THE FORMAL MODEL OF SEMI-INFINITE FLAG MANIFOLDS

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ABSTRACT

The formal model of semi-infinite flag manifold is a variant of an affine flag variety that was recognized from the 1980s but not studied extensively until the late 2010s. In this note, we exhibit constructions and ideas appearing in our recent study of the formal model of semi-infinite flag manifold of a simple algebraic group. Our results have some implications to the theory of rational maps from a projective line to partial flag manifolds, and also on the structures of quantum cohomologies and quantum K-groups of partial flag manifolds.

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1. INTRODUCTION

Compact complex-analytic spaces that admit homogeneous Lie group actions are quite rare in nature, and their classification reduces into three primitive classes: finite groups, tori, and (partial) flag manifolds. The first have discrete topology and the role of geometric consideration is rather small, in general. The second, particularly those admit polarizations, offer a major subject known as abelian varieties. The third, the (partial) flag manifolds of compact simple Lie groups, are ubiquitous in representation theory of semisimple algebraic groups and quantum groups. By the universal nature of general linear groups, flag manifolds of unitary groups are extensively studied from the geometric perspective.

In representation-theoretic considerations, we usually consider flag manifolds as projective algebraic varieties defined over an algebraically closed field (that form a family over Spec $\mathbb{Z}$). This definition naturally extends to an arbitrary Kac–Moody setting, but the resulting objects have at least two variants, thin flag varieties and thick flag manifolds (defined by Kac–Peterson [75] and Kashiwara [40], respectively). In case the Kac–Moody group is of affine type, we have a loop realization of the corresponding Kac–Moody group, essentially identifying the corresponding group with the set of $\Bbbk(\!(z)\!)$-valued points of a simple algebraic group over a field $\Bbbk$. This motivates us to consider yet other versions of flag manifolds of affine type that can be understood as an enhancement of arc schemes of usual flag manifolds. These are the semi-infinite flag manifolds that originate from the ideas of Lusztig [63, §11] and Drinfeld [22]. Lusztig's original idea is to construct varieties that naturally encode representation theory of simple algebraic groups over finite fields. The Lusztig program (see, e.g., [44, 63]) adds representation theory of quantum groups at roots of unity and representation theory of affine Lie algebras at negative rational levels into the picture, and Feigin–Frenkel [19] put representation theory of affine Lie algebras at the critical level into the picture. The semi-infinite flag manifolds itself have two realizations, that we refer to as the ind-model and the formal model. The geometry of the ind-model of semi-infinite flag manifolds, also known as the space of quasimaps from a projective line to a flag manifold, was studied extensively by Braverman, Finkelberg, Mirković, and their collaborators (see [8, 18, 21, 22]).

One instance of the ind-model of semi-infinite flag manifold is the space of principal bundles on an algebraic curve equipped with some reduction. This interpretation realizes some portion of the above representation-theoretic expectations [2, 31]. The formal model of semi-infinite flag manifolds is expected to add a concrete understanding of related representation-theoretic patterns [19, 22, 63]. Unfortunately, such an idea needs to be polished as its implementation faces difficulty due to its essential infinite-dimensionality. This forces us to employ affine Grassmannians instead of semi-infinite flag manifolds in some cases (see [26, 30, 78]) at the moment, that is possible by some tight connections [27, 70].

Meanwhile, it is realized that the semi-infinite flag manifold is a version of the loop space of a flag manifold, and hence it is related to its quantum cohomology [32]. In fact, the ind-model of a semi-infinite flag manifold offers a description of the quantum K-theoretic J-function of a flag manifold [9] that encodes its small quantum K-group.

In both contexts of the above two paragraphs, the Peterson isomorphism [59, 74], that connects the quantum cohomology of a flag manifold with the homology of an affine Grassmannian, should admit an interpretation using a semi-infinite flag manifold. However, such an interpretation is not known today (though we have Corollary 7.3).

The main goal of this note is to explain a realization of the formal model of semi-infinite flag manifold [46,50,52], that is reminiscent to the classical description of the original flag manifolds. Our realization is supported by recent developments in representation theory of affine Lie algebras [14,15,51], that is also reminiscent to the representation theory of simple Lie algebras. It turns out that the study of the formal model of the semi-infinite flag manifold has implications to the corresponding ind-model [50], as well as the study of quantum K-groups of partial flag manifolds and the K-groups of affine Grassmannians [45, 47, 48]. This includes an interpretation (and a proof) of the K-theoretic analogue of the Peterson isomorphism using semi-infinite flag manifolds (Theorem 8.2).

The results presented here describe the formal model of semi-infinite flag manifolds in a down-to-earth fashion, and also provide first nontrivial conclusions deduced from them. However, we have not yet reached our primary goal to understand representation theory from this perspective in a satisfactory fashion. We hope to improve this situation in the near future.

The organization of this note is as follows: We first recall the construction of flag manifolds that is parallel to our later construction in Section 2. We explain the role of quantum groups in the structure theory of Kac–Moody algebras and exhibit two versions of flag varieties of Kac–Moody groups in Section 3. In Section 4, we exhibit some representation theory of affine Lie algebras. Based on it, we explain our construction of the formal model of semi-infinite flag manifolds in Section 5. This enables us to present our idea on the Frobenius splitting of semi-infinite flag manifolds in Section 6. We explain the connection between its Richardson varieties and quasimap spaces in Section 7, and explain how they fit into the study of quantum cohomology of flag manifolds. We exhibit the K-theoretic Peterson isomorphism in Section 8. We discuss the functoriality of the quantum K-groups of partial flag manifolds in Section 9. We finish this note by discussing some perspectives in Section 10.

We assume that every field $\Bbbk$ has characteristic $\neq 2$. A variety is some algebraic-geometric object that admits singularity, and a manifold is a variety that is supposed to be smooth in some sense. An algebraic variety is a separated scheme of finite type defined over a field (i.e., our variety is not necessarily irreducible or reduced). We set $\mathbb{N} := \mathbb{Z}_{\geq 0}$.

2. FLAG MANIFOLDS VIA REPRESENTATION THEORY

Let G be a simply connected semisimple algebraic group over an algebraically closed field $\Bbbk$. Let $T \subset B$ be its maximal torus and a Borel subgroup (maximal solvable subgroup). Let W $(= N_G(T)/T)$ be the Weyl group of G. Let $\mathbb{X}$ be the set of one-dimensional rational T-characters (the set of T-weights), that admits a natural W-action. We set $\mathbb{X}_+ := \sum_{i=1}^{r} \mathbb{N}\varpi_i$, where $\varpi_1, \dots, \varpi_r \in \mathbb{X}$ are fundamental weights with respect to B. The set of isomorphism classes of irreducible rational representations $\{L(\lambda)\}_\lambda$ of G is labeled by $\mathbb{X}_+$ in such a way that each $L(\lambda)$ contains a unique (up to scalar) B-eigenvector

$\mathbf{v}_\lambda$ with its T-weight λ. We refer λ ($\in \mathbb{X}_+$) as the highest weight of $L(\lambda)$. The flag manifold $\mathscr{B} := G/B$ of G is the maximal G-homogeneous space that is projective.

In case $\Bbbk = \mathbb{C}$, we have

$$\mathscr{B} = (\mathscr{Y} \setminus E)/T,$$

where $\mathscr{Y}$ is an affine algebraic variety with $(G \times T)$-action whose ring $\mathbb{C}[\mathscr{Y}]$ of regular functions is written as

$$\mathbb{C}[\mathscr{Y}] \cong \bigoplus_{\lambda \in \mathbb{X}_+} L(\lambda)^* \quad (\text{as } G \times T\text{-modules}), \tag{2.1}$$

and $E \subset \mathscr{Y}$ is the locus where the T-action is not free. Here, the G-action on $\mathbb{C}[\mathscr{Y}]$ is the natural actions on $L(\lambda)$, and the T-action on $\mathbb{C}[\mathscr{Y}]$ comes from the grading $\mathbb{X}_+ \subset \mathbb{X}$ in the RHS of (2.1). These data, together with the condition $E \neq \mathscr{Y}$, essentially determine $\mathbb{C}[\mathscr{Y}]$ as $\mathbb{C}$-algebras generated by $L(\varpi_i)^*$ for $1 \le i \le r$. Consider a point $x_0 \in \mathscr{Y}$ given by $\{\mathbf{v}_\lambda\}_\lambda$, seen as linear maps on $\{L(\lambda)^*\}_\lambda$. The image $[x_0]$ of this point x_0 has its G-stabilizer equal to B. This induces an inclusion

$$G/B \hookrightarrow \mathscr{B} \subset \prod_{i=1}^{r} \mathbb{P}\big(L(\varpi_i)\big)$$

induced from $B/B \mapsto [x_0]$ by the G-action. (One needs additional representation-theoretic analysis to conclude $G/B \cong \mathscr{B}$.) This consideration transfers all geometric statements relevant to $\mathscr{B}$ to algebraic statements on the space in (2.1) in principle, but most of the geometric results on $\mathscr{B}$ and its subvarieties were proved for the first time by other methods (see, e.g., [56]).

Note that the vector space (2.1) does not acquire the structure of a ring when $\mathsf{char}\, \Bbbk = p > 0$. The reason is that we do not have a map $L(\lambda)^* \otimes L(\mu)^* \to L(\lambda+\mu)^*$, or equivalently, $L(\lambda+\mu) \to L(\lambda) \otimes L(\mu)$ for general $\lambda, \mu \in \mathbb{X}_+$. One way to improve the situation is to replace $\{L(\lambda)\}_{\lambda \in \mathbb{X}_+}$ with a suitable family of modules $\{Y(\lambda)\}_{\lambda \in \mathbb{X}_+}$ with larger members such that the G-module map

$$Y(\lambda+\mu) \to Y(\lambda) \otimes Y(\mu) \tag{2.2}$$

exists uniquely (up to constant) for every $\lambda, \mu \in \mathbb{X}_+$. It yields an analogous ring of (2.1) that should be closely related to $\mathscr{B}$. A standard choice of $Y(\lambda)$ ($\lambda \in \mathbb{X}_+$) is the Weyl module $V(\lambda)$ of G, that is, the projective cover of $L(\lambda)$ in the categories of rational G-modules whose composition factors are in $\{L(\mu)\}_{\lambda \ge \mu \in \mathbb{X}_+}$, where $\ge$ is the dominance ordering on $\mathbb{X}$. This produces $\mathscr{B}$ for all characteristics.

Theorem 2.1 (Orthogonality of Weyl modules, **[36, II §4.13]**). *For each $\lambda, \mu \in \mathbb{X}_+$, we have*

$$\mathrm{Ext}^i_G\big(V(\lambda), V(\mu)^*\big) \cong \Bbbk^{\oplus \delta_{i,0}\delta_{\lambda,\mu^*}},$$

where μ^* is the highest weight of $L(\mu)^*$. By taking the Euler–Poincaré characteristic, this Ext*-orthogonality implies the orthogonality of the T-characters of $V(\lambda)$. In particular, the T-characters of $V(\lambda)$ do not depend on $\Bbbk$.*

Note that $L(\lambda) = V(\lambda)$ for $\mathrm{char}\,\Bbbk = 0$ by the semisimplicity of representations, and hence Theorem 2.1 is Schur's lemma in such a case. As $V(\lambda) = \Bbbk \otimes_{\mathbb{Z}} V_{\mathbb{Z}}(\lambda)$ holds for a collection of free $\mathbb{Z}$-modules $V_{\mathbb{Z}}(\lambda)$ ($\lambda \in \mathbb{X}_+$), we find that $\mathscr{B}$ extends to a scheme flat over $\mathbb{Z}$. Another possible choice of $Y(\lambda)$ ($\lambda \in \mathbb{X}_+$), the Verma module $M(\lambda)$ of the (divided power) enveloping algebra of Lie G, produces an open dense B-orbit in $\mathscr{B}$.

3. KAC–MOODY FLAG VARIETIES

Let us keep the setting of the previous section.

3.1. Reminder on Kac–Moody algebras and their quantum groups

Let $\mathfrak{g}_C$ be the Kac–Moody algebra associated to a generalized Cartan matrix (= GCM) C (see [38]). In case $\mathrm{char}\,\Bbbk = 0$, we have the notion of the highest weight integrable representations of $\mathfrak{g}_C$ parametrized by the set of dominant weights P_+ defined similarly to $\mathbb{X}_+$. Let $L(\Lambda)$ denote the highest weight integrable representation of $\mathfrak{g}_C$ corresponding to $\Lambda \in P_+$.

We have the quantum group (or the quantized enveloping algebra) $U_q(\mathfrak{g}_C)$ of $\mathfrak{g}_C$ originally defined by Drinfeld and Jimbo in the 1980s [17, 37]. It is an algebra defined over $\mathbb{Q}(q)$, and the specialization $q \mapsto 1$ recovers the universal enveloping algebra $U(\mathfrak{g}_C)$ of $\mathfrak{g}_C$. Kashiwara [41] and Lusztig [63] defined the canonical/global bases (of the positive/negative parts $U_q^{\pm}(\mathfrak{g}_C)$) of $U_q(\mathfrak{g}_C)$ and their integrable representations that generate their $\mathbb{Q}[q]$-lattices. The construction of Lusztig [64] clarified that quantum groups are, in fact, defined over $\mathbb{Z}[q^{\pm 1}]$ (or even over $\mathbb{N}[q^{\pm 1}]$ if one can say). In the 2010s, the categorification theorems of a quantum group and its integrable representations appeared [39, 53, 76, 77], and there every algebra that admits a categorification has a suitable $\mathbb{Z}[q]$-integral structure with distinguished bases, being the Grothendieck group of a module category of a finitely-generated graded algebras (called KLR algebras or quiver Hecke algebras). Therefore, the following is now widely recognized:

Theorem 3.1 (Lusztig [63, 64, 66] and Kashiwara [41–43]). *Assume that $\Bbbk = \mathbb{C}$. The (lower) global bases of $U_q^{\pm}(\mathfrak{g}_C)$ induce a $\mathbb{Z}$-integral form $U_{\mathbb{Z}}(\mathfrak{g}_C)$ of $U(\mathfrak{g}_C)$ via $q \mapsto 1$. For each $\Lambda \in P_+$, we have a $\mathbb{Z}$-lattice $L(\Lambda)_{\mathbb{Z}}$ of $L(\Lambda)$ obtained from the (lower) global base of the corresponding integrable highest weight module of $U_q(\mathfrak{g}_C)$. In addition, $L(\Lambda)_{\mathbb{Z}}$ is generated by the $U_{\mathbb{Z}}(\mathfrak{g}_C)$-action from a highest weight vector of $L(\Lambda)$.*

By a specialization of $L(\Lambda)_{\mathbb{Z}}$, we obtain a highest weight integrable module $L(\Lambda)$ over an arbitrary field $\Bbbk$. The module $L(\Lambda)$ is no longer irreducible when $\mathrm{char}\,\Bbbk > 0$ (in general), and hence it is a $\mathfrak{g}_C$-analogue of Weyl modules rather than $L(\lambda)$ for G; it is a lack of brevity of the author to choose this notation here. We close this subsection by noting that the integral forms at the end of Section 2 coincide with the integral forms in Theorem 3.1.

3.2. Thin and thick flag varieties

Presentations of the flag varieties for general Kac–Moody groups $\mathscr{G}$ associated to a GCM C are similar to those in the previous section. A triangular decomposition of $\mathfrak{g}_C$ yields an analogous group $\mathscr{I}$ to the Borel subgroup. Let $\mathscr{T}$ be a (standard) maximal torus of $\mathscr{I}$. The highest weight vector in $L(\Lambda)$ is precisely an $\mathscr{I}$-eigenvector with its $\mathscr{T}$-weight Λ. Therefore, the construction in the previous section produces $\mathscr{G}/\mathscr{I}$ via the ring

$$\bigoplus_{\Lambda \in P_+} L(\Lambda)^\vee \subset \bigoplus_{\Lambda \in P_+} L(\Lambda)^*, \tag{3.1}$$

where $L(\Lambda)^*$ is the vector space dual of $L(\Lambda)$, and $L(\Lambda)^\vee$ is the restricted dual of $L(\Lambda)$, defined to be the direct sum of (finite-dimensional) vector space duals offered by the $\mathscr{T}$-weight decomposition of $L(\Lambda)$.

In this case, both vector spaces in (3.1) are naturally rings. This corresponds to the choice of $\mathscr{G}$. The former ring defines $\mathscr{B}_C^{\text{thick}} = \mathscr{G}/\mathscr{I}$ [40, 49, 71] if we take $\mathscr{G}$ to be a version of the Kac–Moody group that is completed with respect to the opposite direction to $\mathscr{I}$. (This is the maximal Kac–Moody group, but the completion is taken in the opposite way as in the literature.) The latter ring can be seen as the projective limit of finitely-generated algebras, and the union of the spectrums of these rings yields $\mathscr{B}_C^{\text{thin}} = \mathscr{G}/\mathscr{I}$ [56, 75] if we take $\mathscr{G}$ as the uncompleted Kac–Moody group (the Kac–Peterson group or the minimal Kac–Moody group), or as the maximal Kac–Moody group completed with respect to the direction of $\mathscr{I}$. In other words, we have variants of flag manifolds of Kac–Moody groups associated to a GCM C as:

$$\bigcup_n \mathscr{B}_{C,n}^{\text{thin}} = \mathscr{B}_C^{\text{thin}} \subset \mathscr{B}_C^{\text{thick}}. \tag{3.2}$$

The scheme $\mathscr{B}_C^{\text{thick}}$ is a union of infinite-dimensional affine spaces, and hence is smooth. However, $\mathscr{B}_C^{\text{thick}}$ is not compact in an essential way [24]. This picture is compatible with the fact that the Kac–Peterson group is defined by one-parameter generators (and relations), and hence $\mathscr{B}_C^{\text{thin}}$ is a union of finite-dimensional subvarieties $\mathscr{B}_{C,n}^{\text{thin}}$ consisting of points presented by a product of at most n generating elements. As such, each scheme $\mathscr{B}_{C,n}^{\text{thin}}$ is singular, and hence $\mathscr{B}_C^{\text{thin}}$ is understood to be singular. In fact, it does not admit an inductive limit description by finite-dimensional smooth pieces [24].

4. GLOBAL WEYL MODULES AND THEIR PROJECTIVITY

Let us consider the untwisted affine Kac–Moody case hereafter, with the same conventions as in the previous sections. In particular, our Kac–Moody groups are extensions of the groups

$$G(\!(z)\!) := G\big(\Bbbk(\!(z)\!)\big) \quad \text{and} \quad G\big[z^{\pm 1}\big] := G\big(\Bbbk\big[z^{\pm 1}\big]\big)$$

by the loop rotation $\mathbb{G}_m$-actions (that we denote by $\mathbb{G}_m^{\text{rot}}$) and the central extension $\mathbb{G}_m$-actions. (These correspond to the maximal/minimal realizations of the Kac–Moody groups in the previous section.) These are not (pro-)algebraic groups, and it sometimes causes difficulty. Nevertheless, each rational representation V of G induces representations

$$V(\!(z)\!) := V \otimes_{\Bbbk} \Bbbk(\!(z)\!) \quad \text{and} \quad V\left[z^{\pm 1}\right] := V \otimes_{\Bbbk} \Bbbk\left[z^{\pm 1}\right]$$

of $G(\!(z)\!)$ and $G[z^{\pm 1}]$, respectively. These representations are not of highest weight, but still integrable representations when we lift them to the central extensions of $G(\!(z)\!)$ and $G[z^{\pm 1}]$ by letting the center $\mathbb{G}_m$ act trivially (i.e., they are level-zero integrable representations viewed as representations of affine Lie algebras).

In addition to the T-action, the representation $V[z^{\pm 1}]$ carries $\mathbb{G}_m^{\mathrm{rot}}$-action. Let δ be the degree-one character of $\mathbb{G}_m^{\mathrm{rot}}$, and set $q := e^{\delta}$. By abuse of notation, we might consider q^n ($n \in \mathbb{Z}$) as the functor that twists the $\mathbb{G}_m^{\mathrm{rot}}$-action by degree n. We define a graded character of a semisimple $(T \times \mathbb{G}_m^{\mathrm{rot}})$-module U as

$$\mathsf{gch}\, U := \sum_{n \in \mathbb{Z}} \sum_{\lambda \in \mathbb{X}} q^n e^{\lambda} \dim \mathrm{Hom}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathbb{C}_{\lambda + n\delta}, U).$$

Then, $\mathsf{gch}\, V[z^{\pm 1}]$ makes sense as all the coefficients are in $\mathbb{Z}$. However, if we take the second symmetric power $S^2(V[z^{\pm 1}])$ of $V[z^{\pm 1}]$ over $\Bbbk$, then it contains an infinity as a coefficient. To avoid such a complication, we sometimes restrict ourselves to the subgroups

$$G[\![z]\!] := G\big(\Bbbk[\![z]\!]\big) \subset G(\!(z)\!) \quad \text{and} \quad G[z] := G\big(\Bbbk[z]\big) \subset G\left[z^{\pm 1}\right].$$

We sometimes use the subgroup $\mathbf{I} \subset G[\![z]\!]$ defined by the pullback of B under the evaluation map $\mathtt{ev}_0 : G[\![z]\!] \to G$ at $z = 0$. The group $\mathbf{I}$ is the Iwahori subgroup obtained from (the completed version of) $\mathscr{I}$ by removing $\mathbb{G}_m^{\mathrm{rot}}$ and quotient out by the central extension.

By the quotient map $\Bbbk[z] \to \Bbbk$ (and $\Bbbk[\![z]\!] \to \Bbbk$) sending $z \mapsto 0$, we can regard every rational G-module V as a $G[z]$-module or a $G[\![z]\!]$-module with (trivial) $\mathbb{G}_m^{\mathrm{rot}}$-action through $\mathtt{ev}_0$. We also have a $G[\![z]\!]$-module structure (without a $\mathbb{G}_m^{\mathrm{rot}}$-action) on $V[\![z]\!] := V \otimes \Bbbk[\![z]\!]$ that surjects onto V.

Definition 4.1 (global Weyl modules). Let $\mathscr{C}(\lambda)$ be the category of rational $G[z]$-modules M that admits a decreasing filtration

$$M = F_0 M \supset F_1 M \supset F_2 M \supset \cdots \quad \text{such that} \bigcap_{k \geq 0} F_k M = \{0\}$$

and each $F_k M / F_{k-1} M$ ($k \geq 1$) belongs to $\{q^m L(\mu)\}_{m \in \mathbb{Z}, \lambda \geq \mu \in \mathbb{X}_+}$. For each $\lambda \in \mathbb{X}_+$, we define the global Weyl module $\mathbb{W}(\lambda)$ of $G[z]$ as the projective cover of $L(\lambda)$ in $\mathscr{C}(\lambda)$.

Note that $\mathbb{W}(\lambda)$ automatically acquires a $\mathbb{G}_m^{\mathrm{rot}}$-action by its universality (as it exists).

Theorem 4.2. *For each $\lambda \in \mathbb{X}_+$ with $\lambda = \sum_{i=1}^r m_i \varpi_i$, we have*

$$\mathrm{End}_{G[z]}\, \mathbb{W}(\lambda) \cong \bigotimes_{i=1}^{r} \Bbbk[x_{i,1}, \ldots, x_{i,m_i}]^{\mathfrak{S}_{m_i}},$$

where each $x_{i,1}, \ldots, x_{i,m_i}$ is of degree one with respect to the $\mathbb{G}_m^{\mathrm{rot}}$-action. In addition, the action of $\mathrm{End}_{G[z]}\, \mathbb{W}(\lambda)$ *on* $\mathbb{W}(\lambda)$ *is free.*

Theorem 4.2 was proved by Fourier–Littelmann [25] (for $\Bbbk = \mathbb{C}$ and G of type ADE), Naoi [72] (for $\Bbbk = \mathbb{C}$ and G of type BCFG), and it was transferred to char $\Bbbk > 0$ in [50] using results from the global bases of quantum affine algebras [4, 42].

By Theorem 4.2, we factor out the positive degree parts of $\mathrm{End}_{G[z]} \mathbb{W}(\lambda)$ to obtain

$$W(\lambda) := \Bbbk \otimes_{\mathrm{End}_{G[z]} \mathbb{W}(\lambda)} \mathbb{W}(\lambda), \quad \lambda \in \mathbb{X}_+.$$

We call it a local Weyl module of $G[z]$.

The following result clarifies that our global/local Weyl modules are the best possible analogues of Weyl modules for G (see Theorem 2.1):

Theorem 4.3 (Chari–Ion [14] for char $\Bbbk = 0$, and [50] + ε for char $\Bbbk > 0$). *For each $\lambda, \mu \in \mathbb{X}_+$, we have*

$$\mathrm{Ext}^i_{G[z]}\big(\mathbb{W}(\lambda), W(\mu)^*\big) \cong \Bbbk^{\oplus \delta_{i,0}\delta_{\lambda,\mu^*}}, \tag{4.1}$$

where μ^* is the highest weight of $L(\mu)^*$. By taking the graded Euler–Poincaré characteristic, (4.1) implies the orthogonality of Macdonald polynomials with respect to the Macdonald pairing specialized to $t = 0$. In particular, gch $W(\lambda)$ *and* gch $\mathbb{W}(\lambda)$ *do not depend on $\Bbbk$.*

The proof of Theorem 4.3 in [50, §3.3] relies on the adjoint property of the Demazure functors observed in [20, Proposition 5.7] and systematically utilized in [15]. The case $\lambda = \mu^*$ and $i > 1$ in Theorem 4.3 is not recorded in [50], and might appear elsewhere.

5. SEMI-INFINITE FLAG MANIFOLDS

We keep the setting of the previous section. In view of the projectivity of $\mathbb{W}(\lambda)$'s in $\mathcal{C}(\lambda)$'s, we find unique degree-zero $G[z]$-module maps

$$\mathbb{W}(\lambda + \mu) \to \mathbb{W}(\lambda) \otimes \mathbb{W}(\mu), \quad \lambda \in \mathbb{X}_+. \tag{5.1}$$

Therefore, the recipe described in Section 2 equips

$$R_G := \bigoplus_{\lambda \in \mathbb{X}_+} \mathbb{W}(\lambda)^\vee$$

with a structure of a commutative algebra compatible with the action of $G[z] \rtimes \mathbb{G}_m^{\mathrm{rot}} \times T$. Since the $\mathbb{G}_m^{\mathrm{rot}}$-degree of R_G is bounded from the above, the $G[z]$-action on R_G automatically extends to the $G[\![z]\!]$-action. We set

$$\mathbf{Q}_G := (\mathrm{Spec}\, R_G \setminus E)/T,$$

where E is a closed subset of Spec R_G on which the T-action is not free. Let us consider the $G(\!(z)\!)$-orbit of

$$\{[\mathbf{v}_{\varpi_i}]\}_{i=1}^r \in \prod_{i=1}^r \mathbb{P}\big(V(\varpi_i)(\!(z)\!)\big), \tag{5.2}$$

viewed as a set of points, that we denote by $\mathcal{Q}_G$. By examining the coefficients of the defining relations of $\mathscr{B}$ with its $\Bbbk(\!(z)\!)$-valued points, we find that the intersection

$$\mathcal{Q}_G \cap \prod_{i=1}^{r} \mathbb{P}\big(V(\varpi_i)[\![z]\!]z^{m_i}\big) \subset \prod_{i=1}^{r} \mathbb{P}\big(V(\varpi_i)[\![z]\!]z^{m_i}\big) \subset \prod_{i=1}^{r} \mathbb{P}\big(V(\varpi_i)(\!(z)\!)\big) \tag{5.3}$$

defines a closed subscheme for any choice of $m_1, \ldots, m_r \in \mathbb{Z}$. We denote this subscheme by $\mathbf{Q}_G(t_\beta)$, where $\beta = \sum_{i=1}^{r} m_i \alpha_i^\vee$ is an element of the dual lattice (coroot lattice) $\mathbb{X}^\vee$ of $\mathbb{X}$ equipped with a basis $\{\alpha_i^\vee\}_{i=1}^r$ such that $\alpha_i^\vee(\varpi_j) = \delta_{i,j}$ (i.e., $\alpha_i^\vee$ is a simple coroot). We note that $\mathbb{P}(V(\varpi_i)[\![z]\!]z^{m_i})$ is a scheme, but it is not of finite type, and $\mathbf{Q}_G(t_\beta)$ is also of infinite type.

Lemma 5.1. *We have $\mathbf{Q}_G(t_\beta) \cong \mathbf{Q}_G(t_\gamma)$ for each pair $\beta, \gamma \in \mathbb{X}^\vee$ as schemes equipped with $G[\![z]\!]$-actions. Hence, the union*

$$\mathbf{Q}_G^{\mathrm{rat}} = \bigcup_{\beta} \mathbf{Q}_G(t_\beta)$$

is a pure ind-scheme of ind-infinite type equipped with the action of $G[\![z]\!] \rtimes \mathbb{G}_m^{\mathrm{rot}}$. Moreover, the set of $G[\![z]\!]$-orbits in $\mathbf{Q}_G^{\mathrm{rat}}$ is in bijection with $\mathbb{X}^\vee$.

In effect, we have an open dense $G[\![z]\!]$-orbit $\mathbf{O}_G(t_\beta) \subset \mathbf{Q}_G(t_\beta)$ that is isomorphic to $G[\![z]\!]/(T \cdot N[\![z]\!])$. By the Bruhat decomposition, we divide $\mathbf{O}_G(t_\beta)$ into the disjoint union of $\mathbf{I}$-orbits as $\bigsqcup_{w \in W} \mathbf{O}(wt_\beta)$ such that $\mathbf{O}(t_\beta) \subset \mathbf{O}_G(t_\beta)$ is open dense. Identifying $\beta \in \mathbb{X}^\vee$ with t_β, we set $W_{\mathrm{af}} := W \ltimes \mathbb{X}^\vee$. We define

$$\mathbf{Q}_G(w) := \overline{\mathbf{O}(w)} \subset \mathbf{Q}_G^{\mathrm{rat}}, \quad w \in W_{\mathrm{af}}.$$

The inclusion relation on $\{\mathbf{Q}_G(w)\}_{w \in W_{\mathrm{af}}}$ is described by the generic Bruhat order [62]. We refer to the partial order on W_{af} induced from this closure ordering by $\le_{\frac{\infty}{2}}$ as in [50, 52] (there we sometimes called $\le_{\frac{\infty}{2}}$ as the semi-infinite Bruhat order).

Theorem 5.2. *The scheme $\mathbf{Q}_G(w)$ is normal for each $w \in W_{\mathrm{af}}$. In addition, the ind-scheme $\mathbf{Q}_G^{\mathrm{rat}}$ is a strict ind-scheme in the sense that each inclusion is a closed immersion. The ind-scheme $\mathbf{Q}_G^{\mathrm{rat}}$ coarsely ind-represents the coset $G(\!(z)\!)/(T \cdot N(\!(z)\!))$.*

The first two statements are proved in [52] when $\mathrm{char}\, \Bbbk = 0$. The proof valid for $\mathrm{char}\, \Bbbk \neq 2$, as well as the last assertion, are contained in [50]. This last assertion says that the (ind-)scheme $\mathbf{Q}_G^{\mathrm{rat}}$ is the universal one that maps to every (ind-)scheme whose points yield $\mathcal{Q}_G$. It follows that if we take a family $\{\mathbb{Y}(\lambda)\}_{\lambda \in \mathbb{X}_+}$ instead of $\{\mathbb{W}(\lambda)\}_{\lambda \in \mathbb{X}_+}$ to define $\mathbf{Q}_G(t_\beta)$, then the corresponding coordinate ring R'_G admits a map to R_G. Let us point out that this can be thought of as a family version of the properties of global Weyl modules discussed in Section 4, and we indeed have several reasonable choices of $\{\mathbb{Y}(\lambda)\}_{\lambda \in \mathbb{X}_+}$ other than $\{\mathbb{W}(\lambda)\}_{\lambda \in \mathbb{X}_+}$ including the coordinate ring of the arc scheme of G/N. For simplicity, we may refer to $\mathbf{Q}_G(t_0)$ as $\mathbf{Q}_G$ below.

The inclusion

$$\mathbf{Q}_G \subset \prod_{i=1}^{r} \mathbb{P}\big(V(\varpi_i)[\![z]\!]\big) \tag{5.4}$$

induces a line bundle $\mathcal{O}_{\mathbf{Q}_G}(\varpi_i)$ on $\mathbf{Q}_G$, that is, the pull-back of $\mathcal{O}(1)$ from $\mathbb{P}(V(\varpi_i)[\![z]\!])$. By taking the tensor products, we have $\mathcal{O}_{\mathbf{Q}_G}(\lambda) := \bigotimes_{i=1}^r \mathcal{O}_{\mathbf{Q}_G}(\varpi_i)^{\otimes n_i}$ for $\lambda = \sum_{i=1}^r n_i \varpi_i$ $(n_i \in \mathbb{Z})$. By Lemma 5.1, we have $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$ $(\lambda \in \mathbb{X})$ on $\mathbf{Q}_G^{\mathrm{rat}}$ that yields $\mathcal{O}_{\mathbf{Q}_G}(\lambda)$ by restriction.

Theorem 5.3 ([52] for $\mathsf{char}\,\Bbbk = 0$, and [50] for $\mathsf{char}\,\Bbbk \neq 2$). *For each $\lambda \in \mathbb{X}$, we have*

$$H^i\left(\mathbf{Q}_G, \mathcal{O}_{\mathbf{Q}_G}(\lambda)\right)^{\vee} \cong \begin{cases} \mathbb{W}(\lambda), & i = 0, \lambda \in \mathbb{X}_+, \\ \{0\}, & \textit{otherwise}. \end{cases}$$

The proof of Theorem 5.3 depends on the freeness of R_G over an infinitely-many-variable polynomial ring, that yields a regular sequence of infinite length. Such a situation never occur for finite type schemes, or infinite type schemes like $\mathcal{B}_C^{\mathrm{thick}}$. In case $G = \mathrm{SL}(2)$, Theorem 5.3 reduces to an exercise in algebraic geometry by $\mathbf{Q}_G \cong \mathbb{P}(\Bbbk^2[\![z]\!])$.

Theorem 5.3 has an ind-model counterpart proved earlier [10]. The Frobenius splitting of $\mathbf{Q}_G$ (explained later) and Theorem 5.3 imply this ind-model counterpart. However, the author is uncertain whether [10] implies Theorem 5.3 (even in case $\mathsf{char}\,\Bbbk = 0$) since the natural ring coming from the ind-model is a completion of R_G, and the completion operation of a ring loses information in general. We have an analogue of Theorem 5.3 for all $\mathbf{I}$-orbit closures, proved for the ind-model in [46, 50] and for the formal model in [50, 52].

6. FROBENIUS SPLITTINGS

We continue to work in the setting of the previous section. We fix a prime $p > 0$. For a scheme $\mathfrak{X}$ over $\mathbb{F}_p$, we have a Frobenius morphism $\mathsf{Fr} : \mathfrak{X} \to \mathfrak{X}$ induced from the pth power map. We have a natural map $\mathsf{Fr}^* \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ that induces a map $\mathcal{O}_{\mathfrak{X}} \to \mathsf{Fr}_* \mathcal{O}_{\mathfrak{X}}$ by adjunction. The Frobenius splitting $\phi : \mathsf{Fr}_* \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$ is an $\mathcal{O}_{\mathfrak{X}}$-module map such that the composition

$$\mathcal{O}_{\mathfrak{X}} \to \mathsf{Fr}_* \mathcal{O}_{\mathfrak{X}} \xrightarrow{\phi} \mathcal{O}_{\mathfrak{X}}$$

is the identity. If $\mathfrak{X}$ is projective (and is of finite type) and $\mathcal{O}_{\mathfrak{X}}$ admits a Frobenius splitting, then $\mathfrak{X}$ is reduced and an ample line bundle has the higher cohomology vanishing [68].

For generality on Frobenius splittings, as well as their applications to $\mathcal{B}$ and $\mathcal{B}_C^{\mathrm{thin}}$, we refer to Brion–Kumar [12] (note that [12] has a finite type assumption, that we drop in case the proof does not require it. In the paragraph above, reducedness does not require the finite type assumption, while the higher cohomology vanishing requires the finite type assumption through the Serre vanishing). Frobenius splitting of $\mathcal{B}$ in $\mathsf{char}\,\Bbbk = p$ is useful in proving that Schubert and Richardson varieties are reduced, normal, and have rational singularities. There are two major ways to construct a Frobenius splitting of $\mathcal{B}$: one is to investigate the global section of the $(1-p)$th power of the canonical bundle, and the other is to use a Bott–Samelson–Demazure–Hansen (=BSDH) resolution of $\mathcal{B}$.

Since $\mathcal{B}_C^{\mathrm{thin}}$ is no longer smooth, we cannot use the canonical bundle to construct a Frobenius splitting. Nevertheless, a (partial) BSDH resolution does the job. The situation of $\mathcal{B}_C^{\mathrm{thick}}$ is a bit worse. The canonical bundle of $\mathcal{B}_C^{\mathrm{thick}}$ makes some sense, but the author does not know whether it has enough power to produce a Frobenius splitting. The scheme $\mathcal{B}_C^{\mathrm{thick}}$

admits a BSDH resolution, but it is a successive $\mathbb{P}^1$-fibration over an infinite-type scheme. Thus, we cannot equip $\mathscr{B}_C^{\text{thick}}$ with a Frobenius splitting by either of the above means at present. Despite this, we can transfer a Frobenius splitting of $\mathscr{B}_C^{\text{thin}}$ to $\mathscr{B}_C^{\text{thick}}$ by using the compatible splitting property of a point [49], following an idea of Mathieu.

Frobenius splitting of $\mathbf{Q}_G^{\text{rat}}$ (or rather each of its ind-piece $\mathbf{Q}_G(w)$) is used below, and hence we need a recipe to produce one. However, the situation of the BSDH resolution is similar to that of $\mathscr{B}_C^{\text{thick}}$, and the canonical bundle on $\mathbf{Q}_G^{\text{rat}}$ simply does not make sense naively (e.g., its T-weight at a point must be infinity). Therefore, we need a new proof strategy. Our strategy in [50] is to regard R_G as a subalgebra of the corresponding coordinate ring of $\mathscr{B}_C^{\text{thick}}$, and prove that a Frobenius splitting of $\mathscr{B}_C^{\text{thick}}$ preserves R_G. For this, we first see that each $\mathbb{W}(m\lambda)$ ($m \in \mathbb{Z}_{>0}$, $\lambda \in \mathbb{X}_+$) is a quotient of $L(m\Lambda)$ for some $\Lambda \in P_+$ by twisting the $G[z^{-1}]$-action into a $G[z]$-action as $z^{-1} \mapsto z$. Let $\pi_m : L(m\Lambda) \to \mathbb{W}(m\lambda)$ be the quotient map. This embeds (a suitable $\mathbb{Z}$-graded subalgebra of) R_G into (3.1) as an algebra with $G[\![z]\!] \ltimes \mathbb{G}_m^{\text{rot}}$-action. We need to show that the map $\phi^\vee$ obtained by dualizing the Frobenius splitting of $\mathscr{B}_C^{\text{thick}}$ induces a map $\phi_{\mathbb{W}}^\vee$ in the following diagram:

$$\begin{tikzcd} L(m\Lambda) \arrow[r, "\phi^\vee"] \arrow[d, "\pi_m"] & L(pm\Lambda) \arrow[r] \arrow[d, "\pi_{pm}"] & L(m\Lambda) \arrow[d, "\pi_m"] \\ \mathbb{W}(m\lambda) \arrow[r, dashed, "\phi_{\mathbb{W}}^\vee"] & \mathbb{W}(pm\lambda) \arrow[r] & \mathbb{W}(m\lambda). \end{tikzcd} \tag{6.1}$$

This is equivalent to seeing that $\phi^\vee(\ker \pi_m) \subset \ker \pi_{pm}$. We use the projectivity of $\mathbb{W}(m\lambda)$ in $\mathfrak{C}(m\lambda)$ to assume that the $G[z]$-module generators of $\ker \pi_m$ have T-weights that do not appear in $\mathbb{W}(m\lambda)$. In view of the fact that $\ker \pi_{pm}$ contains all the T-weight spaces in $L(pm\Lambda)$ whose T-weights do not appear in $\mathbb{W}(pm\lambda)$, we have necessarily $\phi^\vee(\ker \pi_m) \subset \ker \pi_{pm}$ by the T-weight comparison of the generators.

In fact, every $L(\Lambda)$ admits a filtration by global Weyl modules when $\operatorname{char} \Bbbk = 0$ if we twist the action of $G[z]$ on global Weyl modules into $G[z^{-1}]$ [51]. Therefore, we indeed obtain a Frobenius splitting of $\mathbf{Q}_G$ via a novel proof based on the "universality" of the global Weyl module $\mathbb{W}(\lambda)$ explained in Section 4. In conclusion, we have:

Theorem 6.1 ([50, THEOREM B]). *The ind-scheme $\mathbf{Q}_G^{\text{rat}}$ admits a Frobenius splitting that is compatible with all $\mathbf{I}$-orbits when* $\operatorname{char} \Bbbk > 2$.

7. CONNECTION TO THE SPACE OF RATIONAL MAPS

Keep the setting as in Section 5. Let us consider the vector space embedding $\Bbbk(\!(z)\!) \subset \Bbbk[\![z, z^{-1}]\!]$ into the formal power series with unbounded powers. The space $\Bbbk[\![z, z^{-1}]\!]$ no longer forms a ring. Nevertheless, we have an automorphism of $\Bbbk[\![z, z^{-1}]\!]$ by swapping z with z^{-1}. Together with the Chevalley involution of G (an automorphism of G that sends each element of T to its inverse), it induces an involution θ on the ambient space

$$\mathbf{Q}_G^{\text{rat}} \subset \prod_{i=1}^{r} \mathbb{P}\big(V(\varpi_i)[\![z, z^{-1}]\!]\big).$$

We remark that θ induces an automorphism of G such that $B \cap \theta(B) = T$. Let w_0 be the longest element in W.

Theorem 7.1 ([50, THEOREM B]). *For all $w, v \in W_{\mathrm{af}}$, the scheme-theoretic intersection $\mathbf{Q}_G(w) \cap \theta(\mathbf{Q}_G(vw_0))$ is reduced (we denote this intersection by $\mathscr{Q}_G(w, v)$ and call it a Richardson variety of $\mathbf{Q}_G^{\mathrm{rat}}$ below). It is normal when* $\mathsf{char}\, \Bbbk = 0$ *or* $\mathsf{char}\, \Bbbk \gg 0$.

The scheme $\mathscr{Q}_G(w, v)$ is always of finite type, and the case $w, v \in W$ yields a Richardson variety of $\mathscr{B}$. The normality part of the proof of Theorem 7.1 goes as follows: Our Frobenius splitting of $\mathbf{Q}_G^{\mathrm{rat}}$ induces a Frobenius splitting of $\mathscr{Q}_G(w, v)$. In particular, it is reduced and weakly normal in $\mathsf{char}\, \Bbbk > 2$. (Here a weakly normal ring is essentially a normal ring up to topology.) Then, we lift the weak normality to characteristic zero and prove the normality of the intersection by a geometric consideration. Once we deduce the normality in characteristic zero, we can reduce it to $\mathsf{char}\, \Bbbk \gg 0$ by a general result.

Let us exhibit some relevant geometric considerations here. To this end, we assume $\Bbbk = \mathbb{C}$ in the rest of this section. Recall that $H_2(\mathscr{B}, \mathbb{Z}) \cong \mathbb{X}^\vee$. Let $\mathscr{G}\mathscr{B}_{2,\beta}$ (resp. $\mathscr{B}_{2,\beta}$) be the space of genus-zero stable maps with two marked points to $(\mathbb{P}^1 \times \mathscr{B})$ (resp. $\mathscr{B}$) whose image has class $(1, \beta) \in H_2(\mathbb{P}^1 \times \mathscr{B}, \mathbb{Z})$ (resp. $\beta \in H_2(\mathscr{B}, \mathbb{Z})$), regarded as an algebraic variety with rational singularities [28]. We have a subvariety $\mathscr{G}\mathscr{B}^\flat_{2,\beta}$ such that the first marked point lands in $0 \in \mathbb{P}^1$ and the second marked point lands in $\infty \in \mathbb{P}^1$ through the composition

$$\big(C, \{x_1, x_2\}\big) \xrightarrow{f} \mathbb{P}^1 \times \mathscr{B} \xrightarrow{\mathrm{pr}_1} \mathbb{P}^1.$$

Consider the Schubert variety (a B-orbit closure) $\mathscr{B}(w) \subset \mathscr{B}$ corresponding to $w \in W$ and the opposite Schubert variety (a $\theta(B)$-orbit closure) $\mathscr{B}^{\mathrm{op}}(v) \subset \mathscr{B}$ corresponding to $v \in W$.

Let $\mathtt{ev}_i : \mathscr{G}\mathscr{B}^\flat_{2,\beta} \to \mathscr{B}$ $(i = 1, 2)$ denote the evaluation at the point x_i on C. We define

$$\mathscr{G}\mathscr{B}_\beta(w, v) := \mathtt{ev}_1^{-1}\big(\mathscr{B}(w)\big) \cap \mathtt{ev}_2^{-1}\big(\mathscr{B}^{\mathrm{op}}(v)\big).$$

Similarly, let $\mathtt{e}_i : \mathscr{B}_{2,\beta} \to \mathscr{B}$ $(i = 1, 2)$ be the evaluation maps. For all $w, v \in W$ and $\beta \in \mathbb{X}^\vee$, we set $\mathscr{B}_\beta(w, v) := (\mathtt{e}_1^{-1}(\mathscr{B}(w)) \cap \mathtt{e}_2^{-1}(\mathscr{B}^{\mathrm{op}}(v)))$. Let $\mathring{\mathscr{Q}}_G(\beta)$ denote the space of maps from $\mathbb{P}^1$ to $\mathscr{B}$ of degree β. By adding the identity map to $\mathbb{P}^1$, each point of $\mathring{\mathscr{Q}}_G(\beta)$ yields a map $\mathbb{P}^1 \to (\mathbb{P}^1 \times \mathscr{B})$ of degree $(1, \beta)$. In addition, the identification of two $\mathbb{P}^1$'s completely determines the marked points. Hence we have an inclusion $\mathring{\mathscr{Q}}_G(\beta) \subset \mathscr{G}\mathscr{B}^\flat_{2,\beta}$.

Let $\mathscr{Q}_G(\beta)$ $(\beta \in \mathbb{X}^\vee)$ denote the space of quasimaps from $\mathbb{P}^1$ to $\mathscr{B}$ of degree β [22], that is, a natural compactification of $\mathring{\mathscr{Q}}_G(\beta)$ such that

$$\mathscr{Q}_G(\beta) = \bigsqcup_{0 \le \gamma \le \beta} \mathring{\mathscr{Q}}_G(\beta - \gamma) \times (\mathbb{P}^1)^\gamma,$$

where $\gamma \le \beta$ is defined as $\beta - \gamma \in \sum_{i=1}^r \mathbb{Z}_{\ge 0} \alpha_i^\vee$, and

$$(\mathbb{P}^1)^\gamma = \prod_{i=1}^r \big((\mathbb{P}^1)^{m_i} / \mathfrak{S}_{m_i}\big) \quad \text{where } \gamma = \sum_{i=1}^r m_i \alpha_i^\vee.$$

Here $(\mathbb{P}^1)^\gamma$ records the place where the degree of the genuine map drops in which degree components (without ordering). By adding extra $\mathbb{P}^1$ components and (compatible) maps to

$\mathscr{B}$ to $\mathbb{P}^1$ in $(f : \mathbb{P}^1 \to \mathscr{B}) \in \mathring{\mathscr{Q}}_G(\beta - \gamma)$ at the places (and total degrees) recorded by $(\mathbb{P}^1)^\gamma$ (for each $0 \le \gamma \le \beta$), we obtain a map of topological spaces

$$\pi : \mathscr{G}\mathscr{B}^\flat_{2,\beta} \to \mathscr{Q}_G(\beta),$$

that is an identity on $\mathring{\mathscr{Q}}_G(\beta)$. Givental's main lemma asserts that this is a birational morphism of normal algebraic varieties.

Proposition 7.2 ([50, §5.2]). *For each $\beta \in \mathbb{X}^\vee$, we have*

$$\mathscr{Q}_G(\beta) \cong \mathscr{Q}_G(e, t_\beta)$$

as schemes. In addition, π restricts to a birational morphism

$$\pi_{\beta,w,v} : \mathscr{G}\mathscr{B}_\beta(w, v) \to \mathscr{Q}_G(w, vt_\beta), \quad w, v \in W.$$

In particular, we have $\mathscr{G}\mathscr{B}_\beta(w, v) \neq \emptyset$ if and only if $w \le_{\frac{\infty}{2}} vt_\beta$, and its dimension is given by the distance between w and vt_β with respect to $\le_{\frac{\infty}{2}}$.

In other words, the Richardson varieties of $\mathbf{Q}^{\mathrm{rat}}_G$ are precisely the spaces of quasi-maps, possibly with additional conditions imposed by the space of stable maps. According to Buch–Chaput–Mihalcea–Perrin [13], the variety $\mathscr{G}\mathscr{B}_\beta(w, v)$ is irreducible and has rational singularities if it is nonempty. Hence, we find that $\mathscr{Q}_G(w, vt_\beta)$ is irreducible in general. Proposition 7.2 and properties of the maps $\pi_{\beta,w,v}$ are used in our proof of Theorem 7.1.

Proposition 7.2 implies that $\mathscr{Q}_G(w, vt_\beta)$ is the closure (in $\mathscr{Q}_G(\beta)$) of the space of maps from $\mathbb{P}^1$ to $\mathscr{B}$ such that $0, \infty \in \mathbb{P}^1$ land in $\mathscr{B}(w)$ and $\mathscr{B}^{\mathrm{op}}(v)$, respectively. By examining the natural map $\mathscr{G}\mathscr{B}_\beta(w, v) \to \mathscr{B}_\beta(w, v)$ (obtained by forgetting the map to $\mathbb{P}^1$), we obtain:

Corollary 7.3. *For all $w, v \in W$ and $0 \neq \beta \in \mathbb{X}^\vee$, we have*

$$\dim \mathscr{B}_\beta(w, v) = \dim \mathscr{G}\mathscr{B}_\beta(w, v) - 1 \quad \textit{if } \mathscr{G}\mathscr{B}_\beta(w, v) \neq \emptyset,$$

and $\mathscr{B}_\beta(w, v) \neq \emptyset$ if and only if $\mathscr{G}\mathscr{B}_\beta(w, v) \neq \emptyset$. Moreover, we have

$$\mathscr{B}_\beta(w, v) \neq \emptyset \quad \textit{and} \quad \dim \mathscr{B}_\beta(w, v) = 0$$

if and only if $w \le_{\frac{\infty}{2}} vt_\beta$ are adjacent with respect to $\le_{\frac{\infty}{2}}$. In such a case, $\mathscr{B}_\beta(w, v)$ is a point.

Thanks to the dimension axiom in quantum correlators [54, (2.5)], Corollary 7.3 describes which (primary) two-point cohomological Gromov–Witten invariant of $\mathscr{B}$ with respect to the Schubert bases is nonzero (we can also tell its exact value). By the divisor axiom [54, §2.2.4] and the classical Chevalley formula [16], we find the Chevalley formula in quantum cohomology of $\mathscr{B}$ from this [29]. This clarifies the role of $\mathscr{Q}_G(w, vt_\beta)$ in the study of quantum cohomology of $\mathscr{B}$ from our perspective.

Theorem 7.4 ([47]). *Let $\beta \in \mathbb{X}^\vee$ and $w, v \in W$. The variety $\mathscr{Q}_G(w, vt_\beta)$ has rational singularities.*

Theorem 7.4 is proved by Braverman–Finkelberg [9,10] for the case $w = e$, $v = w_0$ by an analysis of Zastava spaces, which does not extend to general w, v. Theorem 7.4 is the most subtle technical point in [47] and its induction steps become possible by Theorem 7.1.

8. K-THEORETIC PETERSON ISOMORPHISM

We follow the setting of the previous section with $\Bbbk = \mathbb{C}$. We understand that the K-groups appearing here contain a suitable class of line bundles supported on subvarieties equipped with some group actions, and its scalar is extended from $\mathbb{Z}$ to $\mathbb{C}$. Let $\mathrm{Gr}_G := G(\!(z)\!)/G[\![z]\!]$ be the affine Grassmannian of G. The set of $\mathbf{I}$-orbits in Gr_G is in bijection with $\mathbb{X}^\vee$, while the set of $G[\![z]\!]$-orbits of Gr_G is in bijection with $\mathbb{X}^\vee_< \subset \mathbb{X}^\vee$ formed by the set of antidominant coroots. For $\beta \in \mathbb{X}^\vee$, we set $\mathring{\mathrm{Gr}}_G(\beta) \subset \mathrm{Gr}_G$ as the corresponding $\mathbf{I}$-orbit and set $\mathrm{Gr}_G(\beta) := \overline{\mathring{\mathrm{Gr}}_G(\beta)} \subset \mathrm{Gr}_G$. We normalize so that $\mathrm{Gr}_G(\beta)$ is G-stable when $\beta \in \mathbb{X}^\vee_<$, and we have $\dim \mathrm{Gr}_G(\beta) = -2|\beta|$ in such a case, where $|\beta| := \sum_{i=1}^r \beta(\varpi_i)$.

We define

$$K_T(\mathrm{Gr}_G) := \bigcup_{\beta \in \mathbb{X}^\vee} K_T\big(\mathrm{Gr}_G(\beta)\big) \quad \text{and} \quad K_G(\mathrm{Gr}_G) := \bigcup_{\beta \in \mathbb{X}^\vee_<} K_G\big(\mathrm{Gr}_G(\beta)\big).$$

These spaces are equipped with the convolution product, defined by the diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G(\!(z)\!) \times \mathrm{Gr}_G \xrightarrow{q} G(\!(z)\!) \times_{\mathbf{I}} \mathrm{Gr}_G \xrightarrow{\mathrm{mult}} \mathrm{Gr}_G$$

as follows: For all cycles $a, b \in K_T(\mathrm{Gr}_G) \cong K_{\mathbf{I}}(\mathrm{Gr}_G)$, we find a left $\mathbf{I}$-equivariant class (a, b) on $G(\!(z)\!) \times_{\mathbf{I}} \mathrm{Gr}_G$ such that

$$p^*(a \boxtimes b) = q^*(a, b)$$

and set

$$a \odot' b := \sum_{i \geq 0} (-1)^i \big[\mathbb{R}^i \,\mathrm{mult}_*(a, b)\big] \in K_{\mathbf{I}}(\mathrm{Gr}_G).$$

This yields an associative product structure on $K_T(\mathrm{Gr}_G)$ that contains a zero divisor. If we restrict ourselves to $K_G(\mathrm{Gr}_G)$, then the algebra structure given by $\odot'$ becomes commutative and integrally closed. Using an isomorphism $K_T(\mathrm{pt}) \otimes_{K_G(\mathrm{pt})} K_G(\mathrm{Gr}_G) \cong K_T(\mathrm{Gr}_G)$ of $K_T(\mathrm{pt})$-modules, we find a multiplication $\odot$ of $K_T(\mathrm{Gr}_G)$ that extends $\odot'$ on $K_G(\mathrm{Gr}_G)$ as a $K_T(\mathrm{pt})$-algebra. This product $\odot$ coincides with a K-theoretic analogue of the Pontrjagin product (by the calculations in [47, §2.2]). In addition, we have

$$[\mathcal{O}_{\mathrm{Gr}_G(\beta+\gamma)}] = [\mathcal{O}_{\mathrm{Gr}_G(\beta)}] \odot [\mathcal{O}_{\mathrm{Gr}_G(\gamma)}] \quad \text{for} \quad \beta, \gamma \in \mathbb{X}^\vee_<.$$

This yields a multiplicative system in $K_T(\mathrm{Gr}_G)$, whose localization is denoted by $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$.

The (localized) small T-equivariant quantum K-group of $\mathscr{B}$ is defined as a vector space

$$qK_T(\mathscr{B})_{\mathrm{loc}} := K_T(\mathscr{B}) \otimes \mathbb{C}\mathbb{X}^\vee \quad \big(\equiv K_T(\mathscr{B}) \otimes_{\mathbb{C}} \mathbb{C} H_2(\mathscr{B}, \mathbb{Z})\big).$$

We denote the variable corresponding to $\beta \in \mathbb{X}^\vee$ as Q^β. The quantum K-theoretic product $\star$ is a binary operation on $qK_T(\mathscr{B})_{\mathrm{loc}}$, defined by Givental [33] and Lee [61], whose value (a priori) belongs to a completion of $qK_T(\mathscr{B})_{\mathrm{loc}}$. It is one of the consequence of our analysis that $\star$ preserves $qK_T(\mathscr{B})_{\mathrm{loc}}$. This is usually referred to as the *finiteness* of the quantum K-theoretic product (for $\mathscr{B}$) in the literature [1,13], and is one of the most fundamental questions in the study of $qK_T(\mathscr{B})$. Lam–Li–Mihalcea–Shimozono [58] conjectured that:

Theorem 8.1 ([47]). *We have an isomorphism of commutative algebras*

$$K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \xrightarrow{\cong} qK_T(\mathscr{B})_{\mathrm{loc}}$$

such that

$$[\mathcal{O}_{\mathrm{Gr}_G(w\beta)}] \odot [\mathcal{O}_{\mathrm{Gr}_G(\gamma)}]^{-1} \mapsto [\mathcal{O}_{\mathscr{B}(w)}] Q^{\beta-\gamma}$$

for $\beta, \gamma \in \mathbb{X}^\vee_<$ such that $\beta(\varpi_i) < 0$ for every $1 \le i \le r$.

Note that a presentation of the ring $qK_T(\mathscr{B})$ for $G = \mathrm{SL}(n)$ can be read-off from Givental–Lee [34], and a presentation of the ring $K_T(\mathrm{Gr}_G)$ is obtained in Bezrukavnikov–Finkelberg–Mirković [6]. However, these are not enough to yield Theorem 8.1 (for $G = \mathrm{SL}(n)$) as the correspondence between Schubert bases is unclear.

We have an action of the nilpotent version $\mathscr{H}^{\mathrm{nil}}$ of the double affine Hecke algebra (associated to G) on $K_T(\mathrm{Gr}_G)$, coming from Kostant–Kumar [55]. In [47], we defined the T-equivariant K-group $K_T(\mathbf{Q}_G^{\mathrm{rat}})$ of $\mathbf{Q}_G^{\mathrm{rat}}$ based on the construction of the $(T \times \mathbb{G}_m^{\mathrm{rot}})$-equivariant K-group of $\mathbf{Q}_G^{\mathrm{rat}}$ in [52]. The $\mathbf{I}$-action on $\mathbf{Q}_G^{\mathrm{rat}}$ induces a $\mathscr{H}^{\mathrm{nil}}$-action on $K_T(\mathbf{Q}_G^{\mathrm{rat}})$.

The object $K_T(\mathbf{Q}_G^{\mathrm{rat}})$ needs a completion in order to admit an action of the line bundle twists by $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$ ($\lambda \in \mathbb{X}$). It reflects the fact that the right-hand side of Theorem 5.3 (i.e., a global Weyl module) is infinite-dimensional in general, and hence the effect of $\otimes \mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\varpi_i)$ ($1 \le i \le r$) requires infinitely many terms to describe.

Our main idea in the proof of Theorem 8.1 is to put $\mathbf{Q}_G^{\mathrm{rat}}$ into the picture:

Theorem 8.2 ([47, THEOREM C]). *We have a commutative diagram*

$$\begin{tikzcd} & K_T(\mathbf{Q}_G^{\mathrm{rat}}) & \\ K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \arrow[ur, hook, "\Phi"] \arrow[rr] & & qK_T(\mathscr{B})_{\mathrm{loc}} \arrow[ul, hook', "\Psi"'] \end{tikzcd}$$

that respects the Schubert bases in each object. In addition, the map Ψ is an embedding of representations of $\mathscr{H}^{\mathrm{nil}}$, and the map Ψ intertwines the tensor product with $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(-\varpi_i)$ in $K_T(\mathbf{Q}_G^{\mathrm{rat}})$ and the quantum product of $\mathcal{O}_{\mathscr{B}}(-\varpi_i)$ on $qK_T(\mathscr{B})_{\mathrm{loc}}$ for each $1 \le i \le r$.

The completion of $K_T(\mathbf{Q}_G^{\mathrm{rat}})$ is compatible with the standard completion of $qK_T(\mathscr{B})$ via the map Ψ. Theorem 8.2 implies that the inverse of the operation $\star\mathcal{O}_{\mathscr{B}}(-\varpi_i)$ makes sense only after the completion of $qK_T(\mathscr{B})_{\mathrm{loc}}$.

Since the quantum K-theoretic correlators (see [33,61]) satisfy neither the dimension axiom nor divisor axiom as in the theory of quantum cohomology, the proof of Theorem 8.2

must be necessarily different from Corollary 7.3. Our construction of the map Ψ is based on the following two observations:

- an interpretation of the ($\mathbb{G}_m$-equivariant) quantum K-theoretic correlator
$$\chi\big(\mathcal{Q}(w, w_0 t_\beta), \mathcal{O}_{\mathcal{Q}(w,w_0t_\beta)}(\lambda)\big) = \chi\big(\mathcal{GB}_\beta(w, w_0), \pi^*_{\beta,w,w_0}\mathcal{O}_{\mathcal{Q}(w,w_0t_\beta)}(\lambda)\big), \tag{8.1}$$
valued in $\mathbb{C}[T][q^{\pm 1}] = \mathbb{C}[T \times \mathbb{G}_m]$, for each $w \in W$, $\beta \in \mathbb{X}^\vee$, $\lambda \in \mathbb{X}_+$;
- an interpretation of its asymptotic behavior
$$\lim_{\beta \to \infty} \chi\big(\mathcal{Q}(w, w_0 t_\beta), \mathcal{O}_{\mathcal{Q}(w,w_0t_\beta)}(\lambda)\big) = \chi\big(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)\big) \in \mathbb{C}(\!(q^{-1})\!)[T] \tag{8.2}$$
for each $w \in W$, $\lambda \in \mathbb{X}_+$ as an element of $K_T(\mathbf{Q}_G^{\mathrm{rat}})$.

Here we can further interpret $\chi(\mathcal{GB}_\beta(w, w_0), \pi^*_{\beta,w,w_0}\mathcal{O}_{\mathcal{Q}(w,w_0t_\beta)}(\lambda))$ using the shift operators of line bundles in quantum K-theory **[35, PROPOSITION 2.13]**, and hence we obtain an (abstract) presentation of $qK_T(\mathcal{B})$ from (8.1) by the reconstruction theorem **[35, PROPOSITION 2.12]**. The identity (8.1) is a consequence of Theorem 7.4, and (8.2) is a consequence of compatible Frobenius splitting properties of $\mathcal{Q}_G(w, v)$s and $\mathbf{Q}_G^{\mathrm{rat}}$ in $\mathsf{char}\, \Bbbk > 2$ (see the explanation about the proof of Theorem 7.1).

There is a noncommutative version of Theorem 8.2, meaning that we include $\mathbb{G}_m^{\mathrm{rot}}$ (the variable "q" above) in each item **[49]**.

9. FUNCTORIALITY OF QUANTUM K-GROUPS

We continue to work in the setting as in the previous section. In **[50]**, we have presented analogues of Theorems 5.2, 5.3, and 7.1 for partial flag manifolds of G. Let us find a standard parabolic subgroup $B \subset P \subset G$ and consider $\mathcal{B}_P := G/P$. Our parabolic version of the semi-infinite flag manifold $\mathbf{Q}_{G,P}^{\mathrm{rat}}$ has its set of $\Bbbk$-valued points $G(\!(z)\!)/(T \cdot [P, P](\!(z)\!))$. The fiber of the natural map
$$\pi_P : \mathbf{Q}_G^{\mathrm{rat}} \to \mathbf{Q}_{G,P}^{\mathrm{rat}}$$
is isomorphic to the semi-infinite flag manifold of $[L, L]$, where $L \subset P$ is the maximal semisimple subgroup of P that contains T (the standard Levi subgroup). We also have a higher cohomology vanishing of equivariant line bundles on $\mathbf{Q}_{G,P}^{\mathrm{rat}}$ (or rather $\pi_P(\mathbf{Q}_G)$) as in Theorem 5.3. These are enough to yield a morphism
$$K_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathbf{Q}_G^{\mathrm{rat}}) \to K_{T\times\mathbb{G}_m^{\mathrm{rot}}}(\mathbf{Q}_{G,P}^{\mathrm{rat}})$$
obtained by the push-forward by π_P (up to technical reservations neglected here and below).

By transferring Theorem 7.4 to Richardson varieties of $\mathbf{Q}_{G,P}^{\mathrm{rat}}$, we find a map
$$\Psi_P : qK_T(\mathcal{B}_P)_{\mathrm{loc}} \to K_T(\mathbf{Q}_{G,P}^{\mathrm{rat}}),$$

that intertwines appropriate line bundle twists (and analogous quantum multiplications). This yields a diagram

$$\begin{tikzcd} qK_T(\mathscr{B})_{\mathrm{loc}} \arrow[r, "\Psi"] \arrow[d, dashed] & K_T(\mathbf{Q}_G) \arrow[d, "(\pi_P)_*"] \\ qK_T(\mathscr{B}_P)_{\mathrm{loc}} \arrow[r, "\Psi_P"] & K_T(\mathbf{Q}_{G,P}) \end{tikzcd}$$

where we set $\mathbf{Q}_{G,P} := \pi_P(\mathbf{Q}_G)$.

The resulting map $qK_T(\mathscr{B}) \to qK_T(\mathscr{B}_P)$ is, in fact, an algebra map [48], and is easy to describe. Note that we cannot have an analogous map between ordinary K-groups because of the higher direct images. It turns out this map sends $Q^{\alpha_i^\vee}$ to 1 for a simple coroot $\alpha_i^\vee$ belonging to L, and hence is not compatible with a naive generalization of the corresponding map in the Peterson isomorphism in homology [59].

We also have a restriction map $qK_T(\mathscr{B}) \to qK_T(\mathscr{B}^L)$, where $\mathscr{B}^L := L/(L \cap B)$ is the flag manifold of a standard Levi subgroup. This map extends to algebra maps [45]

$$K_{G \times \mathbb{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G) \to K_{L \times \mathbb{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_L) \to K_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_T)$$

anticipated in Finkelberg and Tsymbaliuk [23].

10. SOME PERSPECTIVES

Compared with the theory of flag manifolds, many precise results and constructions for $\mathbf{Q}_G^{\mathrm{rat}}$ are still missing. The most accessible set of problems might be to spell out analogues of numerous explicit formulas in classical Schubert calculus purely combinatorially by admitting geometric conclusions from [3,45,47,48,52] partly explained in the previous two sections. We close this note by briefly discussing some of other problems.

10.1. Categorifications of the coordinate rings

The homogeneous coordinate rings of Schubert varieties of a usual flag manifold, that are B-stable quotient rings of (2.1), can be seen as the Grothendieck groups of suitable categories equipped with cluster structures ([60]; see also Section 3.1). Hence, it is natural to expect categorifications of the homogeneous coordinate rings of $\mathbf{Q}_G^{\mathrm{rat}}(w)$ and $\mathscr{B}_C^{\mathrm{thick}}$. See also [21] and [43] for related problems and partial answers.

10.2. Peterson isomorphism in quantum cohomology

The Peterson isomorphism in quantum cohomology [59,74] is an analogue of Theorem 8.1 for homology. We may apply Corollary 7.3 to [69] (that is an essential ingredient in [59]) to utilize $\mathbf{Q}_G^{\mathrm{rat}}$ in its proof (that looks similar to the original strategy in [74]). However, we do not know an analogue of Theorem 8.2 as we lack a proper definition of $H^\bullet(\mathbf{Q}_G)$.

10.3. Constructible sheaves on semi-infinite flags

In representation-theoretic analysis on $\mathscr{B}$, we sometimes encounter constructible sheaves that are not N-equivariant. Also, we want some notion of (co)homology of $\mathbf{Q}_G$ in

Section 10.2. Therefore, it is desirable to understand constructible sheaves on $\mathbf{Q}_G$ following [7]. The resulting objects should have connection to [30]. Note that the combinatorics that should be satisfied by the $\mathbf{I}$-equivariant sheaves (equipped with Frobenius endomorphisms) have been worked out in detail [62, 65].

10.4. Tensor product decompositions

The tensor product decomposition of rational representations of G is deeply connected with our whole story due to the presentation (2.1). In [57], the geometry of flag varieties is used to deduce subtle information on the tensor products beyond the classical Littlewood–Richardson rule. It would be interesting to pursue their analogues in $\mathbf{Q}_G^{\mathrm{rat}}$, possibly utilizing some modular interpretation [11] and connecting with [5].

10.5. The cotangent bundle of semi-infinite flags

A version of the cotangent bundle of $\mathbf{Q}_G^{\mathrm{rat}}$ would make it possible to compare our results with the perspectives in [21, 67, 73]. In addition, its quantization should realize some numerics in Section 10.3. The author hopes to say a bit more on this in St. Petersburg.

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