors through i_k as

$$E_{U_k}^\Omega \xrightarrow{\kappa_k'} {}_iE_{V,k}^\Omega \xrightarrow{i_k} E_V^\Omega$$

for each k. Thus, Theorem 3.2 asserts that $(\kappa_k')^* i_k^*\mathcal{L}_\beta^\Omega$ is a direct sum of shifted perverse sheaves in $\mathcal{Q}_{\beta-k\alpha_i}^\Omega$. We set $n := \dim V_i$. Let $P_k \subset GL(n) \cong GL(V_i)$ be the parabolic subgroup so that its Levi part is $GL(n-k) \times GL(k)$ and stabilizes $E^{\Omega}_{U_k} \subset E^{\Omega}_V$. Then, we have a map

$$\pi_k : GL(n) \times_{P_k} E^{\Omega}_{U_k} \longrightarrow E^{\Omega}_V,$$

that is projective over the image. Note that π_k is locally trivial fibration over ${}_iE^{\Omega}_{V,(k)}$ with its fiber isomorphic to $\mathrm{Gr}(k,n)$.

The sheaf $(\pi_k)_*\pi_k^* i_k^* \mathcal{L}^{\Omega}_\beta$ can be regarded as the induction of the sheaf $\kappa_k^* \mathcal{L}^{\Omega}_\beta$, and hence it is a direct sum of shifted perverse sheaves in $\mathcal{Q}^{\Omega}_\beta$. The above argument tells us that $i_k^* \mathcal{L}^{\Omega}_\beta$ is a direct summand of $(\pi_k)_*\pi_k^* i_k^* \mathcal{L}^{\Omega}_\beta$ when restricted to ${}_iE^{\Omega}_{V,(k)}$. Therefore, we conclude that $i_k^* \mathcal{L}^{\Omega}_\beta$ is a direct sum of shifted perverse sheaves in $\mathcal{Q}^{\Omega}_\beta$ supported on ${}_iE^{\Omega}_{V,k}$ restricted to ${}_iE^{\Omega}_{V,(k)}$ as required. □

For each $k > 0$, we define

$${}_iR^{\Omega}_{\beta,k} := \mathrm{Ext}^{\bullet}_{G_V}(j^*_{V,k}\mathcal{L}^{\Omega}_V, j^*_{V,k}\mathcal{L}^{\Omega}_V),$$

where $j_{V,k} : E^{\Omega}_V \backslash {}_iE^{\Omega}_{V,k} \hookrightarrow E^{\Omega}_V$. By definition, we have ${}_iR^{\Omega}_{\beta,1} = {}_iR^{\Omega}_\beta$. By convention, we have $j_{V,k} = \mathrm{id}$ for $k > \dim V_i$, and we have ${}_iR^{\Omega}_{\beta,k} = R^{\Omega}_\beta$ in this case. We also define ${}^iR^{s_i\Omega}_{s_i\beta,k}$ in a similar fashion, that we use only as "an analogous" situation.

Theorem 3.4. *For each $k > 0$, we have an algebra isomorphism*

$${}_iR^{\Omega}_{\beta,k} \cong R_\beta/(R_\beta e_i(k) R_\beta).$$

Moreover, ${}_iR^{\Omega}_{\beta,k+1} e_i(k) {}_iR^{\Omega}_{\beta,k+1}$ is projective as a ${}_iR^{\Omega}_{\beta,k+1}$-module. Similarly, the algebra ${}^iR^{s_i\Omega}_{s_i\beta,k}$ is isomorphic to $R^{s_i\Omega}_{s_i\beta}/(R^{s_i\Omega}_{s_i\beta} e^*_i(k) R^{s_i\Omega}_{s_i\beta})$, and ${}^iR^{s_i\Omega}_{s_i\beta,k+1} e^*_i(k) {}^iR^{s_i\Omega}_{s_i\beta,k+1}$ is projective as a ${}^iR^{s_i\Omega}_{s_i\beta,k+1}$-module. In particular, the algebras ${}_iR^{\Omega}_{\beta,k}$ and ${}^iR^{s_i\Omega}_{s_i\beta,k}$ do not depend on the choice of Ω.

Proof. Since the case of ${}^iR^{s_i\Omega}_{s_i\beta,k}$ is completely parallel, we concentrate into the case of ${}_iR^{\Omega}_{\beta,k}$. The case $k \gg 0$ is clear, and hence we prove the assertion by the downward induction on k. In particular, we assume that

$${}_iR^{\Omega}_{\beta,k+1} \cong R_\beta/(R_\beta e_i(k+1) R_\beta)$$

to prove our assertion. We denote ${}_iR^{\Omega}_{\beta,k+1}$ by $R_{\beta,k+1}$ for simplicity.

We have

$$\begin{aligned}\mathrm{Ext}^{\bullet}_{G_V}(j^*_{V,k}\mathcal{L}^{\Omega}_V, j^*_{V,k}\mathcal{L}^{\Omega}_V) &\cong \mathrm{Ext}^{\bullet}_{G_V}(j^!_{V,k}\mathcal{L}^{\Omega}_V, j^!_{V,k}\mathcal{L}^{\Omega}_V)\\ &\cong \mathrm{Ext}^{\bullet}_{G_V}((j_{V,k})_! j^!_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V).\end{aligned}$$

We set $E_k := (E^{\Omega}_V \backslash {}_iE^{\Omega}_{V,k+1})$. By assumption, we can restrict ourselves to E_k to compute the Ext-groups. Hence, we freely assume that our maps are restricted to E_k unless otherwise stated (during this proof).

We have a distinguished triangle

$$(j_{V,k})_! j^!_{V,k}\mathcal{L}^{\Omega}_V \to \mathcal{L}^{\Omega}_V \to (i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V \xrightarrow{+1}, \tag{3.2}$$

where $i_{V,k} : {}_iE^{\Omega}_{V,(k)} \hookrightarrow E_k$ is the complement inclusion. This yields an exact seqeunce

$$\begin{aligned}\mathrm{Ext}^{\bullet}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \to \mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) &\xrightarrow{\psi} \mathrm{Ext}^{\bullet}_{G_V}((j_{V,k})_! j^!_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)\\ &\to \mathrm{Ext}^{\bullet+1}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)\end{aligned}$$

as $R_{\beta,k+1}$-modules. Note that L_b is the coefficient of $\mathsf{IC}^{\Omega}(b)$ in $\mathcal{L}^{\Omega}_{\beta}$, and hence its support is contained in ${}_iE^{\Omega}_{V,k}$ when $\epsilon_i(k)L_b \neq \{0\}$. In particular, the simple R^{Ω}_{β}-module L_b contributes to $\mathrm{Ext}^{\bullet}_{G_V}(j^!_{V,k}\mathcal{L}^{\Omega}_V, j^!_{V,k}\mathcal{L}^{\Omega}_V)$ by (graded) Jordan-Hölder multiplicity zero when $\epsilon_i(k)L_b \neq \{0\}$. It follows that $R_{\beta,k+1}e_i(k)R_{\beta,k+1} \subset \ker \psi$.

The action of $H^{\bullet}_{G_V}(\mathrm{pt})$ on $\mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$ is through the center of R_{β} (see e.g. [17]), and it is torsion-free. Hence, the action of $H^{\bullet}_{GL(V_i)}(\mathrm{pt})$ and $H^{\bullet}_{G_V}(\mathrm{pt})$ on $R_{\beta,k+1} \cong \mathrm{Ext}^{\bullet}_{G_V}(j^!_{V,k+1}\mathcal{L}^{\Omega}_V, j^!_{V,k+1}\mathcal{L}^{\Omega}_V)$ factors through the center of $R_{\beta,k+1}$.

Since $(i_{V,k})_* = (i_{V,k})_!$, we have

$$\mathrm{Ext}^{\bullet}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \cong \mathrm{Ext}^{\bullet}_{G_V}(i^*_{V,k}\mathcal{L}^{\Omega}_V, i^!_{V,k}\mathcal{L}^{\Omega}_V).$$

By our convention, $i^*_{V,k}\mathcal{L}^{\Omega}_V$ and $i^!_{V,k}\mathcal{L}^{\Omega}_V$ are supported on ${}_iE^{\Omega}_{V,(k)}$. In addition, we have ${}_iE^{\Omega}_{V,(k)} \cong GL(V_i) \times_{P_k} ({}_iE^{\Omega}_{U_k} \cap {}_iE^{\Omega}_{V,(k)})$ for a parabolic subgroup $P_k \subset GL(V_i)$ borrowed from the proof of Proposition 3.3. Here, the subgroup $GL(k) \subset P_k$ acts on ${}_iE^{\Omega}_{U_k}$ trivially. From this and the induction equivalence ([2, §2.6.3]), we obtain a free action of $H^{\bullet}_{GL(k)}(\mathrm{pt})$ on $\mathrm{Ext}^{\bullet}_{GL(V_i)}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$. The image of the pullback map $H^{\bullet}_{GL(V_i)}(\mathrm{pt}) \to H^{\bullet}_{GL(k)}(\mathrm{pt})$ contain k-algebraically independent elements (over the base field $\mathbb{C}$). From these, we conclude that the $H^{\bullet}_{GL(V_i)}(\mathrm{pt})$-action on $\mathrm{Ext}^{\bullet}_{GL(V_i)}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$ contains at least k algebraically independent elements that acts torsion-freely.

On the other hand, the action of $H^{\bullet}_{GL(V_i)}(\mathrm{pt})$ on $\mathrm{Ext}^{\bullet}_{GL(V_i)}(j^!_{V,k}\mathcal{L}^{\Omega}_V, j^!_{V,k}\mathcal{L}^{\Omega}_V)$ arises from the $GL(V_i)$-action on some algebraic stratification of E_{k-1} (see e.g. Chriss-Ginzburg [3, Definition 3.2.23 and §8.4]) so that the stalks of elements of $\mathcal{Q}^{\Omega}_{\beta}$ are constant (by the construction of $\mathcal{Q}^{\Omega}_{\beta}$; note that our stratification is finite). In other words, we have a finite G_V-stable stratification

$$E^{\Omega}_V \backslash {}_iE^{\Omega}_{V,k} = \bigsqcup_{\lambda \in \Lambda} S_{\lambda}$$

and a complex of locally constant sheaves $\mathcal{E}_{\lambda}$ (obtained by a successive application of recollements) over S_{λ} so that $\mathrm{Ext}^{\bullet}_{GL(V_i)}(j^!_{V,k}\mathcal{L}^{\Omega}_V, j^!_{V,k}\mathcal{L}^{\Omega}_V)$ is written as a finite successive distinguished triangles using $H^{\bullet}_{GL(V_i)}(S_{\lambda}, \mathcal{E}_{\lambda})$. Moreover, $\mathrm{Ext}^{\bullet}_{GL(V_i)}(j^!_{V,k}\mathcal{L}^{\Omega}_V, j^!_{V,k}\mathcal{L}^{\Omega}_V)$ must be a finitely generated $H^{\bullet}_{GL(V_i)}(\mathrm{pt})$-module as a result of the the fact that $\mathcal{L}^{\Omega}_V$ is a finite direct sum of constructible complexes over E^{Ω}_V.

The rank of the stabilizer of the $GL(V_i)$-action on a point of E_{k-1} is always $< k$. As a consequence, the action of $H^{\bullet}_{GL(V_i)}(\mathrm{pt})$ on $H^{\bullet}_{GL(V_i)}(S_{\lambda}, \mathcal{E}_{\lambda})$ (for every $\lambda \in \Lambda$) cannot carry k-algebraically independent elements that act torsion-freely. Therefore, the same holds for $\mathrm{Ext}^{\bullet}_{GL(V_i)}(j^!_{V,k}\mathcal{L}^{\Omega}_V, j^!_{V,k}\mathcal{L}^{\Omega}_V)$. Thus, the map

$$\mathrm{Ext}^{\bullet}_{GL(V_i)}((j_{V,k})_! j^!_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \to \mathrm{Ext}^{\bullet+1}_{GL(V_i)}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$$

must be nullity as we do not have enough number of algebraically independent elements of $H^{\bullet}_{GL(V_i)}(\mathrm{pt})$ that acts on the LHS without torsion. By imposing the G_V-equivariance, we obtain a map

$$H^{\bullet}_{G_V\,/\,GL(V_i)}(\mathrm{pt}) \otimes \mathrm{Ext}^{\bullet}_{GL(V_i)}((j_{V,k})_! j^!_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \\ \to H^{\bullet}_{G_V\,/\,GL(V_i)}(\mathrm{pt}) \otimes \mathrm{Ext}^{\bullet+1}_{GL(V_i)}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$$

that induces a map

$$\mathrm{Ext}^{\bullet}_{G_V}((j_{V,k})_! j^!_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \to \mathrm{Ext}^{\bullet+1}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$$

through the Serre spectral sequences (applied after pulling back to the classifying space of G_V). This map must be also nullity as it is induced from the nullity.

Hence, we conclude a short exact seqeuence

$$0 \to \mathrm{Ext}^{\bullet}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \to R_{\beta,k+1} \xrightarrow{\psi} R_{\beta,k} \to 0$$

as left $R_{\beta,k+1}$-modules.

By Proposition 3.3, the sheaf $(i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V$ is a direct sum of shifted perverse sheaves on E_k, that is supported on ${}_iE^{\Omega}_{V,k}$ (or ${}_iE^{\Omega}_{V,(k)}$). It follows that the graded $R_{\beta,k+1}$-module $\mathrm{Ext}^{\bullet}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V)$ is the direct sum of projective covers of L_b with $\epsilon_i(b) = k$. Since $R_{\beta,k+1}e_i(k)R_{\beta,k+1}$ is the maximal left R_β-submodule of $R_{\beta,k+1}$ generated by irreducible constituents $\{L_b\}_{\epsilon_i(b)=k}$, we deduce

$$R_{\beta,k+1}e_i(k)R_{\beta,k+1} \cong \mathrm{Ext}^{\bullet}_{G_V}((i_{V,k})_* i^*_{V,k}\mathcal{L}^{\Omega}_V, \mathcal{L}^{\Omega}_V) \cong \mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_V, (i_{V,k})_* i^!_{V,k}\mathcal{L}^{\Omega}_V),$$

where the latter modules are actually calculated on E_k. Therefore, we conclude the assertions for $R_{\beta,k}$ as required.

This proceeds the induction step, and we conclude the assertion. □

Corollary 3.5. *The set of isomorphism classes of graded simple modules of ${}_iR^{\Omega}_\beta$ and ${}^iR^{s_i\Omega}_{s_i\beta}$ are $\{L_b\,\langle j\rangle\}_{\epsilon_i(b)=0, j\in\mathbb{Z}}$ and $\{L_b\,\langle j\rangle\}_{\epsilon^*_i(b)=0, j\in\mathbb{Z}}$, respectively.* □

Theorem 3.6 ([12])**.** *The maps q^i_V and $p^i_{V'}$ give rise to a bijective correspondence between perverse sheaves corresponding to $\{b \in B(\infty)_\beta \mid \epsilon_i(b) = 0\}$ and $\{b \in B(\infty)_{s_i\beta} \mid \epsilon^*_i(b) = 0\}$.*

Proof. In view of Theorem 3.4, the combination of [12, Theorem 8.6] and Proposition 2.4 implies the result (see also [11, Proposition 38.1.6]). □

Proposition 3.7 ([6])**.** *In the setting of Proposition 3.1, two graded algebras ${}_iR^{\Omega}_\beta$ and ${}^iR^{s_i\Omega}_{s_i\beta}$ are Morita equivalent to each other. In addition, this Morita equivalence is independent of the choice of Ω (as long as i is a sink).*

Proof. Although the original setting in [6, Proposition 3.5] is only for types ADE, the arguments carry over to this case in view of Theorem 3.6. □

For each $b \in B(\infty)_{s_i\beta}$, we denote by $T_i(b) \in B(\infty)_\beta \sqcup \{\emptyset\}$ the element so that

$$(p^i_{V'})^* \mathsf{IC}^{s_i\Omega}(b)[(\dim V_i)^2] \cong (q^i_V)^* \mathsf{IC}^{\Omega}(T_i(b))[(\dim V'_i)^2],$$

(we understand that $T_i(b) = \emptyset$ if $\mathrm{supp}\,\mathsf{IC}^{s_i\Omega}(b) \cap \mathrm{Im}\, p^i_{V'} = \emptyset$). Note that $T_i(b) = \emptyset$ if and only if $\epsilon^*_i(b) > 0$. In addition, we have $\epsilon_i(T_i(b)) = 0$ if $T_i(b) \neq \emptyset$. We set $T_i^{-1}(b') := b$ if $b' = T_i(b) \neq \emptyset$.

Thanks to Theorem 3.4, we can drop Ω or $s_i\Omega$ from ${}_iR_\beta^\Omega$ and ${}^iR_\beta^{s_i\Omega}$. We define a left exact functor

$$\mathbb{T}_i^* : R_\beta\text{-}\mathsf{gmod} \twoheadrightarrow {}_iR_\beta\text{-}\mathsf{gmod} \xrightarrow{\cong} {}^iR_{s_i\beta}\text{-}\mathsf{gmod} \hookrightarrow R_{s_i\beta}\text{-}\mathsf{gmod},$$

where the first functor is $\mathrm{Hom}_{R_\beta}({}_iR_\beta, \bullet)$, the second functor is Proposition 3.7, and the third functor is the pullback. Similarly, we define a right exact functor

$$\mathbb{T}_i : R_\beta\text{-}\mathsf{gmod} \twoheadrightarrow {}^iR_\beta\text{-}\mathsf{gmod} \xrightarrow{\cong} {}_iR_{s_i\beta}\text{-}\mathsf{gmod} \hookrightarrow R_{s_i\beta}\text{-}\mathsf{gmod},$$

where the first functor is ${}^iR_\beta \otimes_{R_\beta} \bullet$. We call these functors the Saito reflection functors ([6, §3]). By the latter part of Proposition 3.7, we see that these functors are independent of the choices involved.

Theorem 3.8 ([6] Theorem 3.6)**.** *Let $i \in I$. We have:*

1. *For each $b \in B(\infty)_\beta$, we have*

$$\mathbb{T}_i L_b = \begin{cases} L_{T_i(b)} & (\epsilon_i^*(b) = 0) \\ \{0\} & (\epsilon_i^*(b) > 0) \end{cases}, \quad \text{and} \quad \mathbb{T}_i^* L_b = \begin{cases} L_{T_i^{-1}(b)} & (\epsilon_i(b) = 0) \\ \{0\} & (\epsilon_i(b) > 0) \end{cases};$$

2. *The functors $(\mathbb{T}_i, \mathbb{T}_i^*)$ form an adjoint pair;*

3. *For each $M \in {}^iR_\beta\text{-}\mathsf{gmod}$ and $N \in {}_iR_{s_i\beta}\text{-}\mathsf{gmod}$, we have*

$$\mathrm{ext}^*_{R_{s_i\beta}}(\mathbb{T}_i M, N) \cong \mathrm{ext}^*_{R_\beta}(M, \mathbb{T}_i^* N).$$

Proof. The proof of the first assertion is the same as [6, Theorem 3.6] if we replace standard modules with projective modules, that involves only simple perverse sheaves. The proof of the second assertion is exactly the same as [6, Theorem 3.6]. The third assertion requires the second part of Theorem 3.4 instead of [6, Corollary 1.6] (and also we need to repeat projective resolutions inductively on ϵ_i and ϵ_i^* in a downward fashion). □

Theorem 3.9. *Let $i, j \in I$. We have:*

- *If $i \not\leftrightarrow j$, then we have $\mathbb{T}_i\mathbb{T}_j \cong \mathbb{T}_j\mathbb{T}_i$;*
- *If $\#\{h \in \Omega \mid \{h', h''\} = \{i, j\}\} = 1$, then we have $\mathbb{T}_i\mathbb{T}_j\mathbb{T}_i \cong \mathbb{T}_j\mathbb{T}_i\mathbb{T}_j$.*

The same is true for $\mathbb{T}_i^*$ and $\mathbb{T}_j^*$.

Proof. By [12, §9.4], the functor $\mathbb{T}_i$ induces an isomorphism described in [11, Lemma 38.1.3] (see also [18]). Hence, [11, Theorem 39.4.3] (cf. Theorem 3.8 1)) implies that the both sides give the same correspondence between simple modules. As each of $\mathbb{T}_i$ transplants the simple modules and annihilates all the submodule that contains some specific simple modules (that induces an equivalence between some Serre subcategories), the same is true for their composition. Therefore, we conclude the result. □

4 Monoidality of the Saito reflection functor

We work in the setting of §2. The goal of this section is to prove the following:

Theorem 4.1. *Let $i \in I$, and let $\beta_1, \beta_2 \in Q^+$ so that $s_i\beta_1, s_i\beta_2 \in Q^+$. There exists a natural transformation*

$$\mathbb{T}_i^*(\bullet \star \bullet) \longrightarrow \mathbb{T}_i^*(\bullet) \star \mathbb{T}_i^*(\bullet)$$

as functors from the category of ${}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2}$-modules that gives rise to an isomorphism of functors. The same holds for $\mathbb{T}_i$ if we consider functors from the category of ${}^iR_{s_i\beta_1} \boxtimes {}^iR_{s_i\beta_2}$-modules.

Remark 4.2. Theorem 4.1, or rather its $\mathbb{T}$-version, corrects a mistake in the proof of [6, Lemma 4.2 2)]. Note that another correction was made for the arXiv version of [6].

The rest of this section is devoted to the proof of Theorem 4.1, and the main body of the proof is at the end of this section.

Let $\beta_1, \beta_2 \in Q^+$ and set $\beta := \beta_1 + \beta_2$. The induction functor $\star$ is represented by a bimodule $R_\beta e_{\beta_1,\beta_2}$, where

$$e_{\beta_1,\beta_2} = \sum_{\mathbf{m}_1 \in Y^{\beta_1}, \mathbf{m}_2 \in Y^{\beta_2}} e(\mathbf{m}_1) \boxtimes e(\mathbf{m}_2).$$

We fix an orientation Ω, and we might drop the superscript Ω freely if the meaning is clear from the context. We fix I-graded vector spaces $V(1)$ and $V(2)$ so that $\underline{\dim}\, V(i) = \beta_i = \sum_{j \in I} d_j(i)\alpha_i$ for $i = 1, 2$, and $V := V(1) \oplus V(2)$.

We consider two varieties with natural G_V-actions:

$$\mathrm{Gr}^{\Omega}_{V(1),V(2)}(V) := \left\{ (F, x, \psi_1, \psi_2) \,\middle|\, \begin{smallmatrix} F \subset V \,:\, I\text{-graded vector subspace} \\ x \in E_V, \text{ s.t. } xF \subset F \\ \psi_1 : V/F \cong V(1), \psi_2 : F \cong V(2) \end{smallmatrix} \right\},$$

$$\mathrm{Gr}^{\Omega}_{\beta_1,\beta_2}(V) := \left\{ (F, x) \,\middle|\, \begin{smallmatrix} F \subset V \,:\, I\text{-graded vector subspace} \\ x \in E_V, \text{ s.t. } xF \subset F \\ \underline{\dim}\, F = \beta_2 \end{smallmatrix} \right\}.$$

We have a $G_{V(1)} \times G_{V(2)}$-torsor structure $\vartheta^\Omega : \mathrm{Gr}_{V(1),V(2)}(V) \longrightarrow \mathrm{Gr}_{\beta_1,\beta_2}(V)$ given by forgetting ψ_1 and ψ_2. We have two maps

$\mathsf{p}^\Omega : \mathrm{Gr}_{\beta_1,\beta_2}(V) \ni (F, x) \mapsto x \in E_V$ and

$\mathsf{q}^\Omega : \mathrm{Gr}_{V(1),V(2)}(V) \ni (F, x, \psi_1, \psi_2) \mapsto (\psi_1(x \mod F), \psi_2(x|_F)) \in E_{V(1)} \oplus E_{V(2)}$.

Notice that ϑ and q are smooth of relative dimensions $\dim G_{V(1)} + \dim G_{V(2)}$ and $\frac{1}{2}(\dim G_V + \dim G_{V(1)} + \dim G_{V(2)}) + \sum_{h \in \Omega} d_1(h')d_2(h'')$, respectively. The map p is projective. We set $N^{\beta}_{\beta_1,\beta_2} := \frac{1}{2}(\dim G_V - \dim G_{V(1)} - \dim G_{V(2)}) + \sum_{h \in \Omega} d_1(h')d_2(h'')$. For $G_{V(i)}$-equivariant constructible sheaves $\mathcal{F}_i$ on $E_{V(i)}$ for $i = 1, 2$, we define their convolution product as

$$\mathcal{F}_1 \odot \mathcal{F}_2 := \mathsf{p}_! \mathcal{F}_{12}[N^{\beta}_{\beta_1,\beta_2}], \text{ where } \vartheta^* \mathcal{F}_{12} \cong \mathsf{q}^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \text{ in } D^b_{G_V}(\mathrm{Gr}_{V(1),V(2)}(V)),$$

where we also make a renormalization of the weight so that $\mathcal{F}_1 \odot \mathcal{F}_2$ is pure of weight 0 whenever $\mathcal{F}_1, \mathcal{F}_2$ have pure weights whose sum is 0 (that is possible as

$\mathsf{p}_* = \mathsf{p}_!$). We might drop $[N^{\beta}_{\beta_1,\beta_2}]$ and the renormalization of the weight from the convention in the below for the sake of simplicity.

By construction, the convolution of $\mathcal{L}^{\Omega}_{\beta_1}$ and $\mathcal{L}^{\Omega}_{\beta_2}$ yields the direct summand of $\mathcal{L}^{\Omega}_{\beta}$ corresponding to the idempotent e_{β_1,β_2}. Hence, we have

$$R_\beta e_{\beta_1,\beta_2} \cong \mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_{\beta_1} \odot \mathcal{L}^{\Omega}_{\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \tag{4.1}$$

as $(R_\beta, R_{\beta_1} \boxtimes R_{\beta_2})$-bimodule.

Let $\mathcal{L}^{\Omega}_{\beta_1,\beta_2}$ be a complex so that $\vartheta^* \mathcal{L}^{\Omega}_{\beta_1,\beta_2} \cong \mathsf{q}^*(\mathcal{L}^{\Omega}_{\beta_1} \boxtimes \mathcal{L}^{\Omega}_{\beta_2})$. Then, we have

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_{\beta_1} \odot \mathcal{L}^{\Omega}_{\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \cong \mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathsf{p}^! \mathcal{L}^{\Omega}_{\beta}).$$

Since we have (by the induction equivalence [2, §2.6.3])

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta_1,\beta_2}) \cong \mathrm{Ext}^{\bullet}_{G_{V(1)} \times G_{V(2)}}(\mathcal{L}^{\Omega}_{\beta_1} \boxtimes \mathcal{L}^{\Omega}_{\beta_2}, \mathcal{L}^{\Omega}_{\beta_1} \boxtimes \mathcal{L}^{\Omega}_{\beta_2}),$$

we have a (right) $R_{\beta_1} \boxtimes R_{\beta_2}$-module structure of $R_\beta e_{\beta_1,\beta_2}$.

From now on, we assume that $i \in I$ is a sink of Ω and employ the setting of §3. We find $\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}$ so that

$$\vartheta^* \mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2} \cong \mathsf{q}^*((\jmath_{V(1)})_! \jmath^!_{V(1)} \mathcal{L}^{\Omega}_{\beta_1} \boxtimes (\jmath_{V(2)})_! \jmath^!_{V(2)} \mathcal{L}^{\Omega}_{\beta_2}),$$

and $\mathcal{O} := \vartheta(\mathsf{q}^{-1}({}_iE^{\Omega}_{V(1)} \times {}_iE^{\Omega}_{V(2)}))$. The graded vector space

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_! \mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \cong \mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathsf{p}^! \mathcal{L}^{\Omega}_{\beta})$$

admits an $(R_\beta, {}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2})$-bimodule structure.

By restricting each components to the open set ${}_iE^{\Omega}_V$ by $\jmath^*_V = \jmath^!_V$, we deduce that $\mathrm{Ext}^{\bullet}_{G_V}(\jmath^*_V \mathsf{p}_! \mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \jmath^*_V \mathcal{L}^{\Omega}_{\beta})$ is a left ${}_iR_\beta$-module. Applying adjunctions, this module is isomorphic to

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_! \mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, (\jmath_V)_* \jmath^*_V \mathcal{L}^{\Omega}_{\beta}) \cong \mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathsf{p}^! (\jmath_V)_* \jmath^*_V \mathcal{L}^{\Omega}_{\beta}), \tag{4.2}$$

which admits a right ${}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2}$-structure. Hence, (4.2) is an $({}_iR_\beta, {}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2})$-bimodule.

We fix I-graded vector spaces $V'(1)$ and $V'(2)$ so that $\underline{\dim}\, V'(i) = s_i \beta_i$ for $i = 1, 2$. A similar construction as above implies that we have a sheaf $\mathcal{L}^{s_i\Omega,\flat}_{s_i\beta_1,s_i\beta_2}$ so that

$$\vartheta^* \mathcal{L}^{s_i\Omega,\flat}_{s_i\beta_1,s_i\beta_2} \cong \mathsf{q}^*((\jmath_{V'(1)})_! \jmath^!_{V'(1)} \mathcal{L}^{s_i\Omega}_{s_i\beta_1} \boxtimes (\jmath_{V'(2)})_! \jmath^!_{V'(2)} \mathcal{L}^{s_i\Omega}_{s_i\beta_2}).$$

It yields an $({}^iR_{s_i\beta}, {}^iR_{s_i\beta_1} \boxtimes {}^iR_{s_i\beta_2})$-bimodule

$$\mathrm{Ext}^{\bullet}_{G_{V'}}(\mathsf{p}_! \mathcal{L}^{s_i\Omega,\flat}_{s_i\beta_1,s_i\beta_2}, (\jmath_{V'})_* \jmath^*_{V'} \mathcal{L}^{s_i\Omega}_{s_i\beta}) \cong \mathrm{Ext}^{\bullet}_{G_{V'}}(\mathcal{L}^{s_i\Omega,\flat}_{s_i\beta_1,s_i\beta_2}, \mathsf{p}^! (\jmath_{V'})_* \jmath^*_{V'} \mathcal{L}^{s_i\Omega}_{s_i\beta}). \tag{4.3}$$

Theorem 4.3. *Under the above setting, the image of the natural restriction map*

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_! \mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \longrightarrow \mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_! \mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta})$$

is a submodule of the RHS, and is equal to

$${}_iR_\beta \otimes_{R_{\beta_1} \boxtimes R_{\beta_2}} ({}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2}).$$

In addition, it is the pure part of weight zero in $\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_! \mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta})$. *The same is true if we replace* Ω *with* $s_i\Omega$, β_j *by* $s_i\beta_j$, *and* ${}_iR_{\beta_j}$ *with* ${}^iR_{s_i\beta_j}$ $(j = \emptyset, 1, 2)$.

Proof. Since the proofs of the both assertions are similar, we prove only the case of Ω. We have $\mathcal{O} \subset \mathsf{p}^{-1}({}_iE_V^{\Omega})$, and hence the restriction map factors through the restriction to ${}_iE_V^{\Omega}$. By unwinding the definition, we have a factorization

$$\begin{aligned}\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}) &\twoheadrightarrow \mathrm{Ext}^{\bullet}_{G_V}(j_V^!\mathsf{p}_!\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, j_V^!\mathcal{L}^{\Omega}_{\beta})\\ &\cong \mathrm{Ext}^{\bullet}_{G_V}((j_V)_!j_V^!\mathsf{p}_!\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \xrightarrow{\rho} \mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta})\end{aligned}$$

of $(R_\beta, R_{\beta_1} \boxtimes R_{\beta_2})$-bimodule map, where the first map (that is surjection by Theorem 3.4) is the restriction to the open set, the second isomorphism is the adjunction, and the third morphism is obtained by the base change using $j_V^! = j_V^*$ and the composition.

We have a distinguished triangle

$$(j_{V(2)})_!j^!_{V(2)}\mathcal{L}^{\Omega}_{\beta_2} \to \mathcal{L}^{\Omega}_{\beta_2} \to \mathsf{Ker} \xrightarrow{+1}.$$

Since $\mathcal{L}^{\Omega}_{\beta_2}$ is pure of weight 0, it follows that $(j_{V(2)})_!j^!_{V(2)}\mathcal{L}^{\Omega}_{\beta_2}$ must have weight ≤ 0 ([1, Stabilités 5.1.14]). Taking account into the fact that $(j_{V(2)})_!j^!_{V(2)}\mathcal{L}^{\Omega}_{\beta_2}$ and $\mathcal{L}^{\Omega}_{\beta_2}$ share the same stalk along ${}_iE^{\Omega}_{V(2)}$ and the stalk of $(j_{V(2)})_!j^!_{V(2)}\mathcal{L}^{\Omega}_{\beta_2}$ vanishes outside of ${}_iE^{\Omega}_{V(2)}$, we conclude that Ker has weight ≤ 0. We set $\mathcal{K} := \mathsf{p}_!\mathcal{K}'$, where $\vartheta^*\mathcal{K}' \cong \mathsf{q}^*(\mathcal{L}^{\Omega}_{\beta_1} \boxtimes \mathsf{Ker})$.

From now on, we make all computations over ${}_iE_V^{\Omega}$ by using $j_V^* = j_V^!$. The above construction gives us a distinguished triangle

$$\mathsf{p}_!\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2} \to \mathcal{L}^{\Omega}_{\beta_1} \odot \mathcal{L}^{\Omega}_{\beta_2} \to \mathcal{K} \xrightarrow{+1}.$$

Moreover, $\mathcal{K}$ has weight ≤ 0 by $\mathsf{p}_* = \mathsf{p}_!$.

Hence, we deduce an exact sequence of ${}_iR_\beta$-modules

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathcal{K}, \mathcal{L}^{\Omega}_{\beta}) \to \mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \xrightarrow{\rho} \mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}).$$

Note that the middle term has weight 0 by Theorem 3.4 as the both of $\mathcal{L}^{\Omega}_{\beta_1} \odot \mathcal{L}^{\Omega}_{\beta_2}$ and $\mathcal{L}^{\Omega}_{\beta}$ are pure of weight 0. Since $\mathrm{Ext}^{\bullet}_{G_V}(\mathcal{K}, \mathcal{L}^{\Omega}_{\beta})$ has weight ≥ 0 ([1, Stabilités 5.1.14]), we conclude that $\mathrm{Im}\,\rho$ is precisely the weight 0-part of $\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta})$ (see also the arguments in [7]).

Since the $({}_iR_\beta, {}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2})$-action preserves the weight, it follows that $\mathrm{Im}\,\rho$ is an $({}_iR_\beta, {}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2})$-subbimodule of $\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta})$. Since we have $\mathrm{Ext}^{\bullet}_{G_V}(\mathsf{p}_!\mathcal{L}^{\Omega}_{\beta_1,\beta_2}, \mathcal{L}^{\Omega}_{\beta}) \cong {}_iR_\beta e_{\beta_1,\beta_2}$, we have a surjection

$$\pi : {}_iR_\beta e_{\beta_1,\beta_2} \twoheadrightarrow \mathrm{Im}\,\rho.$$

By Proposition 3.3, the sheaf Ker is obtained by successive constructions of cones of shifted perverse sheaves on $\mathcal{Q}^{\Omega}_{\beta_2}$ that are supported outside of ${}_iE^{\Omega}_{V(2)}$. Therefore, we deduce that $\ker\rho$ admits a surjection from the direct sum of R_β-modules of the form

$$P_{b_1} \star P_{b_2} \qquad b_1 \in B(\infty)_{\beta_1}, b_2 \in B(\infty)_{\beta_2}, \epsilon_i(b_2) > 0,$$

that corresponds to $\mathsf{IC}^{\Omega}(b_1) \odot \mathsf{IC}^{\Omega}(b_2)$ with $\epsilon_i(b_2) > 0$. Let us write E the sum of the image of all such R_β-modules in ${}_iR_\beta e_{\beta_1,\beta_2}$ arises as the above induction. In view of the construction of Ker, we have $\ker\pi \subset E$.

On the other hand, E is precisely the kernel of the natural quotient map

$$_{i}R_{\beta}e_{\beta_1,\beta_2} \twoheadrightarrow {}_{i}R_{\beta} \otimes_{R_{\beta_1}\boxtimes R_{\beta_2}} ({}_{i}R_{\beta_1} \boxtimes {}_{i}R_{\beta_2}).$$

As a consequence, we have a quotient map

$$\operatorname{Im}\rho \twoheadrightarrow {}_{i}R_{\beta} \otimes_{R_{\beta_1}\boxtimes R_{\beta_2}} ({}_{i}R_{\beta_1} \boxtimes {}_{i}R_{\beta_2}).$$

The module $\operatorname{Im}\rho$ is a $({}_{i}R_{\beta}, {}_{i}R_{\beta_1} \boxtimes {}_{i}R_{\beta_2})$-bimodule whose bimodule structure is induced from the $({}_{i}R_{\beta}, R_{\beta_1} \boxtimes R_{\beta_2})$-bimodule structure on ${}_{i}R_{\beta}e_{\beta_1,\beta_2}$ by construction (through Theorem 3.4). Thus, $\operatorname{Im}\rho$ admits a surjection from ${}_{i}R_{\beta} \otimes_{R_{\beta_1}\boxtimes R_{\beta_2}} ({}_{i}R_{\beta_1} \boxtimes {}_{i}R_{\beta_2})$, that is the maximal $({}_{i}R_{\beta}, {}_{i}R_{\beta_1} \boxtimes {}_{i}R_{\beta_2})$-bimodule quotient of ${}_{i}R_{\beta}e_{\beta_1,\beta_2}$ (regarded as a $({}_{i}R_{\beta}, R_{\beta_1} \boxtimes R_{\beta_2})$-bimodule). Therefore, we conclude

$$\operatorname{Im}\rho \cong {}_{i}R_{\beta} \otimes_{R_{\beta_1}\boxtimes R_{\beta_2}} ({}_{i}R_{\beta_1} \boxtimes {}_{i}R_{\beta_2})$$

as required. □

Proof of Theorem 4.1. The open subset $\mathcal{O} \subset \mathrm{Gr}^{\Omega}_{\beta_1,\beta_2}(V)$ is the set of points (F, x) so that $x|_{F} \in {}_{i}E^{\Omega}_{V(2)}$ and $x \mod F \in {}_{i}E^{\Omega}_{V(1)}$. We set $\mathcal{O}' := \vartheta(\mathsf{q}^{-1}({}^{i}E^{s_i\Omega}_{V'(1)} \times {}^{i}E^{s_i\Omega}_{V'(2)}))$. The open subset $\mathcal{O}' \subset \mathrm{Gr}^{s_i\Omega}_{s_i\beta_1,s_i\beta_2}(V')$ is the set of points (F', x') so that $x'|_{F'} \in {}^{i}E^{s_i\Omega}_{V'(2)}$ and $x' \mod F' \in {}^{i}E^{s_i\Omega}_{V'(1)}$.

Consider a variety $\mathbb{O}$ with the $G_{V,V'}$-action defined as:

$$\left\{ \{\{W_i, W'_i\}_i, (f_h)_h, (f'_h)_h, \phi, \psi\} \;\middle|\; \begin{array}{c} \{(f_h)_{h\in\Omega}, (f'_h)_{h\in s_i\Omega}, \psi\} \in Z^{\Omega}_{V,V'}, \\ \phi : W_j \cong W'_j \text{ for } j \neq i \\ (\{W_i\}_{i\in I}, (f_h)_{h\in\Omega}) \in \mathrm{Gr}^{\Omega}_{\beta_1,\beta_2}(V), \\ (\{W'_i\}_{i\in I}, (f'_h)_{h\in s_i\Omega}) \in \mathrm{Gr}^{s_i\Omega}_{s_i\beta_1,s_i\beta_2}(V') \\ \psi : W'_i \xrightarrow{\cong} \ker(\bigoplus_{h\in\Omega_i} f_h : \bigoplus_h W_{h'} \to W_i) \end{array} \right\}.$$

In the definition of $\mathbb{O}$, the condition $(f_h|_{\{W_i\}_i})_{h\in\Omega} \in {}_{i}E^{\Omega}_{V(2)}$ guarantees that

$$\dim W'_i = \dim \ker(\bigoplus_{h\in\Omega_i} f_h : \bigoplus_h W_{h'} \to W_i)$$

and similarly the condition $(f'_h|_{\{W_i\}_i})_{h\in s_i\Omega} \in {}^{i}E^{s_i\Omega}_{V'(2)}$ guarantees that

$$\dim W_i = \dim \operatorname{coker}(\bigoplus_{h\in\Omega_i} f_h : W'_i \to \bigoplus_h W_{h'}),$$

that actually asserts the same thing. Since we have an isomorphism

$$\psi : V'_i \xrightarrow{\cong} \ker(\bigoplus_{h\in\Omega_i} f_h : \bigoplus_h V_{h'} \to V_i)$$

from the definition of $Z^{\Omega}_{V,V'}$, taking quotients yield

$$(f_h \mod \{W_i\}_i)_{h\in\Omega} \in {}_{i}E^{\Omega}_{V(1)} \quad \text{and} \quad (f_h \mod \{W'_i\}_i)_{h\in s_i\Omega} \in {}^{i}E^{s_i\Omega}_{V'(1)}.$$

Hence, the quotients of $\mathbb{O}$ by $G_{V'_i}$ and G_{V_i} gives $\widetilde{q}^i_V$ and $\widetilde{p}^i_{V'}$ in the commutative diagram in the below:

$$\begin{tikzcd}
\mathrm{Gr}^{\Omega}_{\beta_1,\beta_2}(V) \arrow[d, "\mathsf{p}^{\Omega}"'] & \mathcal{O} \arrow[l, hook'] \arrow[d, "p"] & \mathbb{O} \arrow[l, two heads, "\widetilde{q}^i_V"'] \arrow[r, two heads, "\widetilde{p}^i_{V'}"] & \mathcal{O}' \arrow[r, hook] \arrow[d, "p'"'] & \mathrm{Gr}^{s_i\Omega}_{s_i\beta_1,s_i\beta_2}(V') \arrow[d, "\mathsf{p}^{s_i\Omega}"'] \\
E^{\Omega}_V & {}_iE^{\Omega}_V \arrow[l, hook', "j_V"'] & Z^{\Omega}_{V,V'} \arrow[l, two heads, "q^i_V"'] \arrow[r, two heads, "p^i_{V'}"] & {}^iE^{s_i\Omega}_{V'} \arrow[r, hook, "\jmath_{V'}"] & E^{s_i\Omega}_{V'}
\end{tikzcd} .$$

Therefore, we have an equivalence of the category of G_V-equivariant sheaves on $\mathcal{O}$, and the category of $G_{V'}$-equivariant sheaves on $\mathcal{O}'$ (cf. [2, §2.6.3]). With an aid of Proposition 3.7, we conclude that

$$\mathrm{Ext}^{\bullet}_{G_V}(\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}, p^!\mathcal{L}^{\Omega}_{\beta}) \cong \mathrm{Ext}^{\bullet}_{G_{V'}}(\mathcal{L}^{s_i\Omega,\flat}_{s_i\beta_1,s_i\beta_2}, (p')^!\mathcal{L}^{s_i\Omega}_{s_i\beta})$$

up to amplifications of direct summands (i.e. we allow to duplicate direct summand of both terms). By Theorem 4.3, the comparison of their weight zero parts identifies

$${}_iR_{\beta} \otimes_{R_{\beta_1} \boxtimes R_{\beta_2}} ({}_iR_{\beta_1} \boxtimes {}_iR_{\beta_2}) \text{ and } {}^iR_{s_i\beta} \otimes_{R_{s_i\beta_1} \boxtimes R_{s_i\beta_2}} ({}^iR_{s_i\beta_1} \boxtimes {}^iR_{s_i\beta_2})$$

through the Morita equivalences in Proposition 3.7. This is actually an identification of bimodules by construction.

In other words, we have an isomorphism

$$\mathbb{T}_i^*(\mathrm{Ext}^{\bullet}_{G_V}(j_V^*(\mathcal{L}^{\Omega,\flat}_{\beta_1,\beta_2}), j_V^*\mathcal{L}^{\Omega}_{\beta})) \cong \mathrm{Ext}^{\bullet}_{G_{V'}}(\jmath_{V'}^*(\mathcal{L}^{s_i\Omega,\flat}_{s_i\beta_1,s_i\beta_2}), \jmath_{V'}^*\mathcal{L}^{s_i\Omega}_{s_i\beta}),$$

where the amplification of direct summands is subsumed in the constructions of $\mathbb{T}_i$. This isomorphism commutes with the Morita equivalence of ${}_iR_{\beta_j}$ and ${}^iR_{s_i\beta_j}$ for $j = 1, 2$ by the above. Hence, taking their weight 0 part yields the desired natural transformation

$$\mathbb{T}_i^*(\bullet \star \bullet) \longrightarrow \mathbb{T}_i^*(\bullet) \star \mathbb{T}_i^*(\bullet)$$

of functors, and it must be an equivalence. The case of $\mathbb{T}_i$ is obtained similarly. $\square$

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References

[1] A. A. Beilinson, J. Bernstein, and P. Deligne. *Faisceaux pervers*, volume 100 of *Astérisque*. Soc. Math. France, Paris, 1982.

[2] Joseph Bernstein and Valery Lunts. *Equivariant sheaves and functors*, volume 1578 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[3] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition.

[4] Masaki Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke Mathematical Journal*, 63(2):465–516, 1991.

[5] Masaki Kashiwara, Myungho Kim, Se jin Oh, and Euiyong Park. Monoidal categories associated with strata of flag manifolds. *Adv. in Math.*, 328:959–1009, 2018. arXiv:1708.04428.

[6] Syu Kato. Poincaré-Birkhoff-Witt bases and Khovanov-Lauda-Rouquier algebras. *Duke Math. J.*, 163(3):619–663, 2014. Correction abridged version: arXiv:1203.5254v5.

[7] Syu Kato. An algebraic study of extension algebras. *Amer. J. Math.*, 139(3):567–615, 2017.

[8] M. Khovanov and A. Lauda. A diagrammatic approach tocategorification of quantum groups I. *Represent. Theory*, 13:309–347, 2009.

[9] Aaron D. Lauda and Monica Vazirani. Crystals from categorified quantum groups. *Adv. Math.*, 228(2):803–861, 2011.

[10] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.

[11] G. Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1993.

[12] G. Lusztig. Canonical bases and Hall algebras. In *Representation theories and algebraic geometry (Montreal, PQ, 1997)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, pages 365–399. Kluwer Acad. Publ., 1998.

[13] George Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *Journal of the American Mathematical Society*, 4(2):365–421, 1991.

[14] Peter J. McNamara. Monoidality of Kato's reflection functors. arXiv:1712.00173, 2017.

[15] Peter J. McNamara. Representation theory of geometric extension algebras. *Journal de l'École polytechnique – Mathématiques*, to appear. arXiv:1701.07949.

[16] R Rouquier. 2-Kac-Moody algebras. arXiv:0812.5023.

[17] Michela Varagnolo and Eric Vasserot. Canonical bases and KLR-algebras. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 659:67–100, 2011.

[18] Jie Xiao and Minghui Zhao. Geometric realizations of Lusztig's symmetries. *J. Algebra*, 475:392–422, 2017.