Darboux coordinates on the BFM spaces*

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Abstract

Bezrukavnikov-Finkelberg-Mirković [Compos. Math. **141** (2005)] identified the equivariant K-group of an affine Grassmannian, that we refer as (the coordinate ring of) a BFM space á là Teleman [Proc. ICM Seoul (2014)], with a version of Toda lattice. We give a new system of generators and relations of a certain localization of this space, that can be seen as a version of its Darboux coordinate. This establishes a conjecture in Finkelberg-Tymbaliuk [Progress in Math. **300** (2019)] that relates the BFM space of a connected reductive algebraic group with those of Levi subgroups.

Introduction

Let G be a connected reductive algebraic group over $\mathbb{C}$. Let B be a Borel subgroup of G and let $H \subset B$ be its maximal torus. Let Gr_G denote the (thin) affine Grassmannian of G. The G-equivariant K-group $K_G(\mathrm{Gr}_G)$ of Gr_G admits the structure of an algebra, and it is identified with the phase space of the relativistic Toda lattice in [4]. In particular, the space $K_G(\mathrm{Gr}_G)$ carries a Poisson bracket. Braverman-Finkelberg-Nakajima [42, 9, 10] constructed a commutative algebra $\mathcal{A}(G, V)$ for each representation V of G, whose spectrum is supposed to be a part of the space of vacua in the corresponding three-dimensional gauge theory. The space Gr_G played an essential rôle there, and we have a Poisson algebra embedding

$$\mathcal{A}(G, V) \hookrightarrow \mathcal{A}(G, \{0\}) = K_G(\mathrm{Gr}_G). \tag{0.1}$$

In addition, Teleman [45] gives a recipe to understand $\mathcal{A}(G, V)$ from $K_G(\mathrm{Gr}_G)$.

Associated to G, we have its flag manifold $\mathscr{B}$. In [26, 25], we have constructed a ring morphism connecting $K_G(\mathrm{Gr}_G)$ with the equivariant quantum K-group $qK_G(\mathscr{B})$ of $\mathscr{B}$ ([19, 36]):

$$K_G(\mathrm{Gr}_G)_{\mathrm{loc}} \cong qK_G(\mathscr{B})_{\mathrm{loc}}, \tag{0.2}$$

where the subscripts "loc" denote certain localizations, whose meaning *differs* in the both sides. This result, commonly referred to as the K-theoretic Peterson isomorphism ([34]), also exhibits an aspect of the rich structures of $K_G(\mathrm{Gr}_G)$.

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Finkelberg-Tymbaliuk [17] extensively studied $K_{GL(n)}(\mathrm{Gr}_{GL(n)})$ and deduced an algebra morphism

$$K_{GL(n)}(\mathrm{Gr}_{GL(n)}) \longrightarrow K_L(\mathrm{Gr}_L) \tag{0.3}$$

for a connected (standard) Levi subgroup $L \subset GL(n)$. As this homomorphism is an incarnation of the coproduct structure of their shifted affine quantum groups (and also as they have similar homomorphisms for homologies [15]), they led to conjecture that (0.3) exists for every connected reductive G and also with the extra $\mathbb{G}_m$-action given by the loop rotation action.

The goal of this paper is to answer this conjecture affirmatively as:

Theorem A ($\doteq$ Theorem 5.1 + Corollary 5.2)**.** *For each connected reductive subgroup $H \subset L \subset G$, we have a chain of injective algebra homomorphisms:*

$$K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G) \hookrightarrow K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L) \hookrightarrow K_{H\times\mathbb{G}_m}(\mathrm{Gr}_H).$$

Since the main portion of Theorem A is the case of simple and simply connected G, we concentrate into this case in the rest of this introduction.

Here $K_{H\times\mathbb{G}_m}(\mathrm{Gr}_H)$ is the (quantized) Heisenberg algebra, and hence this embedding can be seen to equip each $K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L)$ with its Darboux coordinate system. In addition, Corollary 3.10 supplies its modification that describes a certain localization of the ring $K_{L\times\mathbb{G}_m}(\mathrm{Gr}_L)$. This makes $K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)$ into (the quantized phase space of) an integrable system called the relativistic Toda lattice, as described in Bezrukavnikov-Finkelberg-Mirković [4]. In view of the homology version of (0.2) discovered by Peterson [44], it can be understood as the K-theoretic version of the fundamental presentation of (equivariant) quantum cohomology of flag varieties due to Givental-Kim [22] and Kim [30].

In the course of the proof of Theorem A, we exhibit the non-commutative version of the main result in [26]:

Theorem B ($\doteq$ Corollary 3.3 and Theorem 3.7)**.** *We have a commutative diagram, whose bottom arrow is an isomorphism of non-commutative rings:*

$$\begin{array}{ccc} & K_{H\times\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}}) & \\ {\scriptstyle\Phi}\nearrow & & \nwarrow{\scriptstyle\Psi} \\ K_{H\times\mathbb{G}_m}(\mathrm{Gr}_G)_{\mathrm{loc}} & \xrightarrow{\hspace{3em}} & qK_{H\times\mathbb{G}_m}(\mathcal{B})_{\mathrm{loc}} \end{array},$$

where $\mathbf{Q}_G^{\mathrm{rat}}$ is the semi-infinite flag manifold of G ([25]). Moreover, all of these morphisms respect Schubert bases.

Our strategy to prove Theorem A is as follows: We first refine some of the algebraic arguments in [26] to prove Theorem B. Then, we transplant the natural operations of $K_{G\times\mathbb{G}_m}(\mathbf{Q}_G^{\mathrm{rat}})$ and give an algebra generator set of a suitable localization $K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)_{\mathrm{loc}}$ of $K_{G\times\mathbb{G}_m}(\mathrm{Gr}_G)$ in term of the Heisenberg action of $K_{H\times\mathbb{G}_m}(\mathrm{Gr}_H)$. These boil down the proof of Theorem A into a comparison of integral structures. For this comparison, we prove the ($\mathbb{G}_m$-equivariant version of the) following, best expressed in the language of quantum K-groups.

Let $\mathcal{B}^L$ be the flag variety of L. Let $\mathbb{X}^*$ be the weight lattice of H. Let $\{\varpi_i\}_{i\in\mathtt{I}}$ be the set of fundamental weights with respect to $H \subset B$. We have line bundles $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ and $\mathcal{O}_{\mathcal{B}^L}(-\varpi_i)$ on $\mathcal{B}$ and $\mathcal{B}^L$, respectively. Let $Q_+^\vee$ denote the nonnegative span of positive coroots of G, and let $Q_{L,+}^\vee$ denote the nonnegative span of positive coroots of L. We have a natural inclusion $Q_{L,+}^\vee \subset Q_+^\vee$. Let us employ the definition of quantum K-groups as:

$$qK_G(\mathcal{B}) = K_G(\mathcal{B}) \otimes \mathbb{C}[\![Q_+^\vee]\!] \quad \text{and} \quad qK_L(\mathcal{B}^L) = K_L(\mathcal{B}^L) \otimes \mathbb{C}[\![Q_{L,+}^\vee]\!],$$

where $\beta \in Q_+^\vee$ defines a formal variable $Q^\beta \in \mathbb{C}[\![Q_+^\vee]\!]$. These spaces are equipped with the commutative ring structures whose multiplications are denoted by $\star$. The multiplication $\star$ coincides with the usual multiplications rules of $K_G(\mathcal{B})$ or $K_L(\mathcal{B}^L)$ by setting $Q^\beta = 0$ for all $\beta \neq 0$.

Theorem C ($\doteq$ Theorem 4.1)**.** *There exists a surjective morphism of rings*

$$qK_G(\mathcal{B}) \longrightarrow\!\!\!\!\!\rightarrow qK_L(\mathcal{B}^L)$$

obtained by setting $Q^\beta \equiv 0$ for $\beta \in Q_+^\vee \setminus Q_{L,+}^\vee$. This morphism sends the quantum multiplication of $\mathcal{O}_{\mathcal{B}}(-\varpi_i)$ to the quantum multiplication by $\mathcal{O}_{\mathcal{B}^L}(-\varpi_i)$ for each $i \in \mathtt{I}$.

We remark that the classical analogue of Theorem C is an isomorphism, sometimes referred to as the "induction equivalence". We present a direct proof in the main body of this paper, that yields an interesting representation theoretic consequence (Corollary 4.4), though it holds in much greater generality (Theorem A.1). Theorems C and [27, Theorem A] upgrade the key observations in Leoung-Li [37] to the K-theoretic settings.

Example D. Assume that $G = SL(n, \mathbb{C})$. Let us choose the fundamental weights $\varpi_1, \ldots, \varpi_{n-1}$ and simple coroots $\alpha_1^\vee, \ldots, \alpha_{n-1}^\vee$ in accordance with the table in the end of Bourbaki [6]. We understand that $\varpi_n = 0$. Let $V = \mathbb{C}^n$ be the dual vector representation of G. According to Givental-Lee [20], we have

$$\operatorname{ch} V = [\mathcal{O}_{\mathcal{B}}(-\varpi_1)] + \sum_{i=1}^{n-1} a^{\varpi_i}([\mathcal{O}_{\mathcal{B}}(-\varpi_{i+1})]) \in qK_G(\mathcal{B}),$$

where we have $a^{\varpi_i} = (1 - Q^{\alpha_i^\vee})([\mathcal{O}_{\mathcal{B}}(-\varpi_i)]\star)^{-1} \in \operatorname{End} qK_G(\mathcal{B})$. Let $L \subset G$ be a Levi subgroup. If we specialize $Q^{\alpha_i^\vee} = 0$ when $\alpha_i^\vee \notin Q_{L,+}^\vee$, then the effect of $\operatorname{ch} V$ restricts to that of $qK_L(\mathcal{B}^L)$. When $\alpha_i^\vee \notin Q_{L,+}^\vee$, the effect a^{ϖ_i} becomes a character twist on $qK_L(\mathcal{B}^L)$.

Here we warn that the definition of quantum K-groups, as well as the normalizations in Theorem C and Example D are different from the main body of the paper for the sake of simplicity of expositions.

The organization of this paper is as follows: After recalling preliminary stuffs in §1, we provide a certain collection of elements in the equivariant K-groups of semi-infinite flag manifolds (Theorem 2.15) in §2. These collections are the "reduced version" of line bundles, and the only non-trivial point is that we can divide the classes of line bundles properly. Using these elements, we provide (Proposition 3.8) a new system of generators of $K_G(\mathrm{Gr}_G)_{\mathrm{loc}}$ in §3. In order to transplant elements from semi-infinite flag manifolds to affine Grassmannian, we prove Theorem B (Theorem 3.7). In §4, we prove Theorem C (Theorem 4.1), that is an essential tool to compute the "leading terms" of the the maps in Theorem A. Using them, we prove Theorem A in §5. In Appendix A, we present an another proof of Theorem C (Theorem A.1) that applies in much greater generality.

1 Preliminaries

A vector space is always a $\mathbb{C}$-vector space, and a graded vector space refers to a $\mathbb{Z}$-graded vector space whose graded pieces are finite-dimensional and its grading is bounded from the above. Tensor products are taken over $\mathbb{C}$ unless stated otherwise. We define the graded dimension of a graded vector space as

$$\mathrm{gdim}\, M := \sum_{i \in \mathbb{Z}} q^i \dim_{\mathbb{C}} M_i \in \mathbb{Q}(\!(q^{-1})\!).$$

We set $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$. As a rule, we suppress $\emptyset$ and associated parenthesis from notation. This particularly applies to $\emptyset = \mathtt{J} \subset \mathtt{I}$ frequently used to specify parabolic subgroups.

1.1 Groups, root systems, and Weyl groups

Basically, material presented in this subsection can be found in [13, 33].

Let G be a connected, reductive algebraic group over $\mathbb{C}$ such that $[G, G]$ is a simply connected group of rank r and we have a complementary torus H' such that $G \cong [G, G] \times H'$. Let B and H be a Borel subgroup and a maximal torus of G such that $H \subset B$. We set N $(= [B, B])$ to be the unipotent radical of B. We denote the Lie algebra of an algebraic group by the corresponding German small letter. We have a (finite) Weyl group $W := N_G(H)/H$. For an algebraic group E, we denote its set of $\mathbb{C}[z]$-valued points by $E[z]$, its set of $\mathbb{C}[\![z]\!]$-valued points by $E[\![z]\!]$, and its set of $\mathbb{C}(z)$-valued points by $E(z)$. Let $\mathbf{I} \subset G[\![z]\!]$ be the preimage of $B \subset G$ via the evaluation at $z = 0$ (the Iwahori subgroup of $G[\![z]\!]$).

Let $\mathbb{X}^* := \mathrm{Hom}_{gr}(H, \mathbb{G}_m)$ be the weight lattice of H, and let $\mathbb{X}^*(G)$ denote the subgroup of $\mathbb{X}^*$ whose elements define characters of G. We set $\mathbb{X}_*$ and $\mathbb{X}_*(G)$ as the dual lattices of $\mathbb{X}^*$ and $\mathbb{X}^*(G)$, respectively. We denote the natural pairings of lattices by $\langle \bullet, \bullet \rangle$.

Let $\Delta \subset \mathbb{X}^*$ be the set of roots, let $\Delta_+ \subset \Delta$ be the set of roots that yield root subspaces in $\mathfrak{b}$, and let $\Pi \subset \Delta_+$ be the set of simple roots. We set $\Delta_- := -\Delta_+$. Let $Q^\vee \subset \mathbb{X}_*$ be the $\mathbb{Z}$-span of coroots. We define $\Pi^\vee \subset Q^\vee$ to be the set of positive simple coroots, and let $Q^\vee_+ \subset Q^\vee$ be the set of non-negative integer span of $\Pi^\vee$. For $\beta, \gamma \in \mathbb{X}_*$, we define $\beta \geq \gamma$ if and only if $\beta - \gamma \in Q^\vee_+$. Let $\mathtt{I} := \{1, 2, \ldots, r\}$. We fix bijections $\mathtt{I} \cong \Pi \cong \Pi^\vee$ such that $i \in \mathtt{I}$ corresponds to $\alpha_i \in \Pi$, its coroot $\alpha_i^\vee \in \Pi^\vee$, and a simple reflection $s_i \in W$ corresponding to α_i. We also have a reflection $s_\alpha \in W$ corresponding to $\alpha \in \Delta_+$. For each $\mathtt{J} \subset \mathtt{I}$, we set $\mathbb{X}^*_+(\mathtt{J}) := \{\lambda \in \mathbb{X}^* \mid \langle \alpha_i^\vee, \lambda \rangle \geq 0, \ \ \forall i \in \mathtt{J}\}$. Let $\{\varpi_i\}_{i \in \mathtt{I}} \subset \mathbb{X}^*_+$ be the set of fundamental weights (i.e. $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{i,j}$) and we set $\rho := \sum_{i \in \mathtt{I}} \varpi_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in \mathbb{X}^*_+$.

For a subset $\mathtt{J} \subset \mathtt{I}$, we define $P^{\mathtt{J}}$ to be the standard parabolic subgroup of G corresponding to $\mathtt{J}$. I.e. we have $\mathfrak{b} \subset \mathfrak{p}^{\mathtt{J}} \subset \mathfrak{g}$ and $\mathfrak{p}^{\mathtt{J}}$ contains the root subspace corresponding to $-\alpha_i$ $(i \in \mathtt{I})$ if and only if $i \in \mathtt{J}$. Then, the set of characters of $P^{\mathtt{J}}$ is identified with $\mathbb{X}^*_0(\mathtt{J}) := \mathbb{X}^*(G) \oplus \Lambda^{(\mathtt{I} \setminus \mathtt{J})}$, where we set $\Lambda^{\mathtt{J}} := \sum_{i \in \mathtt{J}} \mathbb{Z} \varpi_i$. We also set

$$\Lambda^{\mathtt{J}}_{++} := \sum_{j \in \mathtt{J}} \mathbb{Z}_{>0} \varpi_j \subset \Lambda^{\mathtt{J}}_+ := \sum_{j \in \mathtt{J}} \mathbb{Z}_{\geq 0} \varpi_j \subset \mathbb{X}^*, \quad Q^\vee_{\mathtt{J},+} := \sum_{j \in \mathtt{J}} \mathbb{Z}_{\geq 0} \alpha_j^\vee \subset Q^\vee_{\mathtt{J}} := \sum_{j \in \mathtt{J}} \mathbb{Z} \alpha_j^\vee.$$

We define $W^{\mathtt{J}} \subset W$ to be the subgroup generated by $\{s_i\}_{i \in \mathtt{J}}$. It is the Weyl group of the maximal reductive subgroup $L^{\mathtt{J}}$ of $P^{\mathtt{J}}$ that contains H (we refer $L^{\mathtt{J}}$ as the standard Levi subgroup of $P^{\mathtt{J}}$ in the below).

Let $\lambda \in \mathbb{X}^*$. We consider the subset

$$\Sigma(\lambda) := \text{convex span of } \{W\lambda\} \subset \mathbb{X}^* \otimes_{\mathbb{Z}} \mathbb{R}.$$

We set $\Sigma_*(\lambda) := \Sigma(\lambda) \setminus \{W\lambda\}$.

We set $\mathbf{G} := G \times \mathbb{G}_m$, $\mathbf{L}^{\mathtt{J}} := L^{\mathtt{J}} \times \mathbb{G}_m$, and $\mathbf{H} := H \times \mathbb{G}_m$ for the simplicity of notation.

Let $\Delta_{\mathrm{af}} := \Delta \times \mathbb{Z}\delta \cup \{m\delta\}_{m \neq 0}$ be the untwisted affine root system of Δ with its positive part $\Delta_+ \subset \Delta_{\mathrm{af},+}$. We set $\alpha_0 := -\vartheta + \delta$, $\Pi_{\mathrm{af}} := \Pi \cup \{\alpha_0\}$, and $\mathtt{I}_{\mathrm{af}} := \mathtt{I} \cup \{0\}$, where ϑ is the highest root of Δ_+. We set $W_{\mathrm{af}} := W \ltimes Q^\vee$ and call it the affine Weyl group. It is a reflection group generated by $\{s_i \mid i \in \mathtt{I}_{\mathrm{af}}\}$, where s_0 is the reflection with respect to α_0. Let $\ell : W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ be the length function and let $w_0^{\mathtt{J}} \in W$ be the longest element in $W^{\mathtt{J}} \subset W_{\mathrm{af}}$. We set $\widetilde{W}_{\mathrm{af}} := W \ltimes \mathbb{X}_*$ and call it the extended affine Weyl group. We have $t_\beta \in \mathbb{X}_* \subset \widetilde{W}_{\mathrm{af}}$ for each $\beta \in \mathbb{X}_*$ such that $t_\beta \in W_{\mathrm{af}}$ for $\beta \in Q^\vee$, $u t_\beta u^{-1} = t_{u\beta}$ for each $u \in W$, and $t_{-\vartheta^\vee} := s_\vartheta s_0$ (for the coroot $\vartheta^\vee$ of ϑ). By setting

$$\ell(w t_\gamma) = \ell(t_\gamma w) = \ell(w)$$

for $w \in W_{\mathrm{af}}$ and $\gamma \in \mathbb{X}_*(G)$, we extend the length function to $\widetilde{W}_{\mathrm{af}}$ (that is possible by $\mathbb{X}_* \cong \mathbb{X}_*(G) \times Q^\vee$).

Let $\leq$ be the Bruhat order of W_{af}. In other words, $w \leq v$ holds if and only if a subexpression of a reduced decomposition of v yields a reduced decomposition of w (see [5]). We define the generic (semi-infinite) Bruhat order $\leq_{\frac{\infty}{2}}$ as:

$$w \leq_{\frac{\infty}{2}} v \Leftrightarrow w t_\beta \leq v t_\beta \quad \text{for every } \beta \in Q^\vee \text{ such that } \langle \beta, \alpha_i \rangle \ll 0 \text{ for } i \in \mathtt{I}. \tag{1.1}$$

By [38], this defines a preorder on W_{af}. Here we remark that $w \leq v$ if and only if $w \geq_{\frac{\infty}{2}} v$ for $w, v \in W$.

Theorem 1.1 (Peterson [44] Lecture 13)**.** *Let $w \in W_{\mathrm{af}}$ be such that $w \leq_{\frac{\infty}{2}} e$. We have $w = u t_\beta$ for some $u \in W$ and $\beta \in Q^\vee_+$.* □

For $w, v \in \widetilde{W}_{\mathrm{af}}$, we write $w \geq_{\frac{\infty}{2}} v$ if and only if there exists $\gamma \in \mathbb{X}_*(G)$ such that $w t_\gamma, v t_\gamma \in W_{\mathrm{af}}$ and $w t_\gamma \geq_{\frac{\infty}{2}} v t_\gamma$.

Let $\widetilde{W}_{\mathrm{af}}^-$ denote the set of minimal length representatives of $\widetilde{W}_{\mathrm{af}}/W$ in $\widetilde{W}_{\mathrm{af}}$. We set

$$\mathbb{X}_*^-(\mathtt{J}) := \{\beta \in \mathbb{X}_* \mid \langle \beta, \alpha_i \rangle < 0, \forall i \in \mathtt{J}\}$$

and

$$\mathbb{X}_*^{\leq}(\mathtt{J}) := \{\beta \in \mathbb{X}_* \mid \langle \beta, \alpha_i \rangle \leq 0, \forall i \in \mathtt{J}\}.$$

We have $\mathbb{X}_*^-(\mathtt{J}) \subset \mathbb{X}_*^-(\mathtt{J}')$ and $\mathbb{X}_*^{\leq}(\mathtt{J}) \subset \mathbb{X}_*^{\leq}(\mathtt{J}')$ when $\mathtt{J}' \subset \mathtt{J}$.

Theorem 1.2 (see e.g. Macdonald [39])**.** *For $\beta \in \mathbb{X}_*^-$, it holds:*

1. *We have $\ell(u t_\beta) = \ell(t_\beta) - \ell(u)$ and $\ell(t_\beta u) = \ell(t_\beta) + \ell(u)$ for every $u \in W$;*

2. *For each $u \in W$ and $\beta' \in \mathbb{X}_*^{\leq}$, we have*

$$\ell(t_{u\beta}) = \ell(u t_\beta u^{-1}) = \ell(t_\beta) \quad \textit{and} \quad \ell(t_{u(\beta+\beta')}) = \ell(t_{u\beta}) + \ell(t_{u\beta'}) = 2\langle \beta + \beta', \rho \rangle;$$

3. *Each $w \in \widetilde{W}_{\mathrm{af}}^{-}$ is decomposed into $w = ut_\gamma$ for some $u \in W$ and $\gamma \in \mathbb{X}_*^{\le}$ such that $\ell(w) = \ell(t_\gamma) - \ell(u)$.*

Proof. The first assertions follow from [39, (2.4.1)]. The second assertions follow from 1) and [39, (2.4.2)]. The third assertion is a consequence of [39, (2.4.3)]. $\square$

For each $\lambda \in \mathbb{X}_+^*(\mathtt{J})$, we denote a finite-dimensional simple $P^{\mathtt{J}}$-module with a non-zero B-eigenvector $\mathbf{v}_\lambda$ of H-weight λ by $V^{\mathtt{J}}(\lambda)$. Let $R(G)$ be the (complexified) representation ring of G. We have an identification $R(G) = (\mathbb{C}[H])^W \subset \mathbb{C}\mathbb{X}^*$ by taking characters. For a semi-simple H-module V, we set

$$\operatorname{ch} V := \sum_{\lambda \in \mathbb{X}^*} e^{\lambda} \cdot \dim_{\mathbb{C}} \operatorname{Hom}_H(\mathbb{C}_\lambda, V).$$

If V is a $\mathbb{Z}$-graded H-module in addition, then we set

$$\operatorname{gch} V := \sum_{\lambda \in \mathbb{X}, n \in \mathbb{Z}} q^n e^{\lambda} \cdot \dim_{\mathbb{C}} \operatorname{Hom}_H(\mathbb{C}_\lambda, V_n).$$

For a $\mathbf{H}$-equivariant coherent sheaf on a projective $\mathbf{H}$-variety $\mathcal{X}$, let $\chi(\mathcal{X}, \mathcal{F}) \in \mathbb{C}[\mathbf{H}]$ denote its equivariant Euler-Poincaré characteristic. We set $\mathbb{X}_{\mathrm{af}}^* := \mathbb{X}^* \oplus \mathbb{Z}\delta$ and understand that $e^{\delta} = q \in \mathbb{C}\mathbb{X}_{\mathrm{af}}^* = \mathbb{C}[\mathbf{H}]$.

For $\mathtt{J}' \subset \mathtt{J} \subset \mathtt{I}$, we identify $W^{\mathtt{J}}/W^{\mathtt{J}'}$ with its minimal coset representative in $W^{\mathtt{J}}$. We set $\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}} := P^{\mathtt{J}}/P^{\mathtt{J}'}$ and call it the partial flag manifold of $L^{\mathtt{J}}$. It is equipped with the Bruhat decomposition

$$\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}} = \bigsqcup_{w \in W^{\mathtt{J}}/W^{\mathtt{J}'}} \mathbb{O}_{\mathtt{J}'}^{\mathtt{J}}(w)$$

into B-orbits such that $\operatorname{codim}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}} \mathbb{O}_{\mathtt{J}'}^{\mathtt{J}}(w) = \ell(w)$ for each $w \in W^{\mathtt{J}}/W^{\mathtt{J}'}$. We set $\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}(w) := \overline{\mathbb{O}_{\mathtt{J}'}^{\mathtt{J}}(w)} \subset \mathcal{B}^{\mathtt{J}}$.

We have a notion of H-equivariant K-group $K_H(\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}})$ of $\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}$ with coefficients in $\mathbb{C}$ (see e.g. [32]). Explicitly, we have

$$K_H(\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}) = \bigoplus_{w \in W^{\mathtt{J}}/W^{\mathtt{J}'}} \mathbb{C}[H]\, [\mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}(w)}]. \tag{1.2}$$

For each $\lambda \in w_0^{\mathtt{J}} \mathbb{X}_0^*(\mathtt{J}')$, we have a line bundle $\mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}}(\lambda)$ such that

$$\operatorname{ch} H^0(\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}, \mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}}(\lambda)) = \operatorname{ch} V^{\mathtt{J}}(\lambda), \quad \mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}}} \mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}}(-\mu) \cong \mathcal{O}_{\mathcal{B}_{\mathtt{J}'}^{\mathtt{J}}}(\lambda - \mu)$$

holds for $\lambda, \mu \in w_0^{\mathtt{J}} \mathbb{X}_0^*(\mathtt{J}') \cap \mathbb{X}_+^*(\mathtt{J})$.

1.2 The nil-DAHA and its spherical version

Definition 1.3. The nil-DAHA $\mathcal{H}_q$ or $\mathcal{H}_q(G)$ of type G is a $\mathbb{C}_q$-algebra generated by $\{e^{\lambda}\}_{\lambda \in \mathbb{X}^*} \cup \{D_i\}_{i \in \mathtt{I}_{\mathrm{af}}} \cup \{T_\gamma\}_{\gamma \in \mathbb{X}_*(G)}$ subject to the following relations:

1. $e^{\lambda+\mu} = e^{\lambda} \cdot e^{\mu}$ for $\lambda, \mu \in \mathbb{X}^*$;
2. $D_i^2 = D_i$ for each $i \in \mathtt{I}_{\mathrm{af}}$;

3. For each distinct $i, j \in \mathtt{I}_{\rm af}$, we set $m_{i,j} \in \mathbb{Z}_{>0}$ as the minimum number such that $(s_i s_j)^{m_{i,j}} = 1$. Then, we have

$$\overbrace{D_i D_j \cdots}^{m_{i,j}\text{-terms}} = \overbrace{D_j D_i \cdots}^{m_{i,j}\text{-terms}};$$

4. For each $\lambda \in \mathbb{X}^*$ and $i \in \mathtt{I}_{\rm af}$, we have

$$D_i e^{\lambda} - e^{s_i \lambda} D_i = \frac{e^{\lambda} - e^{s_i \lambda}}{1 - e^{\alpha_i}}, \qquad \text{where} \qquad e^{\alpha_0} = q \cdot e^{-\vartheta^{\vee}};$$

5. $T_{\gamma} T_{\gamma'} = T_{\gamma'} T_{\gamma}$ for each $\gamma, \gamma' \in \mathbb{X}_*(G)$;

6. $T_{\gamma} D_i = D_i T_{\gamma}$ for each $i \in \mathtt{I}_{\rm af}$ and $\gamma \in \mathbb{X}_*(G)$;

7. $T_{\gamma} e^{\lambda} = q^{\langle \gamma, \lambda \rangle} e^{\lambda} T_{\gamma}$ for each $\lambda \in \mathbb{X}^*$ and $\gamma \in \mathbb{X}_*(G)$.

We also consider the $\mathbb{C}_q$-subalgebras $\mathcal{H}_q^0, \mathcal{H}_q(\mathtt{J}) \subset \mathcal{H}_q$ generated by $\{D_i \mid i \in \mathtt{I}_{\rm af}\}$ and $\{e^{\lambda}, D_i \mid \lambda \in \mathbb{X}^*, i \in \mathtt{J}\}$ (for $\mathtt{J} \subset \mathtt{I}_{\rm af}$), respectively.

Let $\mathcal{S}_q' := \mathbb{C}[\mathbf{H}] \otimes \mathbb{C} W_{\rm af}$ be the smash product algebra, whose multiplication reads as:

$$(e^{\lambda} \otimes w)(e^{\mu} \otimes v) = e^{\lambda + w\mu} \otimes wv \qquad \lambda, \mu \in \mathbb{X}^*_{\rm af}, w, v \in W_{\rm af}.$$

We add $1 \otimes t_{\gamma} \in \mathbb{C} \otimes \mathbb{C}\widetilde{W}_{\rm af}$ ($\gamma \in \mathbb{X}_*(G)$) such that

$$(e^{\lambda} \otimes t_{\gamma})(e^{\mu} \otimes t_{\gamma'}) = q^{\langle \gamma, \mu \rangle} e^{\lambda + \mu} \otimes t_{\gamma + \gamma'} \qquad \lambda, \mu \in \mathbb{X}^*_{\rm af}, \gamma, \gamma' \in \mathbb{X}_*(G)$$

to $\mathcal{S}_q'$ to obtain the smash product algebra $\mathcal{S}_q := \mathbb{C}[\mathbf{H}] \otimes \mathbb{C}\widetilde{W}_{\rm af}$. Let $\mathbb{C}(\mathbf{H})$ denote the fraction field of (the Laurent polynomial algebra) $\mathbb{C}[\mathbf{H}]$. We have a scalar extension

$$\mathcal{R}_q := \mathbb{C}(\mathbf{H}) \otimes_{\mathbb{C}[\mathbf{H}]} \mathcal{S}_q = \mathbb{C}(\mathbf{H}) \otimes_{\mathbb{C}} \mathbb{C}\widetilde{W}_{\rm af}.$$

The following is a very slight extension of [35] §2.2 (and hence we omit its proof):

Theorem 1.4 (cf. [35] §2.2)**.** *We have an embedding of algebras $\imath^* : \mathcal{H}_q \hookrightarrow \mathcal{R}_q$:*

$$e^{\lambda} \mapsto e^{\lambda} \otimes 1, \; D_i \mapsto \frac{1}{1 - e^{\alpha_i}} \otimes 1 - \frac{e^{\alpha_i}}{1 - e^{\alpha_i}} \otimes s_i, \; T_{\gamma} \mapsto 1 \otimes t_{\gamma}.$$

for each $\lambda \in \mathbb{X}^*_{\rm af}, i \in \mathtt{I}_{\rm af}$, and $\gamma \in \mathbb{X}_*(G)$.

Corollary 1.5 (Leibniz fule for D_i)**.** *Let $i \in \mathtt{I}_{\rm af}$ and $\lambda \in \mathbb{X}^*_{\rm af}$. We have*

$$D_i \cdot e^{\lambda} = \frac{e^{\lambda} - e^{s_i \lambda}}{1 - e^{\alpha_i}} + e^{s_i \lambda} \cdot D_i \quad \text{in} \quad \mathcal{R}_q.$$

Since we have a natural action of $\mathcal{R}_q$ on $\mathbb{C}(\mathbf{H})$, we obtain an action of $\mathcal{H}_q$ on $\mathbb{C}(\mathbf{H})$ (in a way it preserves $\mathbb{C}[\mathbf{H}]$), that we call the polynomial representation.

For $w \in t_{\gamma} W_{\rm af}$ ($\gamma \in \mathbb{X}_*(G)$), we find a reduced expression $w = t_{\gamma} s_{i_1} \cdots s_{i_{\ell}}$ ($i_1, \ldots, i_{\ell} \in \mathtt{I}_{\rm af}$) and set

$$D_w := T_{\gamma} D_{s_{i_1}} D_{s_{i_2}} \cdots D_{s_{i_{\ell}}} \in \mathcal{H}_q.$$

By Definition 1.3 3), the element D_w is independent of the choice of a reduced expression. By Definition 1.3 2), we have $D_i D_{w_0} = D_{w_0}$ for each $i \in \mathtt{I}$, and hence $D_{w_0}^2 = D_{w_0}$. We have an explicit form

$$D_{w_0} = 1 \otimes \left(\sum_{w \in W} w \right) \cdot \frac{e^{-\rho}}{\prod_{\alpha \in \Delta^+} (e^{-\alpha/2} - e^{\alpha/2})} \otimes 1 \in \mathcal{A}_q \tag{1.3}$$

obtained from the (left W-invariance of the) Weyl character formula. We set

$$\mathcal{H}_q^{\mathrm{sph}} \equiv \mathcal{H}_q^{\mathrm{sph}}(G) := D_{w_0} \mathcal{H}_q D_{w_0}$$

and call it the spherical nil-DAHA of type G.

Theorem 1.6 (see e.g. Kostant-Kumar [32]). *We have a $\mathcal{H}_q(\mathtt{I})$-action on $K_{\mathbf{H}}(\mathcal{B})$ with the following properties:*

1. *For each $\lambda \in \mathbb{X}^*$, the left multiplication by $e^\lambda \in \mathcal{H}_q(\mathtt{I})$ is equal to the H-character twist of $K_{\mathbf{H}}(\mathcal{B})$ by e^λ;*
2. *For each $i \in \mathtt{I}$, we have*
$$D_i([\mathcal{O}_{\mathcal{B}(w)}]) = \begin{cases} [\mathcal{O}_{\mathcal{B}(s_i w)}] & (s_i w < w) \\ [\mathcal{O}_{\mathcal{B}(w)}] & (s_i w > w) \end{cases};$$
3. *For $\lambda \in \mathbb{X}^*$, the twist by $\mathcal{O}_{\mathcal{B}}(\lambda)$ defines a $\mathcal{H}_q(\mathtt{I})$-module automorphism;*
4. *We have $K_{\mathbf{G}}(\mathcal{B}) = D_{w_0} K_{\mathbf{H}}(\mathcal{B})$;*
5. *We have $K_{\mathbf{H}}(\mathcal{B}) = \mathcal{H}_q(\mathtt{I}) \cdot [\mathcal{O}_{\mathcal{B}}] = \mathbb{C}_q[H] \cdot K_{\mathbf{G}}(\mathcal{B}) \subset K_{\mathbf{H}}(\mathcal{B})$.*

Corollary 1.7. *For each $\mathtt{J}' \subset \mathtt{J} \subset \mathtt{I}$, we have a $\mathcal{H}_q(\mathtt{J}')$-module map*

$$K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}}) \longrightarrow K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}'})$$

that sends $[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}(\lambda)]$ to $[\mathcal{O}_{\mathcal{B}^{\mathtt{J}'}}(\lambda)]$ for every $\lambda \in \mathbb{X}^*$.

Proof. We have an algebra map $K_{\mathbf{L}^{\mathtt{J}}}(\mathcal{B}^{\mathtt{J}}) \longrightarrow K_{\mathbf{L}^{\mathtt{J}'}}(\mathcal{B}^{\mathtt{J}'})$ that sends $[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}(\lambda)]$ to $[\mathcal{O}_{\mathcal{B}^{\mathtt{J}'}}(\lambda)]$ for every $\lambda \in \mathbb{X}^*$. It is invariant under the action of D_j for $j \in \mathtt{J}'$ by Theorem 1.6 3). By extending the scalar, we obtain a map $K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}}) \longrightarrow K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}'})$. By the Leibniz rule, this map commutes with the D_i-actions for each $i \in \mathtt{J}'$. Thus, it gives rise to a $\mathcal{H}_q(\mathtt{J}')$-module map as required. □

Corollary 1.8 ([32]). *For each $\mathtt{J}' \subset \mathtt{J} \subset \mathtt{I}$, the pullback defines a subspace*

$$K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}}_{\mathtt{J}'}) \cong K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}}) D_{w_0^{\mathtt{J}'}} \subset K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}}).$$

1.3 Quasi-map spaces

Here we recall basics of quasi-map spaces from [16, 14].

We have W-equivariant isomorphism $H_2(\mathcal{B}, \mathbb{Z}) \cong Q^\vee$. This identifies the (integral points of the) effective cone of $\mathcal{B}$ with $Q^\vee_+$. A quasi-map (f, D) is a map $f : \mathbb{P}^1 \to \mathcal{B}$ together with an $\mathtt{I}$-colored effective divisor

$$D = \sum_{i \in \mathtt{I}, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha_i^\vee) \alpha_i^\vee \otimes [x] \in Q^\vee \otimes_{\mathbb{Z}} \mathrm{Div}\, \mathbb{P}^1 \quad \text{with} \quad m_x(\alpha^\vee) \in \mathbb{Z}_{\geq 0}.$$

We call D the defect of (f, D). We define the total defect of (f, D) by

$$|D| := \sum_{i \in \mathtt{I}, x \in \mathbb{P}^1(\mathbb{C})} m_x(\alpha_i^\vee)\alpha_i^\vee \in Q_+^\vee.$$

For each $\beta \in Q_+^\vee$, we set

$$\mathcal{Q}(\mathcal{B}, \beta) := \{f : \mathbb{P}^1 \to X \mid \text{ quasi-map s.t. } f_*[\mathbb{P}^1] + |D| = \beta\},$$

where $f_*[\mathbb{P}^1]$ is the class of the image of $\mathbb{P}^1$ multiplied by the degree of $\mathbb{P}^1 \to \operatorname{Im} f$. We denote $\mathcal{Q}(\mathcal{B}, \beta)$ by $\mathcal{Q}_G(\beta)$ or $\mathcal{Q}(\beta)$ for simplicity.

Definition 1.9 (Drinfeld-Plücker data)**.** Consider a collection $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in \Lambda_+}$ of inclusions $\psi_\lambda : \mathcal{L}^\lambda \hookrightarrow V(\lambda) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}$ of line bundles $\mathcal{L}^\lambda$ over $\mathbb{P}^1$. The data $\mathcal{L}$ is called a Drinfeld-Plücker data (DP-data) if the canonical inclusion of G-modules

$$\eta_{\lambda,\mu} : V(\lambda + \mu) \hookrightarrow V(\lambda) \otimes V(\mu)$$

induces an isomorphism

$$\eta_{\lambda,\mu} \otimes \mathrm{id} : \psi_{\lambda+\mu}(\mathcal{L}^{\lambda+\mu}) \xrightarrow{\cong} \psi_\lambda(\mathcal{L}^\lambda) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \psi_\mu(\mathcal{L}^\mu)$$

for every $\lambda, \mu \in \Lambda_+$.

Theorem 1.10 (Drinfeld, see Finkelberg-Mirković [16])**.** *The variety $\mathcal{Q}(\beta)$ is isomorphic to the variety formed by isomorphism classes of the DP-data $\mathcal{L} = \{(\psi_\lambda, \mathcal{L}^\lambda)\}_{\lambda \in \Lambda_+}$ such that* $\deg \mathcal{L}^\lambda = \langle w_0\beta, \lambda\rangle$*. In addition, $\mathcal{Q}(\beta)$ is an irreducible variety of dimension* $\dim \mathcal{B} + 2\langle \beta, \rho\rangle$*.*

Theorem 1.11 (Braverman-Finkelberg [8])**.** *The variety $\mathcal{Q}(\beta)$ is a normal variety with rational singularities.*

For each $\lambda \in \mathbb{X}^*$, and $\beta \in Q_+^\vee$, we have a G-equivariant line bundle $\mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)$ obtained by the tensor product of the pull-backs $\mathcal{O}_{\mathcal{Q}(\beta)}(\varpi_i)$ of the i-th $\mathcal{O}(1)$ via the embedding

$$\mathcal{Q}(\beta) \hookrightarrow \prod_{i \in \mathtt{I}} \mathbb{P}(V(\varpi_i) \otimes_{\mathbb{C}} \mathbb{C}[z]_{\le -\langle w_0\beta, \varpi_i\rangle}) \tag{1.4}$$

and a G-character. We have $\chi(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}}(\lambda)) \in \mathbb{C}[\mathbf{H}]$ for $\beta \in Q^\vee, \lambda \in \mathbb{X}^*$, where the grading q is understood to count the degree of z detected by the $\mathbb{G}_m$-action. Here we understand that $\chi(\mathcal{Q}(\beta), \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)) = 0$ if $\beta \notin Q_+^\vee$.

We have an embedding $\mathcal{B} \subset \mathcal{Q}(\beta)$ such that the line bundles $\mathcal{O}(\lambda)$ $(\lambda \in \mathbb{X}^*)$ correspond to each other by restrictions ([8, 24]).

1.4 Graph and map spaces and their line bundles

We refer [31, 18, 20] for the precise definitions of the notions appearing in this subsection.

For each non-negative integer n and $\beta \in Q_+^\vee$, we set $\mathcal{GB}_{n,\beta}$ to be the space of stable maps of genus zero curves with n-marked points to $(\mathbb{P}^1 \times \mathcal{B})$ of bidegree $(1, \beta)$, that is also called the graph space of $\mathcal{B}$. A point of $\mathcal{GB}_{n,\beta}$ is a genus zero curve C with n-marked points $\{x_1, \ldots, x_n\}$, together with a map to $\mathbb{P}^1$ of degree one. Hence, we have a unique $\mathbb{P}^1$-component of C that maps isomorphically onto $\mathbb{P}^1$. We call this component the main component of C and denote it by C_0. For a genus zero curve C, let $|C|$ denote the number of its irreducible components. The space $\mathcal{GB}_{n,\beta}$ is a normal projective variety by [18, Theorem 2] that have at worst quotient singularities arising from the automorphism of curves (in particular, they have rational singularities). The natural $\mathbf{H}$-action on $(\mathbb{P}^1 \times \mathcal{B})$ induces a natural $\mathbf{H}$-action on $\mathcal{GB}_{n,\beta}$. Moreover, $\mathcal{GB}_{0,\beta}$ has only finitely many isolated $\mathbf{H}$-fixed points, and thus we can apply the formalism of Atiyah-Bott-Lefschetz localization (cf. [20, p200L26] and [8, Proof of Lemma 5]).

We have a morphism $\pi_{n,\beta} : \mathcal{GB}_{n,\beta} \to \mathcal{Q}(\beta)$ that factors through $\mathcal{GB}_{0,\beta}$ (Givental's main lemma [21]; see [14, §8] and [18, §1.3]). Let $\widetilde{\mathtt{ev}}_j : \mathcal{GB}_{n,\beta} \to \mathbb{P}^1 \times \mathcal{B}$ $(1 \le j \le n)$ be the evaluation at the j-th marked point, and let $\mathtt{ev}_j : \mathcal{GB}_{n,\beta} \to \mathcal{B}$ be its composition with the second projection. The variety $\mathcal{GB}_{n,\beta}$ is irreducible (as a special feature of flag varieties, see [18, §1.2] and [29]).

Let $\mathcal{X}(\beta) \subset \mathcal{GB}_{2,\beta}$ denote the subscheme such that the first marked point projects to $0 \in \mathbb{P}^1$, and the second marked point projects to $\infty \in \mathbb{P}^1$ through the first projection of $\mathbb{P}^1 \times \mathcal{B}$. By abuse of notation, we write the restriction of $\mathtt{ev}_i$ $(i = 1, 2)$ to $\mathcal{X}(\beta)$ by the same letter. Let $\pi_\beta : \mathcal{X}(\beta) \to \mathcal{Q}(\beta)$ be the restriction of $\pi_{2,\beta}$ to $\mathcal{X}(\beta)$. In view of Theorem 1.11, the morphism π_β is the rational resolution of singularities in an orbifold sense.

For each $\lambda \in \mathbb{X}^*$, we have a line bundle $\mathcal{O}_{\mathcal{X}(\beta)}(\lambda) := \pi_\beta^* \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda)$. In case we want to stress the group G, we write $\mathcal{X}_G(\beta)$ instead of $\mathcal{X}(\beta)$.

1.5 Equivariant quantum K-group of $\mathcal{B}$

We introduce a polynomial ring $\mathbb{C}Q_+^\vee$ and the formal power series ring $\mathbb{C}[\![Q_+^\vee]\!]$ with their variables $Q_i = Q^{\alpha_i^\vee}$ $(i \in \mathtt{I})$. We set $Q^\beta := \prod_{i \in \mathtt{I}} Q_i^{\langle \beta, \varpi_i \rangle}$ for each $\beta \in Q^\vee$. We define the $\mathbf{G}$-equivariant (small) quantum D_q-module of $\mathcal{B}$ as:

$$qK_{\mathbf{G}}(\mathcal{B}) := K_{\mathbf{G}}(\mathcal{B}) \otimes \mathbb{C}Q_+^\vee. \tag{1.5}$$

Note that the specialization $q = 1$ yields

$$qK_G(\mathcal{B}) := K_G(\mathcal{B}) \otimes \mathbb{C}Q_+^\vee. \tag{1.6}$$

Let $qK_{\mathbf{G}}(\mathcal{B})^\wedge$ and $qK_G(\mathcal{B})^\wedge$ denote the completions of $qK_{\mathbf{G}}(\mathcal{B})$ and $qK_G(\mathcal{B})$ with respect to the variables $\{Q_i\}_{i \in \mathtt{I}}$.

Let $\langle \bullet, \bullet \rangle^{\mathtt{GW}}$ be the $R(\mathbf{G})$-linear pairing on $qK_{\mathbf{G}}(\mathcal{B})^\wedge$ defined as:

$$\langle a, b \rangle^{\mathtt{GW}} := \sum_{\beta \in Q_+^\vee} \chi(\mathcal{X}(\beta), \mathrm{ev}_1^* a \otimes \mathrm{ev}_1^* b) Q^\beta \in \mathbb{C}[\mathbf{H}][\![Q_+^\vee]\!] \qquad a, b \in qK_{\mathbf{G}}(\mathcal{B})^\wedge.$$

Since the specialization $Q^\beta = 0$ $(\beta \neq 0)$ recovers the ($\mathbf{G}$-equivariant) Euler-Poincaré pairing of $\mathcal{B}$, we know that $\langle \bullet, \bullet \rangle^{\mathtt{GW}}$ is non-degenerate. For each $\lambda \in \mathbb{X}^*$, the bilinear functional

$$\langle a, b \rangle_\lambda^{\mathtt{GW}} := \sum_{\beta \in Q_+^\vee} \chi(\mathcal{X}(\beta), \pi_\beta^* \mathcal{O}_{\mathcal{Q}(\beta)}(\lambda) \otimes \mathrm{ev}_1^* a \otimes \mathrm{ev}_1^* b) Q^\beta \in \mathbb{C}[\mathbf{H}][\![Q_+^\vee]\!]$$

induces a(n unique) linear operator $A^\lambda(\bullet)$ on $qK_{\mathbf{G}}(\mathcal{B})^\wedge$ such that

$$\left\langle A^\lambda a, b \right\rangle^{\mathtt{GW}} = \langle a, b \rangle_\lambda^{\mathtt{GW}} \qquad a, b \in qK_{\mathbf{G}}(\mathcal{B})^\wedge.$$

We remark that the operator A^λ is the character twist when $\lambda \in \mathbb{X}^*(G)$. In case we want to stress the dependence on G, we write $\langle \bullet, \bullet \rangle^{\mathtt{GW}}_G$ and A^λ_G instead of $\langle \bullet, \bullet \rangle^{\mathtt{GW}}$ and A^λ, respectively.

Theorem 1.12 (Iritani-Milanov-Tonita [23] and [26]). *We have:*

1. *For $\lambda, \mu \in \mathbb{X}^*$, we have $A^\lambda \circ A^\mu = A^{\lambda+\mu}$ in $\mathrm{End}_{R(\mathbf{G})}(qK_{\mathbf{G}}(\mathcal{B})^\wedge)$;*
2. *For $\lambda \in \mathbb{X}^*$ and $c \in K_{\mathbf{G}}(\mathcal{B}) \otimes 1 \subset qK_{\mathbf{G}}(\mathcal{B})$, we have*
$$A^\lambda c \equiv \mathcal{O}_{\mathcal{B}}(\lambda) \otimes_{\mathcal{O}_{\mathcal{B}}} c \mod (Q_i \mid i \in \mathtt{I});$$
3. *The $q = 1$ specialization of the operator $A^{-\varpi_i}$ $(i \in \mathtt{I})$ is the quantum multiplication by $[\mathcal{O}_{\mathcal{B}}(-\varpi_i)]$ on $qK_G(\mathcal{B})$;*
4. *The $R(G)$-action, the $\mathbb{C}Q^\vee$-action, together with the quantum multiplications by $[\mathcal{O}_{\mathcal{B}}(-\varpi_i)]$ $(i \in \mathtt{I})$, generates $qK_G(\mathcal{B})$ as a ring;*
5. *For $f \in \mathbb{C}_q[A^\lambda, Q^\beta \mid \lambda \in \mathbb{X}^*, \beta \in Q^\vee_+]$, we have $f[\mathcal{O}_{\mathcal{B}}] = 0$ in $qK_{\mathbf{G}}(\mathcal{B})$ if and only if*
$$\langle f[\mathcal{O}_{\mathcal{B}}], [\mathcal{O}_{\mathcal{B}}] \rangle^{\mathtt{GW}}_\lambda = 0 \qquad \lambda \in \Lambda_+.$$

Proof. The first two assertions follows from [23] Proposition 2.13 and Proposition 2.10, respectively. The third assertion is [1, Lemma 6] (or [26, Theorem 4.2]). The fourth assertion is a consequence of the finiteness of quantum K-groups, seen in [1, Proposition 9] and [26, Corollary 3.3]. The fifth assertion can be read off from the proof of [26, Theorem 3.11]. □

2 Preparatory results

2.1 Affine Grassmanians

We define our (thin) affine Grassmannian and (thin) flag manifold by

$$\mathrm{Gr}_G := G(\!(z)\!)/G[\![z]\!] \quad \text{and} \quad X_G := G(\!(z)\!)/\mathbf{I},$$

respectively. We have a natural map $\pi : X_G \to \mathrm{Gr}_G$ whose fiber is isomorphic to $\mathcal{B}$. By [3, §4.6] (cf. [40, §2]), the sets of connected components of Gr_G and X_G are in bijection with $\mathbb{X}_*(G)$. Here we note that our assumption on G guarantees that all connected components of Gr_G are mutually isomorphic as ind-varieties with $G[\![z]\!]$-actions.

Theorem 2.1 (Bruhat decomposition, [33] Corollary 6.1.20). *We have $\mathbf{I}$-orbit decompositions*

$$\mathrm{Gr}_G = \bigsqcup_{\beta \in \mathbb{X}_*} \mathring{\mathrm{Gr}}_G(\beta) \quad \textit{and} \quad X = \bigsqcup_{w \in \widetilde{W}_{\mathrm{af}}} \mathbb{O}^{\mathrm{af}}_G(w)$$

with the following properties:

1. *we have $\mathbb{O}^{\mathrm{af}}_G(v) \subset \overline{\mathbb{O}^{\mathrm{af}}_G(w)}$ if and only if $v \le w$;*
2. *$\pi(\mathbb{O}^{\mathrm{af}}_G(w)) \subset \mathring{\mathrm{Gr}}_G(\beta)$ if and only if $w \in t_\beta W$.* □

Let us set $\mathrm{Gr}_G(\beta) := \overline{\mathring{\mathrm{Gr}}_G(\beta)}$ and $X_w := \overline{\mathbb{O}^{\mathrm{af}}_G(w)}$ for $\beta \in \mathbb{X}_*$ and $w \in \widetilde{W}_{\mathrm{af}}$. For $w \in \widetilde{W}_{\mathrm{af}}^-$, we also set $\mathrm{Gr}_G(w) := \mathrm{Gr}_G(\beta)$ for $\beta \in \mathbb{X}_*$ such that $w \in t_\beta W$.

We set

$$K_{\mathbf{H}}(\mathrm{Gr}_G) := \bigoplus_{\beta \in \mathbb{X}_*} \mathbb{C}[\mathbf{H}]\,[\mathcal{O}_{\mathrm{Gr}_G(\beta)}] \quad \text{and} \quad K_{\mathbf{H}}(X_G) := \bigoplus_{w \in \widetilde{W}_{\mathrm{af}}} \mathbb{C}[\mathbf{H}]\,[\mathcal{O}_{X_w}].$$

The following is an affine version of Theorem 1.6:

Theorem 2.2 (Kostant-Kumar [32])**.** *The vector space $K_{\mathbf{H}}(X_G)$ affords a regular representation of $\mathcal{H}_q$ such that:*

1. *the subalgebra $\mathbb{C}[\mathbf{H}] \subset \mathcal{H}_q$ acts by the multiplication of the coefficients;*
2. *we have $D_i[\mathcal{O}_{X_w}] = [\mathcal{O}_{X_{s_i w}}]$ $(s_i w > w)$ or $[\mathcal{O}_{X_w}]$ $(s_i w < w)$.* □

Being a regular representation, we sometimes identify $K_{\mathbf{H}}(X_G)$ with $\mathcal{H}_q$ (through $e^\lambda[\mathcal{O}_{X_w}] \leftrightarrow e^\lambda D_w$ for $\lambda \in \mathbb{X}^{\mathrm{af}}_*, w \in \widetilde{W}_{\mathrm{af}}$) and consider product of two elements in $\mathcal{H}_q \cup K_{\mathbf{H}}(X_G)$. We may denote this product on $K_{\mathbf{H}}(X_G)$ by $\odot_q$.

Theorem 2.3 (Kostant-Kumar [32])**.** *The pullback defines an inclusion map $\pi^* : K_{\mathbf{H}}(\mathrm{Gr}_G) \hookrightarrow K_{\mathbf{H}}(X_G)$ such that*

$$\pi^*[\mathcal{O}_{\mathrm{Gr}_G(\beta)}] = [X_{t_\beta}] D_{w_0} \qquad \beta \in Q^\vee.$$

In particular, $\mathrm{Im}\,\pi^* = \mathcal{H}_q \odot_q D_{w_0}$ *is a $\mathcal{H}_q$-submodule.* □

Theorem 2.4. *Let $w \in \widetilde{W}_{\mathrm{af}}^-$ and let $\beta \in \mathbb{X}_*^-$. We have*

$$\pi^*[\mathcal{O}_{\mathrm{Gr}_G(w)}] \odot_q \pi^*[\mathcal{O}_{\mathrm{Gr}_G(\beta)}] = \pi^*[\mathcal{O}_{\mathrm{Gr}_G(w t_\beta)}].$$

Proof. We have $\ell(t_\beta) = \ell(w_0) + \ell(w_0 t_\beta)$ by Theorem 1.2 1). We have $w = u t_\gamma$ for some $u \in W$ and $\gamma \in \mathbb{X}_*^{\leq}$ such that $\ell(w) = \ell(t_\gamma) - \ell(u)$ by Theorem 1.2 3). Now we have $\ell(u t_{\gamma+\beta}) = \ell(w) + \ell(t_\beta)$ by Theorem 1.2 2). From these, the assertion follows by Theorem 2.2 and Theorem 2.3. □

Theorem 2.4 implies that the set

$$\{\pi^*[\mathcal{O}_{\mathrm{Gr}_G(\beta)}] \mid \beta \in \mathbb{X}_*^-\} \subset (K_{\mathbf{H}}(\mathrm{Gr}_G), \odot_q)$$

forms a multiplicative system with respect to the right action. We denote by $K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ the localization of $K_{\mathbf{H}}(\mathrm{Gr}_G)$ with respect to this right action. The action of an element $[\mathcal{O}_{\mathrm{Gr}_G(\beta)}]$ on $K_{\mathbf{H}}(\mathrm{Gr}_G)$ in Theorem 2.4 is torsion-free, and hence we have an embedding $K_{\mathbf{H}}(\mathrm{Gr}_G) \subset K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}}$. Since the left $\mathcal{H}_q$-module structure on $(K_{\mathbf{H}}(\mathrm{Gr}_G), \odot_q)$ commutes with this right action, we conclude that $K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ is a $\mathcal{H}_q$-module that contains $K_{\mathbf{H}}(\mathrm{Gr}_G)$.

Corollary 2.5. *Let $i \in \mathtt{I}$. For $\beta \in \mathbb{X}_*^-$, we set*

$$\mathbf{h}_i := \pi^*[\mathcal{O}_{\mathrm{Gr}_G(s_i t_\beta)}] \odot_q \pi^*[\mathcal{O}_{\mathrm{Gr}_G(t_\beta)}]^{-1}.$$

Then, the element $\mathbf{h}_i$ is independent of the choice of β.

Proof. By Theorem 2.4, we have

$$\begin{aligned}[\mathcal{O}_{\mathrm{Gr}_G(s_it_{\gamma+\beta})}]\odot_q[\mathcal{O}_{\mathrm{Gr}_G(t_{\gamma+\beta})}]^{-1} &= [\mathcal{O}_{\mathrm{Gr}_G(s_it_{\beta})}]\odot_q[\mathcal{O}_{\mathrm{Gr}_G(t_{\gamma})}]\odot_q[\mathcal{O}_{\mathrm{Gr}_G(t_{\gamma})}]^{-1}\odot_q[\mathcal{O}_{\mathrm{Gr}_G(t_{\beta})}]^{-1}\\ &= [\mathcal{O}_{\mathrm{Gr}_G(s_it_{\beta})}]\odot_q[\mathcal{O}_{\mathrm{Gr}_G(t_{\beta})}]^{-1}\end{aligned}$$

for $\gamma \in \mathbb{X}_*^-$. Hence, we conclude the assertion. □

In the below, we may drop π^* in the notation and consider

$$K_{\mathbf{G}}(\mathrm{Gr}_G) = D_{w_0}K_{\mathbf{H}}(\mathrm{Gr}_G) \cong D_{w_0}K_{\mathbf{H}}(X_G)D_{w_0} \subset K_{\mathbf{H}}(X_G)$$

as a subalgebra of $K_{\mathbf{H}}(X_G)$. Note that $[\mathcal{O}_{\mathrm{Gr}_G(\beta)}] \in K_{\mathbf{G}}(\mathrm{Gr}_G)$ for $\beta \in \mathbb{X}_*^-$. In addition, $[\mathcal{O}_{\mathrm{Gr}_G(0)}]$ is the multiplicative unit of $K_{\mathbf{G}}(\mathrm{Gr}_G)$, and we sometimes denote it by 1. It is clear that $K_{\mathbf{G}}(\mathrm{Gr}_G)$ affords a regular representation of $\mathcal{H}_q^{\mathrm{sph}}$.

For each $\gamma \in \mathbb{X}_*$, we can write $\gamma = \beta_1 - \beta_2$, where $\beta_1, \beta_2 \in \mathbb{X}_*^-$. In particular, we have an element

$$\mathtt{t}_\gamma := [\mathcal{O}_{\mathrm{Gr}_G(t_{\beta_1})}]\odot_q[\mathcal{O}_{\mathrm{Gr}_G(t_{\beta_2})}]^{-1}.$$

Lemma 2.6. *For each $\gamma \in Q^\vee$, the element $\mathtt{t}_\gamma \in K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ is independent of the choices involved.*

Proof. Similar to the proof of Corollary 2.5. The detail is left to the readers. □

2.2 Semi-infinite flag manifolds

In this subsection, we assume that G is a simple algebraic group. This assumption implies $\Lambda = \mathbb{X}^*$, $Q^\vee = \mathbb{X}_*$, and $W_{\mathrm{af}} = \widetilde{W}_{\mathrm{af}}$. In [25], we have exhibited an ind-scheme $\mathbf{Q}_G^{\mathrm{rat}}$ of ind-infinite type that is universal among these whose set $\mathbb{C}$-valued points are $G(\!(z)\!)/(H \cdot N(\!(z)\!))$. It is equipped with a $G(\!(z)\!)$-equivariant line bundle $\mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda)$ for each $\lambda \in \mathbb{X}^*$. Here we normalized the label of line bundles such that $\Gamma(\mathbf{Q}_G^{\mathrm{rat}}, \mathcal{O}_{\mathbf{Q}_G^{\mathrm{rat}}}(\lambda))$ is co-generated by its H-weight λ-part as a $B(\!(z)\!)$-module.

Theorem 2.7 ([16, 14])**.** *We have an $\mathbf{I}$-orbit decomposition*

$$\mathbf{Q}_G^{\mathrm{rat}} = \bigsqcup_{w \in W_{\mathrm{af}}} \mathbb{O}(w)$$

with the following properties:

1. *each $\mathbb{O}(w)$ has infinite dimension and infinite codimension in $\mathbf{Q}_G^{\mathrm{rat}}$;*
2. *the right action of $\gamma \in Q^\vee$ on $\mathbf{Q}_G^{\mathrm{rat}}$ yields the translation $\mathbb{O}(w) \mapsto \mathbb{O}(wt_\gamma)$;*
3. *we have $\mathbb{O}(w) \subset \overline{\mathbb{O}(v)}$ if and only if $w \le_{\frac{\infty}{2}} v$.* □

We define a $\mathbb{C}[\mathbf{H}]$-module $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$ as:

$$K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}}) := \{\sum_{w \in W_{\mathrm{af}}} a_w[\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}[\mathbf{H}],\ \exists \beta_0 \in Q^\vee \text{ s.t. } a_{ut_\beta} = 0,\ \forall u \in W, \beta \not> \beta_0\},$$

where the sum in the definition is understood to be formal (i.e. we allow infinite sums). We define its subset

$$K_{\mathbf{H}}(\mathbf{Q}_G(t_\beta)) := \{ \sum_{w \in W_{\mathrm{af}}} a_w[\mathcal{O}_{\mathbf{Q}_G(w)}] \mid a_w \in \mathbb{C}[\mathbf{H}] \text{ s.t. } a_{ut_\gamma} = 0, \forall u \in W, \gamma \not\geq \beta\}$$

for each $\beta \in Q^\vee$. Employing the family $\{K_{\mathbf{H}}(\mathbf{Q}_G(t_\beta))\}_{\beta \in Q^\vee}$ of subsets of $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$ as an open base of 0, we obtain a topology on $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$.

Theorem 2.8 ([28] Theorem 6.5)**.** *The vector space $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$ affords a representation of $\mathcal{H}_q$ such that:*

1. *the subalgebra $\mathbb{C}[\mathbf{H}] \subset \mathcal{H}_q$ acts by the multiplication as $\mathbb{C}[\mathbf{H}]$-modules;*
2. *we have*

$$D_i([\mathcal{O}_{\mathbf{Q}_G(w)}]) = \begin{cases} [\mathcal{O}_{\mathbf{Q}_G(s_i w)}] & (s_i w >_{\frac{\infty}{2}} w) \\ [\mathcal{O}_{\mathbf{Q}_G(w)}] & (s_i w <_{\frac{\infty}{2}} w) \end{cases}.$$

For each $\beta \in Q^\vee$, we set

$$K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}}) := D_{w_0}(K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})) \quad \text{and} \quad K_{\mathbf{G}}(\mathbf{Q}_G(t_\beta)) := D_{w_0}(K_{\mathbf{H}}(\mathbf{Q}_G(t_\beta))).$$

From the description of Theorem 2.8, we deduce that the right $Q^\vee$-action gives $\mathcal{H}_q$-module endomorphisms of $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$. We denote this endomorphism for $\beta \in Q^\vee$ by Q^β. It gives rise to an endomorphism of $K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$. We set $\mathbb{C}_q(\!(Q^\vee)\!) := \mathbb{C}_q Q^\vee \otimes_{\mathbb{C}_q Q_+^\vee} \mathbb{C}_q[\![Q_+^\vee]\!]$. The commutative rings $\mathbb{C}_q Q^\vee$ and $\mathbb{C}_q(\!(Q^\vee)\!)$ act on $K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$ from the right.

Theorem 2.9. *For each $\lambda \in \Lambda$, the $\mathbb{C}[\mathbf{H}]$-linear extension of the assignment*

$$[\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)] \in K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}}) \quad w \in W_{\mathrm{af}}$$

defines a $\mathcal{H}_q$-module automorphism (that we call $\Xi(\lambda)$). In addition, we have:

1. *$\Xi(\lambda) \circ \Xi(\mu) = \Xi(\lambda + \mu)$ for $\lambda, \mu \in \Lambda$;*
2. *$[\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)] = e^{w\lambda}[\mathcal{O}_{\mathbf{Q}_G(w)}] + \sum_{v <_{\frac{\infty}{2}} w} a_w^v(\lambda)[\mathcal{O}_{\mathbf{Q}_G(v)}]$ for $a_w^v \in \mathbb{C}[\mathbf{H}]$;*
3. *The coefficients a_w^v belongs to a $\mathbb{C}_q$-span of $\{e^\mu\}_{\mu \in \Sigma(\lambda)}$;*
4. *$[\mathcal{O}_{\mathcal{B}(w)}(\lambda)] = e^{w\lambda}[\mathcal{O}_{\mathcal{B}(w)}] + \sum_{w<v \in W} a_w^v(\lambda)[\mathcal{O}_{\mathcal{B}(v)}]$ for each $w \in W$.*

Proof. The existence of the $\mathcal{H}_q$-module structure and the assertion in the first item follow from [28, Theorem 6.4] (though the definition of the K-groups are slightly different). The second item follows by [28, Theorem 5.10] since a path with the equal initial/final directions is unique, and the path interpretation of coefficients a_w^v automatically imposes order relation $v <_{\frac{\infty}{2}} w$ (see [28, §2.3]). The third item follows from the fact that a_w^v is obtained as a q-weighted count of the character of the global Weyl modules, whose set of H-weights are contained in $\Sigma(\lambda)$ (see e.g. [24, §1.2]).

We prove the fourth item. The open dense $G[\![z]\!]$-orbit $\mathbb{O}$ of $\mathbf{Q}_G(e)$ is the affine fibration over $\mathcal{B}$, and its fiber is a homogeneous space of $\ker(G[\![z]\!] \to G)$.

Since the restriction from $\mathbf{Q}_G(e)$ to $\mathcal{B}$ passes $\mathbb{C}_\mu \otimes \mathcal{O}_{\mathbf{Q}_G(e)}(\lambda)$ to $\mathbb{C}_\mu \otimes \mathcal{O}_{\mathcal{B}}(\lambda)$ $(\lambda, \mu \in \Lambda)$, this restriction yields a $\mathbb{C}[\mathbf{H}]$-linear map

$$K_{\mathbf{H}}(\mathbf{Q}_G(e)) \longrightarrow K_{\mathbf{H}}(\mathbb{O}) \xrightarrow{\cong} K_{\mathbf{H}}(\mathcal{B}),$$

with its kernel spanned by $[\mathcal{O}_{\mathbf{Q}(ut_\beta)}]$ for $u \in W$ and $\beta \neq 0$. This also maps $[\mathcal{O}_{\mathbf{Q}(u)}]$ to $[\mathcal{O}_{\mathcal{B}(u)}]$ for each $u \in W$. Since $v \notin W$ and $v \le_{\frac{\infty}{2}} e$ implies $v = ut_\beta$ with $u \in W$ and $0 \neq \beta \in Q^\vee_+$, we conclude the assertion in the third item. □

Lemma 2.10 ([26] Lemma 1.14)**.** *For each $i \in \mathtt{I}$, we have*

$$[\mathcal{O}_{\mathbf{Q}_G(s_i)}] = [\mathcal{O}_{\mathbf{Q}_G(e)}] - e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}_G(e)}(-\varpi_i)].$$

We consider a $\mathbb{C}[\mathbf{H}]$-module endomorphism H_i $(i \in \mathtt{I})$ of $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$ as:

$$H_i : [\mathcal{O}_{\mathbf{Q}_G(w)}] \mapsto [\mathcal{O}_{\mathbf{Q}_G(w)}] - e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}_G(w)}(-\varpi_i)] \qquad w \in W_{\mathrm{af}}.$$

Lemma 2.11. *For $i, j \in \mathtt{I}$, we have*

$$\Xi(\varpi_i) \circ Q^{\alpha_j^\vee} = q^{-\langle \alpha_j^\vee, \varpi_i \rangle} Q^{\alpha_j^\vee} \circ \Xi(\varpi_i) \in \mathrm{End}_{\mathcal{H}_q} K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}}).$$

Proof. For each $w \in W_{\mathrm{af}}$, we have

$$\Xi(\varpi_i)([\mathcal{O}_{\mathbf{Q}_G(w)}]) = \sum_{v \in W_{\mathrm{af}}} a_v^w [\mathcal{O}_{\mathbf{Q}_G(v)}], \qquad \text{where} \quad a_v^w \in \mathbb{C}[\mathbf{H}] \text{ and}$$

$$\mathrm{gch}\,\Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda + \varpi_i)) = \sum_{v \in W_{\mathrm{af}}} a_v^w \mathrm{gch}\,\Gamma(\mathbf{Q}_G(v), \mathcal{O}_{\mathbf{Q}_G(v)}(\lambda))$$

for each $\lambda \in \Lambda_+$. Since we have

$$\mathrm{gch}\,\Gamma(\mathbf{Q}_G(wt_\gamma), \mathcal{O}_{\mathbf{Q}_G(wt_\gamma)}(\lambda)) = q^{-\langle \gamma, \lambda \rangle} \mathrm{gch}\,\Gamma(\mathbf{Q}_G(w), \mathcal{O}_{\mathbf{Q}_G(w)}(\lambda))$$

for each $\gamma \in Q^\vee$ and $\lambda \in \Lambda$ by [25, Corollary A.4], we deduce that

$$\Xi(\varpi_i) \circ Q^{\alpha_j^\vee}([\mathcal{O}_{\mathbf{Q}_G(w)}]) = q^{-\langle \alpha_j^\vee, \varpi_i \rangle} Q^{\alpha_j^\vee} \circ \Xi(\varpi_i)([\mathcal{O}_{\mathbf{Q}_G(w)}]).$$

Thus, the $\mathbb{C}[\mathbf{H}]$-linearity of the composition maps implies the result. □

The following result is a version of the Demazure character formula for semi-infinite flag manifolds [24, Theorem A]:

Theorem 2.12. *Let $w \in W$ and $\lambda \in \Lambda$. We have*

$$D_{t_{w\beta}}[\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)] = [\mathcal{O}_{\mathbf{Q}_G(wt_\beta)}(\lambda)] = q^{-\langle \beta, \lambda \rangle} Q^\beta [\mathcal{O}_{\mathbf{Q}_G(w)}(\lambda)]$$

for every $\beta \in Q^\vee_<$. Moreover, $\{\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda)\}_{\lambda \in \Lambda}$ is a $\mathbb{C}_q(\!(Q^\vee)\!)$-free basis of $K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$.

Proof. The first assertion for $\lambda \in \Lambda_+$ is [24, Theorem 4.13] (it lifts to the formal version by [28]). In view of Theorem 2.9, it prolongs to all $\lambda \in \Lambda$. This proves the first assertion.

We prove the second assertion. Note that $\bigoplus_{u \in W} \mathbb{C}[\mathbf{H}][\mathcal{O}_{\mathbf{Q}_G(u)}] \subset K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$ is stable by the $\mathcal{H}_q(\mathtt{I})$-action, and it is isomorphic to $K_{\mathbf{H}}(\mathcal{B})$ as $\mathcal{H}_q(\mathtt{I})$-modules by the comparison of the actions. In view of Theorem 2.9 2) and 4), it follows that the coefficient of $[\mathcal{O}_{\mathbf{Q}_G(e)}]$ distinguishes two elements in the D_{w_0}-invariants of $\bigoplus_{u \in W} \mathbb{C}[\mathbf{H}][\mathcal{O}_{\mathbf{Q}_G(u)}]$. Since we allow formal sums with respect to $Q^\vee_+$, we conclude that $\{\mathcal{O}_{\mathbf{Q}_G(e)}(\lambda)\}_{\lambda \in \Lambda}$ defines a $\mathbb{C}_q[\![Q^\vee]\!]$-free basis of $K_{\mathbf{G}}(\mathbf{Q}_G(e))$. Now the assertion follows by the $Q^\vee$-translations. □

Lemma 2.13. *For each $i \in \mathtt{I}_{\mathrm{af}}$, $\lambda \in \mathbb{X}^*$, and $w \in W_{\mathrm{af}}$, we have*

$$D_i(e^{\lambda}[\mathcal{O}_{\mathbf{Q}_G(w)}]) \equiv \begin{cases} e^{\lambda}[\mathcal{O}_{\mathbf{Q}_G(w)}] + e^{s_i\lambda}[\mathcal{O}_{\mathbf{Q}_G(s_i w)}] & \langle \alpha_i^{\vee}, \lambda \rangle < 0, s_i w >_{\frac{\infty}{2}} w \\ e^{\lambda}[\mathcal{O}_{\mathbf{Q}_G(s_i w)}] & \langle \alpha_i^{\vee}, \lambda \rangle = 0, s_i w >_{\frac{\infty}{2}} w \\ -e^{s_i\lambda}[\mathcal{O}_{\mathbf{Q}_G(w)}] + e^{s_i\lambda}[\mathcal{O}_{\mathbf{Q}_G(s_i w)}] & \langle \alpha_i^{\vee}, \lambda \rangle > 0, s_i w >_{\frac{\infty}{2}} w \\ (e^{\lambda} + e^{s_i\lambda})[\mathcal{O}_{\mathbf{Q}_G(w)}] & \langle \alpha_i^{\vee}, \lambda \rangle < 0, s_i w <_{\frac{\infty}{2}} w \\ e^{\lambda}[\mathcal{O}_{\mathbf{Q}_G(w)}] & \langle \alpha_i^{\vee}, \lambda \rangle = 0, s_i w <_{\frac{\infty}{2}} w \\ 0 & \langle \alpha_i^{\vee}, \lambda \rangle > 0, s_i w <_{\frac{\infty}{2}} w \end{cases}$$

modulo the $\mathbb{C}_q$-span of $\{e^{\mu}[\mathcal{O}_{\mathbf{Q}_G(v)}] \mid \mu \in \Sigma_*(\lambda), v \in W_{\mathrm{af}}\}$.

Proof. The assertion follows from the behavior of the Hecke operators (i.e. $D_i - 1$) seen in (the $t = 0$ version of the $t^{1/2}$-twist of) [12, Proposition 3.3]. One can also directly prove using Corollary 1.5 and the convexity results in [12, §1]. $\square$

Let $\lambda \in \Lambda$. We consider two subspaces

$$K_{\preceq\lambda} := \mathrm{Span}_{\mathbb{C}_q}\{e^{\mu}[\mathcal{O}_{\mathbf{Q}_G(w)}] \mid w \in W_{\mathrm{af}}, \mu \in \Sigma(\lambda)\} \subset K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$$
$$K_{\prec\lambda} := \mathrm{Span}_{\mathbb{C}_q}\{e^{\mu}[\mathcal{O}_{\mathbf{Q}_G(w)}] \mid w \in W_{\mathrm{af}}, \mu \in \Sigma_*(\lambda)\} \subset K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}}).$$

Here we stress that our span consists of finite sums.

Corollary 2.14. *For each $\lambda \in \Lambda$, the spaces $K_{\prec\lambda} \subset K_{\preceq\lambda}$ are $\mathcal{H}_q^0$-submodules of $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$.*

Proof. Combine Theorem 2.8, Corollary 1.5, and Lemma 2.13. $\square$

Theorem 2.15. *For each $\lambda \in \Lambda$, we have a unique element $C(\lambda) \in K_{\preceq\lambda}$ with the following properties:*

1. *We have $C(\lambda) \equiv D_{w_0}(e^{w_0\lambda}[\mathcal{O}_{\mathbf{Q}_G(w_0)}]) \mod K_{\prec\lambda}$;*
2. *For each $\beta \in Q^{\vee}_{<}$, we have $D_{t_\beta}C(\lambda) = q^{-\langle \beta, \lambda \rangle}C(\lambda)Q^{\beta}$.*

Proof of Theorem 2.15. We prove the assertion by induction on the inclusion relation between $\Sigma(\lambda)$. We assume that $D_{w_0}K_{\prec\lambda}$ is spanned by the joint eigenvectors with respect to the action of $\{D_{t_\beta}\}_{\beta \in Q^{\vee}_{<}}$, and construct $C(\lambda) \in D_{w_0}K_{\preceq\lambda}$. Thanks to Theorem 2.12 and Theorem 2.9, the element $C(\lambda)$ exists (in fact uniquely) as an element in $K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$.

The case $\lambda = 0$ is clear by setting $C(0) := D_{w_0}([\mathcal{O}_{\mathbf{Q}_G(w_0)}]) = [\mathcal{O}_{\mathbf{Q}_G(e)}]$ thanks to Lemma 2.13.

We consider the general case by induction. Write $e >_{\frac{\infty}{2}} w = ut_\gamma$ for $u \in W$ and $\gamma \in Q^{\vee}_{+}$. Let $\beta' \in Q^{\vee}_{<}$ be such that $\gamma + \beta' \in Q^{\vee}_{<}$. We have

$$\ell(wt_{\beta'}) = \ell(t_{\beta'}) - \ell(u) - 2\langle \gamma, \rho \rangle \quad \text{and hence} \quad \ell(wt_{\beta'}) < \ell(t_{\beta'})$$

by Theorem 1.2. It follows that

$$\ell(t_{\beta+\beta'}) > \ell(wt_{\beta'}) + \ell(t_\beta) \qquad \beta \in Q^{\vee}_{<}.$$

Consequently, the coefficient of $[\mathcal{O}_{\mathbf{Q}_G(t_\beta)}]$ of $D_{t_\beta}(C(\lambda))$ modulo $K_{\prec\lambda}$ must be determined by the coefficient of $[\mathcal{O}_{\mathbf{Q}_G(e)}]$ in $C(\lambda)$ by Lemma 2.13, that is $e^{t_\beta(\lambda)} = q^{-\langle \beta, \lambda \rangle}e^{\lambda}$. We set

$$C'(\lambda) := D_{w_0}(e^{w_0\lambda}[\mathcal{O}_{\mathbf{Q}_G(w_0)}]).$$

Since $D_{t_\beta}(C'(\lambda))$ is D_{w_0}-invariant, we conclude that

$$D_{t_\beta}(C'(\lambda)) = q^{-\langle \beta, \lambda \rangle} C'(\lambda) Q^\beta \mod K_{\prec \lambda}$$

by Theorem 2.12. In particular, we find that

$$D_{t_\beta}(C'(\lambda)) - q^{-\langle \beta, \lambda \rangle} C'(\lambda) Q^\beta \in K_{\prec \lambda}. \tag{2.1}$$

By the first condition of our assertion and the induction hypothesis, we find that $D_{w_0} K_{\prec \lambda}$ is spanned by $\{C(\mu)\}_{\mu \in \Sigma_*(\lambda)}$ as a $\mathbb{C}_q Q^\vee$-module. These are the D_{t_β}-eigenvectors for each $\beta \in Q^\vee_<$. We expand the LHS of (2.1) as

$$\sum_{\mu \in \Sigma_*(\lambda)} C(\mu) b^\mu_\lambda \qquad b^\mu_\lambda \in \mathbb{C}_q Q^\vee_+.$$

Here we remark that this sum must be finite.

For any choices of $c^\mu_\lambda \in \mathbb{C}(q)[\![Q^\vee_+]\!]$ $(\mu \in \Lambda)$, we have

$$\begin{aligned} D_{t_\beta}(C'(\lambda) - &\sum_{\mu \in \Sigma_*(\lambda)} C(\mu) c^\mu_\lambda) - q^{-\langle \beta, \lambda \rangle} (C'(\lambda) - \sum_{\mu \in \Sigma_*(\lambda)} C(\mu) c^\mu_\lambda) \\ &= \sum_{\mu \in \Sigma_*(\lambda)} C(\mu) (b^\mu_\lambda - q^{-\langle \beta, \mu \rangle} c^\mu_\lambda + q^{-\langle \beta, \lambda \rangle} c^\mu_\lambda). \end{aligned}$$

It follows that the element

$$C'(\lambda) - \sum_{\mu \in \Sigma_*(\lambda)} c^\mu_\lambda C(\mu) \qquad c^\mu_\lambda := \frac{q^{\langle \beta, \mu \rangle}}{1 - q^{\langle \beta, \mu - \lambda \rangle}} b^\mu_\lambda \in \frac{1}{1 - q^{\langle \beta, \mu - \lambda \rangle}} \mathbb{C}_q Q^\vee_+ \tag{2.2}$$

satisfies the desired properties in $\mathbb{C}(q) \otimes_{\mathbb{C}_q} K_{\preceq \lambda}$ (note that we have $\langle \beta, \mu - \lambda \rangle \neq 0$ for every $\mu \in \Sigma_*(\lambda)$ for some choice of β). Here we remark that the coefficients $\{c^\mu_\lambda\}_\mu$ does not depend on the choice of $\beta \in Q^\vee_<$ by the characterization in $\mathbb{C}(q) \otimes_{\mathbb{C}_q} K_{\mathbf{H}}(\mathbf{Q}^{\mathrm{rat}}_G)$ coming from Theorem 2.12. Thus, we conclude that (2.2) belongs to

$$K_{\preceq \lambda} = \big(C(q) \otimes_{\mathbb{C}_q} K_{\preceq \lambda} \big) \cap K_{\mathbf{H}}(\mathbf{Q}^{\mathrm{rat}}_G) \subset \mathbb{C}(q) \otimes_{\mathbb{C}_q} K_{\mathbf{H}}(\mathbf{Q}^{\mathrm{rat}}_G).$$

Therefore, we obtain the desired element $C(\lambda)$ inside $K_{\preceq \lambda}$ by induction. Hence, the induction proceeds and we conclude the result. □

Corollary 2.16. *For each $i \in \mathtt{I}$, we have*

$$[\mathcal{O}_{\mathbf{Q}_G(e)}(\varpi_i)] = C(\varpi_i) \frac{1}{1 - Q^{\alpha_i^\vee}} := \sum_{m \geq 0} C(\varpi_i) Q^{m \alpha_i^\vee}.$$

Proof. Compare $C(\varpi_i)$ with the Pieri-Chevalley rule in [28, Theorem 5.10] through Theorem 2.12. □

Theorem 2.17 ([26] Theorem 3.11 and Remark 3.12)**.** *There exists a $R(\mathbf{G})$-linear embedding*

$$\Psi_G : qK_{\mathbf{G}}(\mathcal{B}) \hookrightarrow K_{\mathbf{G}}(\mathbf{Q}^{\mathrm{rat}}_G)$$

such that:

1. *$\Psi_G(Q^\beta) = [\mathcal{O}_{\mathbf{Q}_G(t_\beta)}]$ for each $\beta \in Q^\vee_+$;*
2. *$\Psi_G(A^\lambda(\bullet)) = \Xi(\lambda)(\Psi_G(\bullet))$ for each $-\lambda \in \Lambda_+$.* □

3 Darboux coordinates of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$

We work in the same settings as in §1.1.

3.1 Non-commutative K-theoretic Peterson isomorphism

Theorem 3.1. *Assume that G is simple. We have a $\mathcal{H}_q^{\mathrm{sph}}$-module embedding*

$$\Phi_G : K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends $[\mathcal{O}_{\mathrm{Gr}_G(0)}]$ to $[\mathcal{O}_{\mathbf{Q}_G(e)}]$, intertwines the right product $\odot_q$ on the LHS to the tensor product on the RHS. More precisely, we have: For each $i \in \mathtt{I}$ and $\xi \in K_{\mathbf{G}}(\mathrm{Gr}_G)$, it holds

$$\Phi(\xi \odot_q (e^{-\varpi_i} - e^{-\varpi_i}\mathbf{h}_i)) = \Xi(-\varpi_i)(\xi).$$

To prove Theorem 3.1, we need:

Lemma 3.2. *We have an isomorphism*

$$\mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}) \cong K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$$

determined by the image of $[\mathcal{O}_{\mathrm{Gr}_G(0)}]$. In particular, every $\mathcal{H}_q^{\mathrm{sph}}$-endomorphism of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ is obtained by the composition of the right multiplication of $K_{\mathbf{G}}(\mathrm{Gr}_G)$ followed by the application of $\mathtt{t}_\gamma$ for some $\gamma \in \mathbb{X}_*$.

Proof. As the torus factor H' of G produces $K_{\mathbf{H}'}(\mathrm{Gr}_{H'}) = K_{\mathbf{H}'}(\mathrm{Gr}_{H'})_{\mathrm{loc}}$ as a ($\mathbb{C}_q$-)tensor factors of $K_{\mathbf{G}}(\mathrm{Gr}_G)$ and $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ that are isomorphic to a Heisenberg algebra, we can factor out such a factor to assume that G is simple.

Since $K_{\mathbf{G}}(\mathrm{Gr}_G)$ affords a regular representation of $\mathcal{H}_q^{\mathrm{sph}}$, we see that

$$\mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G)) \cong K_{\mathbf{G}}(\mathrm{Gr}_G).$$

Here the isomorphism is obtained by the right multiplication and hence $f \in \mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G))$ is determined by $f(1)$.

Let $f \in \mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G))$. By construction of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$, we can take $\beta \in \mathbb{X}_*^-$ such that $f(1) \odot_q \mathtt{t}_\beta \in K_{\mathbf{G}}(\mathrm{Gr}_G)$. It follows that $1 \mapsto f(1) \odot_q \mathtt{t}_\beta$ uniquely gives rise to an element of $\mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G))$. Since the right action of $\mathtt{t}_\beta$ is invertible, we conclude that $f(1) \in K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ already defines an element of $\mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}})$ uniquely as required. □

Proof of Theorem 3.1. Thanks to [26, Proposition 2.13 and Remark 2.14], we have a $\mathcal{H}_q^{\mathrm{sph}}$-module embedding

$$\Phi_G : K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$$

that sends $\mathtt{t}_\beta$ to $[\mathcal{O}_{\mathbf{Q}_G(t_\beta)}]$ as the (left) D_{w_0}-invariant part of the corresponding embedding of $\mathbf{H}$-equivariant K-groups (cf. Corollary 3.3).

From the construction of the map Φ_G through its $\mathbf{H}$-equivariant variants, we see that $K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$ is the completion of the image of Φ_G with respect to the topology given in §2.2. In view of Lemma 3.2, we find that $\Xi(\lambda)$ defines an element of $\mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}})$ if and only if $\Xi(\lambda)([\mathcal{O}_{\mathbf{Q}_G(e)}])$ is a finite linear combination of $\{[\mathcal{O}_{\mathbf{Q}_G(w)}]\}_{w\in W_{\mathrm{af}}}$. This happens for $\lambda = -\varpi_i$ by Lemma 2.10. Namely, we have $\Xi(-\varpi_i) = e^{-\varpi_i}(\mathrm{id} - H_i)$. Again by [26, Proposition 2.13 and Remark 2.14], we conclude that $\Xi(-\varpi_i)$ induces a(n left $\mathcal{H}_q^{\mathrm{sph}}$-module) endomorphism of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ that sends $[\mathcal{O}_{\mathrm{Gr}_G(0)}]$ to $e^{-\varpi_i}(\mathrm{id} - \mathbf{h}_i)$. Therefore, we conclude that the equality in the assertion. □

Corollary 3.3. *Assume that G is simple. We have a $\mathcal{H}_q$-module embedding*

$$\Phi : K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$$

extending Φ_G with the following properties:

1. *we have $\Phi([\mathcal{O}_{\mathrm{Gr}_G(ut_\beta)}]) = [\mathcal{O}_{\mathbf{Q}_G(ut_\beta)}]$ for $u \in W$ and $\beta \in Q^\vee_<$;*
2. *the right multiplication by $\mathtt{t}_\gamma$ corresponds to the right translation by $\gamma \in Q^\gamma$ for each $\gamma \in Q^\vee$;*
3. *For each $i \in \mathtt{I}$ and $\xi \in K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}}$, it holds*

$$\Phi(\xi \odot_q \mathbf{h}_i) = H_i(\xi).$$

Proof. Notice that we have $[\mathcal{O}_{\mathcal{B}}] \in K_{\mathbf{G}}(\mathcal{B})$ in Theorem 1.6, that results in $\mathcal{H}_q(\mathtt{I})K_{\mathbf{G}}(\mathcal{B}) = K_{\mathbf{H}}(\mathcal{B})$ by Theorem 1.6 5). The comparison of Theorem 1.6 with Theorem 2.2 yields

$$\mathcal{H}_q K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} = \mathbb{C}_q[H] K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} = K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}},$$

while the comparison of Theorem 1.6 with Theorem 2.8 yields

$$\mathcal{H}_q K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}}) = \mathbb{C}_q[H] K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}}) = K_{\mathbf{H}}(\mathbf{Q}_G^{\mathrm{rat}})$$

as $\mathcal{H}_q$-modules with the desired properties except for the first item. The first item follows from [26, Proposition 2.13 and Remark 2.14]. □

Corollary 3.4. *Keep the setting of Lemma 3.2. Each $\mathcal{H}_q^{\mathrm{sph}}$-module endomorphism of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ is continuous with respect to the the topology induced from the topology of $K_{\mathbf{H}}(\mathbf{Q}_{[G,G]}^{\mathrm{rat}})$ (defined in §2.2) under $\Phi_{[G,G]}$ (by extending the scalar from $\mathbb{C}_q$ to $K_{\mathbf{H}'}(\mathrm{Gr}_{H'})$).* □

3.2 Darboux generators of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$

For each $i \in \mathtt{I}$, we set

$$\phi_i := e^{-\varpi_i}(\mathrm{id} - \odot_q \mathbf{h}_i) \in K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} \cong \mathrm{End}_{\mathcal{H}_q^{\mathrm{sph}}}(K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}).$$

Lemma 3.5. *Assume that G is simple. There exists a unique $\mathcal{H}_q^{\mathrm{sph}}$-module endomorphism ξ_i on $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ for each $i \in \mathtt{I}$ such that*

$$\xi_i \circ \phi_i = (\mathrm{id} - \mathtt{t}_{\alpha_i^\vee}) \quad \textit{and} \quad \phi_i \circ \xi_i = (\mathrm{id} - q\mathtt{t}_{\alpha_i^\vee}).$$

In addition, we have

$$\xi_i \circ \xi_j = \xi_j \circ \xi_i, \quad \xi_i \circ \phi_j = \phi_j \circ \xi_i, \quad \textit{and} \quad \phi_i \circ \phi_j = \phi_j \circ \phi_i \qquad i \neq j.$$

Proof. We transplant these endomorphisms to $K_{\mathbf{G}}(\mathbf{Q}_G^{\mathrm{rat}})$. We set ϕ_i to be the endomorphism $\Xi(-\varpi_i)$, and ξ_i to be the endomorphism $(1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)$ for each $i \in \mathtt{I}$. A priori, ξ_i only defines a $\mathcal{H}_q^{\mathrm{sph}}$-endomorphism of the completion of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ that is isomorphic to $K_{\mathbf{G}}(\mathbf{Q}^{\mathrm{rat}})$ via (the natural extension of) Φ_G. To see that ξ_i defines an endomorphism of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$, it suffices to see that whether $(1-Q^{\alpha_i^\vee})\Xi(\varpi_i)$ defines an endomorphism of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$. By Corollary 3.4, it suffices to see that

$$(1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)([\mathcal{O}_{\mathbf{Q}_G(e)}]) = [\mathcal{O}_{\mathbf{Q}_G(e)}(\varpi_i)] - [\mathcal{O}_{\mathbf{Q}_G(t_{\alpha_i^\vee})}(\varpi_i)]$$

is a finite linear combination of $\{[\mathcal{O}_{\mathbf{Q}_G(w)}]\}_{w \in W_{\mathrm{af}}}$, that is the content of Corollary 2.16. Now the commutation relation between them follow from Lemma 2.11. □

Corollary 3.6. *Keep the setting of Lemma 3.5. Then, the elements*

$$\Phi_G(\left(\prod_{i \in \mathtt{I}, \langle \alpha_i^\vee, \lambda \rangle < 0} \xi_i^{-\langle \alpha_i^\vee, \lambda \rangle}\right)\left(\prod_{i \in \mathtt{I}, \langle \alpha_i^\vee, \lambda \rangle > 0} \phi_i^{\langle \alpha_i^\vee, \lambda \rangle}\right)[\mathcal{O}_{\mathrm{Gr}_G(0)}]) \qquad \lambda \in \Lambda \quad (3.1)$$

are $\mathbb{C}_q Q^\vee$-linearly independent in $K_{\mathbf{G}}(\mathbf{Q}^{\mathrm{rat}})$. In particular, there is no additional relations among $\{\xi_i, \phi_i\}_{i \in \mathtt{I}}$ (to those presented in Lemma 3.5).

Proof. The elements in (3.1) are non-zero since ϕ_i and ξ_i defines $\Xi(-\varpi_i)$ and $(1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)$ for each $i \in \mathtt{I}$, that are invertible in $K_{\mathbf{G}}(\mathbf{Q}^{\mathrm{rat}})$. In view of Theorem 2.12, these elements belong to different (joint) eigenspaces with respect to the action of D_{t_β} ($\beta \in Q^\vee_<$), and hence they are $\mathbb{C}_q Q^\vee$-linearly independent. If we have an additional relation among $\{\xi_i, \phi_i\}_{i \in \mathtt{I}}$, then it violates the linear independence of (3.1). Consequently, it is impossible and hence the relations presented in Lemma 3.5 is optimal. □

We set $qK_{\mathbf{H}}(\mathcal{B})_{\mathrm{loc}} := \mathbb{C}Q^\vee \otimes_{\mathbb{C}Q^\vee_+} qK_{\mathbf{H}}(\mathcal{B})$.

Theorem 3.7. *Assume that G is simple. We have a $\mathcal{H}_q$-module isomorphism*

$$\Psi^{-1} \circ \Phi : K_{\mathbf{H}}(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow qK_{\mathbf{H}}(\mathcal{B})_{\mathrm{loc}}$$

with the following properties:

1. *We have $(\Psi^{-1} \circ \Phi)([\mathcal{O}_{\mathrm{Gr}_G(u)}]\mathtt{t}_\beta) = [\mathcal{O}_{\mathcal{B}(u)}]Q^\beta$ for $u \in W$ and $\beta \in Q^\vee$;*
2. *For each $i \in \mathtt{I}$ and $\xi \in K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$, it holds*

$$(\Psi^{-1} \circ \Phi)(\phi_i(\xi)) = A^{-\varpi_i}\left((\Psi^{-1} \circ \Phi)(\xi)\right).$$

Proof. The existence of the isomorphism and the first item follows from Corollary 3.3 and [26, Theorem 4.1 and its proof]. The second item is a consequence of the identification of ϕ_i with $\Xi(-\varpi_i)$ under Φ. □

Proposition 3.8. *We have a $\mathbb{C}_q$-algebra embedding*

$$K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_{\mathbf{H}}(\mathrm{Gr}_H)$$

given by $\mathtt{t}_\gamma \mapsto \mathtt{t}_\gamma$ ($\gamma \in \mathbb{X}_*$), $e^\lambda \mapsto e^\lambda$ ($\lambda \in \mathbb{X}^*(G)$), and

$$\phi_i \mapsto e^{-\varpi_i}, \xi_i \mapsto (1 - \mathtt{t}_{\alpha_i^\vee})e^{\varpi_i} \ \ (i \in \mathtt{I}).$$

Remark 3.9. **1)** Taking Theorem 3.1 into account, Proposition 3.8 follows as the symmetrization of a result of Daniel Orr [43, (0.2) and Theorem 5.1] when G is simple of types ADE; **2)** By taking the $q = 1$ specialization, this embedding becomes an embedding of commutative algebras that gives rise to an isomorphism between their fraction fields.

Proof of Proposition 3.8. The element e^λ ($\lambda \in \mathbb{X}^*(G)$) and t_γ ($\gamma \in \mathbb{X}_*(G)$) generates a common subalgebras of the both sides. If we add these elements to the case of $G = [G, G]$, then we obtain the whole embedding. Thus, we can assume that G is simple.

The commutation relation is preserved by a direct calculation. Thus, it remains to see that the elements in Proposition 3.8 generates the whole $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$. We have

$$\left(\prod_{j=0}^{m-1}(1 - q^{-j}Q^{\alpha_i^\vee})\right)\Xi(m\varpi_i) = \left(\prod_{j=0}^{m-1}(1 - q^{-j}Q^{\alpha_i^\vee})\right)\Xi(\varpi_i)^m$$
$$= (1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)\left(\prod_{j=0}^{m-2}(1 - q^{-j}Q^{\alpha_i^\vee})\right)\Xi(\varpi_i)^{m-1}$$
$$= \cdots$$
$$= \left((1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)\right)^m.$$

The Pieri-Chevalley rule [28, Theorem 5.13] is $\mathbb{C}[\mathbf{H}]$-linear, and the action of $\Xi(\varpi_i)$ sends the Schubert class $[\mathcal{O}_{\mathbf{Q}(w)}]$ ($w \in W_{\mathrm{af}}$) to a possibly infinite sum

$$e^\mu[\mathcal{O}_{\mathbf{Q}(v)}] \qquad w \ge_{\frac{\infty}{2}} v \in W_{\mathrm{af}}, \mu \in \Sigma(\varpi_i).$$

In view of Corollary 2.16, the action of $(1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)$ sends the Schubert class $\mathcal{O}_{\mathbf{Q}(e)}$ to a linear combination of

$$e^{v\varpi_i}[\mathcal{O}_{\mathbf{Q}(v)}] \qquad v \in W$$

modulo the formal sum of $e^\mu[\mathcal{O}_{\mathbf{Q}(v)}]$ for $\mu \in \Sigma_*(\varpi_i)$ and $v \in W_{\mathrm{af}}$. In addition, the term of the shape $e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}(v)}]$ must be $e^{\varpi_i}[\mathcal{O}_{\mathbf{Q}(e)}]$ by inspection (using Lemma 2.13).

We have $[Q^{\alpha_i^\vee}, \Xi(\pm\varpi_j)] = 0$ for $i \neq j$ (Lemma 2.11). In view of Theorem 2.12 and the fact that Q^β ($\beta \in Q^\vee$) commutes with the $\mathcal{H}_q$-action, we deduce that

$$\left(\prod_{i \in \mathtt{I}, \langle \alpha_i^\vee, \lambda \rangle < 0} \Xi(-\varpi_i)^{-\langle \alpha_i^\vee, \lambda \rangle}\right) \prod_{i \in \mathtt{I}, \langle \alpha_i^\vee, \lambda \rangle > 0} \left((1 - Q^{\alpha_i^\vee})\Xi(\varpi_i)\right)^{\langle \alpha_i^\vee, \lambda \rangle} [\mathcal{O}_{\mathbf{Q}(e)}] \tag{3.2}$$

is a (joint) eigenfunctions of D_{t_γ} ($\gamma \in Q^\vee_<$). By Theorem 2.15, we deduce that the $\mathbb{C}_q$-coefficient of the term $e^\mu[\mathcal{O}_{\mathbf{Q}(w)}]$ ($w \in W_{\mathrm{af}}$) in (3.2) is non-zero only if $\mu \in \Sigma(\lambda)$, and the class (3.2) is uniquely determined by the $\mathbb{C}_q$-coefficients of $e^\lambda[\mathcal{O}_{\mathbf{Q}(t_\beta)}]$ for all $\beta \in Q^\vee$.

We first examine the case $\lambda \in \Lambda_+$. Since $\lambda \in \Sigma(\lambda)$ is an extremal point, we find that $(\lambda + \varpi_i) \in \Sigma(\lambda + \varpi_i)$ is attained uniquely as the sum of elements from $\Sigma(\lambda)$ and $\Sigma(\varpi_i)$ whenever $\lambda \in \Lambda_+$ (namely the sum of $\lambda \in \Sigma(\lambda)$ and $\varpi_i \in \Sigma(\varpi_i)$). From this, we find that the $\mathbb{C}_q$-coefficient of the term $e^{\lambda}[\mathcal{O}_{\mathbf{Q}(w)}]$ ($w \in W_{\mathrm{af}}$) is just one for $w = e$ and it is zero for $w \neq e$ by induction from the case $\lambda = 0 \in \Lambda_+$. Since the both sides are (joint) eigenfunctions of D_{t_γ} ($\gamma \in Q^\vee_<$) with common (joint) eigenvalues whose coefficients of $e^{\lambda}[\mathcal{O}_{\mathbf{Q}(t_\beta)}]$ ($\beta \in Q^\vee$) are the same, we conclude

$$C_\lambda = \left(\prod_{i\in\mathtt{I}}((1-Q^{\alpha_i^\vee})\Xi(\varpi_i))^{\langle\alpha_i^\vee,\lambda\rangle}\right)[\mathcal{O}_{\mathbf{Q}_G(e)}] \qquad \lambda \in \Lambda_+$$

by Theorem 2.15.

Now we consider general $\lambda \in \Lambda$. Find $\mathtt{J} \subset \mathtt{I}$, $\lambda_+ \in \Lambda_+^{(\mathtt{I}\setminus\mathtt{J})}$, and $\lambda_- \in \Lambda_+^{\mathtt{J}}$ such that $\lambda = \lambda_+ - \lambda_-$. When $\lambda_- = 0$, then the weight e^{λ_+} appears only as a coefficient of $[\mathcal{O}_{\mathbf{Q}(e)}]$ in C_{λ_+} by the previous paragraph. If we want to represent $\lambda \in \Lambda$ by a sum of elements from $\Sigma(\lambda_+)$ and $\Sigma(-\lambda_-) = \Sigma(-w_0^{\mathtt{J}}\lambda_-)$, then we have necessarily $\lambda = \lambda_+ - \lambda_-$ since λ belongs to the same W-orbit as $\lambda_+ - w_0^{\mathtt{J}}\lambda_- \in \Lambda_+$. The coefficient of $e^{-\lambda_-}[\mathcal{O}_{\mathbf{Q}(t_\beta)}]$ in $C_{-\lambda_-}$ is one if $\beta = 0$, and zero if $\beta \neq 0$ by [41, Corollary 3.15] (note that the set of paths $\mathrm{QLS}(\lambda_-)$ contains a unique path whose weight is of the form $q^* e^{\lambda_-}$ since it represents the character of a local Weyl module, and such a path contributes to $[\mathcal{O}_{\mathbf{Q}(e)}]$ only once by the shape of the formula). It follows that the coefficient of $e^{\lambda}[\mathcal{O}_{\mathbf{Q}(t_\beta)}]$ in C_λ is one if $\beta = 0$, and zero if $\beta \neq 0$. Therefore, we conclude that (3.2) must be C_λ for every $\lambda \in \Lambda$.

It follows that

$$\Phi_G^{-1}(C_\lambda) = \left(\prod_{i\in\mathtt{I},\langle\alpha_i^\vee,\lambda\rangle<0} \xi_i^{-\langle\alpha_i^\vee,\lambda\rangle}\right)\left(\prod_{i\in\mathtt{I},\langle\alpha_i^\vee,\lambda\rangle>0} \phi_i^{\langle\alpha_i^\vee,\lambda\rangle}\right)[\mathcal{O}_{\mathrm{Gr}_G(0)}] \in K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}.$$

By Theorem 2.15 and Theorem 3.1 (cf. Corollary 3.3), one sees that $\{\Phi_G^{-1}(C_\lambda)\}_{\lambda\in P}$ forms a $\mathbb{C}_q Q^\vee$-basis of $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$. Thus, the elements in the assertion generates the whole $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$, and we have the desired inclusion. □

Corollary 3.10. *The $\mathbb{C}_q$-algebra $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ is generated by $\mathtt{t}_\gamma$ ($\gamma \in \mathbb{X}_*$), e^λ ($\lambda \in \mathbb{X}^*(G)$), and ϕ_i, ξ_i ($i \in \mathtt{I}$).* □

Corollary 3.11. *We have a $\mathbb{C}_q$-algebra embedding*

$$K_{\mathbf{G}}(\mathrm{Gr}_G) \hookrightarrow K_{\mathbf{H}}(\mathrm{Gr}_H)$$

obtained by the restriction of the domain in Proposition 3.8. □

4 Induction equivalence for flag varieties

We work under the setting of §2.2. In particular, G is simple. The goal of this section is to present the following:

Theorem 4.1. *Let $L = L^{\mathtt{J}}$ be the standard Levi subgroup corresponding to $\mathtt{J} \subset \mathtt{I}$. There is a $\mathbb{C}_q\mathbb{X}^*(G)$-linear surjective map*

$$qK_{\mathbf{G}}(\mathcal{B})^\wedge \longrightarrow qK_{\mathbf{L}}(\mathcal{B}^{\mathtt{J}})^\wedge$$

sending $[\mathcal{O}_{\mathcal{B}}]$ *to* $[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}]$*, and it intertwines the action of* $A^{\pm\varpi_i}$ *to the action of* $A^{\pm\varpi_i}$ *for each* $i \in \mathtt{I}$*. In addition, the kernel of this map is generated by* $Q^{-w_0\alpha_i^\vee}$ *for* $i \in (\mathtt{I} \setminus \mathtt{J})$*.*

Theorem 4.1 is proved in subsection §4.2 via explicit calculation. We record more general result as Theorem A.1.

4.1 Reductions of quasi-map spaces

Lemma 4.2. *Let* $\beta \in -w_0 Q^\vee_{\mathtt{J},+}$*. We have an isomorphism*

$$\mathcal{Q}_G(\beta) \cong G \times_{P^{\mathtt{J}}} \mathcal{Q}_{L^{\mathtt{J}}}(\beta),$$

where the the unipotent radical of $P^{\mathtt{J}}$ *acts on* $\mathcal{Q}_{L^{\mathtt{J}}}(\beta)$ *trivially.*

Proof. The definition of $\mathcal{Q}_G(\beta)$ is to consider a collection of $\mathbb{C}$-lines ℓ_λ in $V(\lambda) \otimes \mathbb{C}[z]$ for each $\lambda \in \Lambda_+$ (cf. [25, Lemma 3.28 and Theorem 3.30]). In particular, such collections must satisfy the same relation as $\mathbb{C}(\!(z)\!)$-lines if we extend the scalar. By (1.4), we have $\ell_{\varpi_i} \in V(\varpi_i) \subset V(\varpi_i) \otimes \mathbb{C}(\!(z)\!)$ for $i \notin \mathtt{J}$. Thanks to the Plücker relations (see e.g. [7, Theorem 1.1.2]), we know that $\ell_{\varpi_i} \in G\mathbf{v}_{\varpi_i}$ for $i \notin \mathtt{J}$. Therefore, a point of $\mathcal{Q}_G(\beta)$ is G-conjugate to a point represented as a collection of $\mathbb{C}$-lines $\{\ell'_\lambda\}_{\lambda \in \Lambda_+}$ such that $\ell'_{\varpi_i} = \mathbb{C}\mathbf{v}_{\varpi_i}$ for $i \notin \mathtt{J}$. By the Plücker relation (considered over the field $\mathbb{C}(\!(z)\!)$), it follows that $\ell'_{\varpi_j} \in L^{\mathtt{J}}(\!(z)\!)\mathbf{v}_{\varpi_j}$ for $j \in \mathtt{J}$ in this case. This forces our point to belong to $\mathcal{Q}_{L^{\mathtt{J}}}(\beta)$, with the trivial action of the unipotent radical of $P^{\mathtt{J}}$. From these, we deduce a surjective homomorphism $G \times_{P^{\mathtt{J}}} \mathcal{Q}_{L^{\mathtt{J}}}(\beta) \to \mathcal{Q}_G(\beta)$. Since the G-orbit of $\{\mathbb{C}\mathbf{v}_{\varpi_i}\}_{i \notin \mathtt{J}}$ is $\mathcal{B}_{\mathtt{J}}$, this map is a homeomorphism between projective normal varieties. It must be an isomorphism by the Zariski main theorem. □

Corollary 4.3. *Keep the setting of Lemma 4.2. For each* $\lambda \in \Lambda_+$*, we have a surjective* ($P^{\mathtt{J}}$*-module) map*

$$H^0(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G(\beta)}(\lambda)) \longrightarrow\!\!\!\!\rightarrow H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(\lambda)).$$

Proof. In view of [25, Theorem 3.33], we have a surjection

$$H^0(\mathbf{Q}_{L^{\mathtt{J}}}(e), \mathcal{O}_{\mathbf{Q}_{L^{\mathtt{J}}}(e)}(\lambda)) \longrightarrow\!\!\!\!\rightarrow H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(\lambda)).$$

In view of [25, Theorem 1.2], the H-weight of $H^0(\mathbf{Q}_{L^{\mathtt{J}}}(e), \mathcal{O}_{\mathbf{Q}_{L^{\mathtt{J}}}(e)}(\lambda))$ is concentrated in $w_0\lambda + Q^\vee_{\mathtt{J},+}$. Since $\mathcal{Q}_{L^{\mathtt{J}}}(\beta)$ is stable under the $L^{\mathtt{J}}$-action, it follows that $H^0(\mathbf{Q}_{L^{\mathtt{J}}}(e), \mathcal{O}_{\mathbf{Q}_{L^{\mathtt{J}}}(e)}(\lambda))$ is a direct sum of finite-dimensional irreducible $L^{\mathtt{J}}$-module. Since $\langle \alpha_i^\vee, \alpha_j \rangle \le 0$ for every $i \in \mathtt{I} \setminus \mathtt{J}$ and $j \in \mathtt{J}$ (and $\lambda \in \Lambda_+$), every finite-dimensional irreducible $L^{\mathtt{J}}$-submodule in $H^0(\mathbf{Q}_{L^{\mathtt{J}}}(e), \mathcal{O}_{\mathbf{Q}_{L^{\mathtt{J}}}(e)}(\lambda))$ is an irreducible $[L^{\mathtt{J}}, L^{\mathtt{J}}]$-module twisted by a weight μ such that $\langle \alpha_i^\vee, \mu \rangle \le 0$ for every $i \in (\mathtt{I} \setminus \mathtt{J})$. It follows that

$$H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(\lambda))^* \hookrightarrow H^0(G/P^{\mathtt{J}}, \mathcal{V})^*,$$

where $\mathcal{V}$ is the G-equivariant vector bundle obtained by inflating the $P^{\mathtt{J}}$-module $H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(\lambda))$. By the Leray spectral sequence, we have

$$H^0(G/P^{\mathtt{J}}, \mathcal{V}) \cong H^0(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G(\beta)}(\lambda)).$$

Therefore, we conclude

$$H^0(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G(\beta)}(\lambda)) \cong H^0(G/P^{\mathtt{J}}, \mathcal{V}) \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(\lambda))$$

as desired. □

Let $\mathfrak{g}[z] := \mathfrak{g} \otimes \mathbb{C}[z]$ be the Lie algebra obtained by scalar extension. Each $\lambda \in \Lambda_+$ defines a $\mathfrak{g}[z]$-module $\mathbb{W}_G(\lambda)$ that is the global Weyl module in the sense of [11]. By expressing $\lambda \in \Lambda_+$ as the sum $\lambda = \lambda^{(1)} + \lambda^{(2)}$ of $\lambda^{(1)} \in \Lambda_+^{\mathtt{J}}$ and $\lambda^{(2)} \in \Lambda^{\mathtt{I} \setminus \mathtt{J}}$, we have the corresponding global Weyl module $\mathbb{W}_{[L^{\mathtt{J}}, L^{\mathtt{J}}]}(\lambda^{(1)})$ of $[\mathfrak{l}^{\mathtt{J}}, \mathfrak{l}^{\mathtt{J}}][z]$ (by taking the external tensor product of the global Weyl modules for all simple factors of $[L^{\mathtt{J}}, L^{\mathtt{J}}]$). We define

$$\mathbb{W}_{L^{\mathtt{J}}}(\lambda) := \mathbb{W}_{[L^{\mathtt{J}}, L^{\mathtt{J}}]}(\lambda^{(1)}) \otimes \mathbb{C}_{\lambda^{(2)}},$$

that is a $([\mathfrak{l}^{\mathtt{J}}, \mathfrak{l}^{\mathtt{J}}][z] + \mathfrak{h})$-module.

Corollary 4.4. *For each $\lambda \in \Lambda_+$, we have an inclusion $\mathbb{W}_{L^{\mathtt{J}}}(\lambda) \subset \mathbb{W}_G(\lambda)$ between global Weyl modules.*

Proof. In view of [8, Proposition 5.1] (cf. [25, Theorem 3.33]), we have

$$\bigcup_{\beta \in -w_0 Q^{\vee}_{\mathtt{J},+}} H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(-w_0\lambda))^* = \mathbb{W}_{L^{\mathtt{J}}}(\lambda). \tag{4.1}$$

By Corollary 4.3, we have

$$H^0(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(-w_0\lambda))^* \hookrightarrow H^0(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G(\beta)}(-w_0\lambda))^* \hookrightarrow \mathbb{W}_G(\lambda).$$

Combined with (4.1), we conclude the result. □

Proposition 4.5. *Let $i \in \mathtt{I}$. Find $i' \in \mathtt{I}$ such that $\alpha_{i'} = w_0 \alpha_i$. The $A^{\pm \varpi_i}$-action on $qK_{\mathbf{G}}(\mathcal{B})$ is the same as the tensor product of $\mathcal{O}_{\mathcal{B}}(\pm\varpi_i)$ on $K_{\mathbf{G}}(\mathcal{B})^\wedge$ modulo $Q_{i'}$.*

Proof. Let $\mathtt{J}' := \mathtt{I} \setminus \{i'\}$. By our definition of $A^{\pm\varpi_i}$, it suffices to see

$$\left\langle A^{\pm\varpi_i} a, b \right\rangle^{\mathtt{GW}}_G \equiv \langle \mathcal{O}_{\mathcal{B}}(\pm\varpi_i) \otimes a, b \rangle^{\mathtt{GW}}_G \mod Q_{i'} \tag{4.2}$$

for every $a, b \in K_{\mathbf{G}}(\mathcal{B})$. Since $K_{\mathbf{G}}(\mathcal{B})$ is generated by A^λ for $-\lambda \in \Lambda_+$ and Q^β ($\beta \in Q^\vee_+$) as $\mathbb{C}_q\mathbb{X}^*(G)$-algebra, we can take $a = A^\mu$ and $b = [\mathcal{O}_{\mathcal{B}}]$. Since $\mathcal{Q}_G(\beta)$ has rational singularities for every $\beta \in Q^\vee_+$ (Theorem 1.11), we have

$$\left\langle A^{\pm\varpi_i + \lambda}[\mathcal{O}_{\mathcal{B}}], [\mathcal{O}_{\mathcal{B}}] \right\rangle^{\mathtt{GW}}_G = \sum_{\beta \in Q^\vee_+} Q^\beta \chi(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G(\beta)}(\pm\varpi_i + \lambda))) \qquad \lambda \in \mathbb{X}^*.$$

In case $\langle \beta, \varpi_{i'} \rangle = 0$, the structure map $\mathcal{Q}_{L^{\mathtt{J}'}}(\beta) \to \mathrm{pt}$ and Lemma 4.2 yield a projection map $\eta : \mathcal{Q}_G(\beta) \to G/P^{\mathtt{J}'} = \mathcal{B}_{\mathtt{J}'}$, that is G-equivariant. This implies

$$\chi(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G(\beta)}(\lambda))) = D_{w_0}(e^{-\langle \alpha_i^\vee, \lambda \rangle \varpi_{i'}} \chi(\mathcal{Q}_L(\beta), \mathcal{O}_{\mathcal{Q}_L(\beta)}(\lambda - \langle \alpha_i^\vee, \lambda \rangle \varpi_i))) \tag{4.3}$$

for each $\lambda \in \mathbb{X}^*$. The twist by $e^{-\varpi_{i'}}$ in the RHS of (4.3) is just a $\mathcal{O}(1)$-line bundle twist of $\mathcal{B}_{\mathtt{J}'}$ pulled back by η. Thus, it arises from the line bundle twist of $\mathcal{O}_{\mathcal{B}}(\varpi_i)$ through $\mathtt{ev}_1$. Therefore, we conclude (4.2) as required. □

4.2 Proof of Theorem 4.1

This subsection is entirely devoted to the proof of Theorem 4.1. We set $\mathtt{J}^\# := \{i \in \mathtt{I} \mid \alpha_i = -w_0\alpha_j,\ j \in \mathtt{I}\setminus\mathtt{J}\}$ and $\mathtt{J}' := \{i \in \mathtt{I} \mid \alpha_i = -w_0\alpha_j,\ j \in \mathtt{J}\}$.

By Theorem 1.12, we know that $qK_{\mathbf{L}^{\mathtt{J}}}(\mathcal{B}^{\mathtt{J}})$ is generated from $[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}]$ by $A^{\pm w_0\varpi_i}$ $(i \in \mathtt{J})$, Q_i $(i \in \mathtt{J}')$, and $\mathbb{X}_0^*(\mathtt{J})$ as an algebra. Suppose that

$$f(e^\mu, x_i, Q) = \sum_{\vec{m}\in\mathbb{Z}^r,\mu\in\mathbb{X}_0^*(\mathtt{J}),\gamma\in Q^\vee_{\mathtt{J},+}} f_{\vec{m},\mu,\beta}e^\mu x^{\vec{m}}Q^\gamma \in \mathbb{C}_q\mathbb{X}_0^*(\mathtt{J})[x_1^{\pm1},\ldots,x_r^{\pm1}][\![Q^\vee_{\mathtt{J},+}]\!],$$

where $x^{\vec{m}} := x_1^{m_1}\cdots x_r^{m_r}$ for $\vec{m} = (m_1,\ldots,m_r)$, satisfies

$$f(e^\mu, A^{\varpi_i}, Q)[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}] = 0 \in qK_{\mathbf{L}^{\mathtt{J}}}(\mathcal{B}^{\mathtt{J}}),$$

where $A^{\pm w_0\varpi_i}$ is interpreted as $e^{\mp\varpi_i}$ for $i \notin \mathtt{J}$. The line bundle $\mathbb{C}_\mu\otimes\mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}(\beta)}(-w_0\lambda)$ for $\beta \in Q^\vee_{\mathtt{J}',+}$, $\mu \in \mathbb{X}_0^*(\mathtt{J})$, and $\lambda \in \Lambda^{\mathtt{J}}$ inflates to $\mathcal{O}_{\mathcal{Q}_G(\beta)}(-w_0(\lambda+\mu))$ by Lemma 4.2 and (1.4). Let

$$\tilde{f}(e^\mu, A^{\varpi_i}, Q) = \sum_{\vec{m}\in\mathbb{Z}^r,\nu\in\mathbb{X}^*(G),\beta\in Q^\vee_{\mathtt{J},+}} \tilde{f}_{\vec{m},\nu,\beta}e^\nu x^{\vec{m}}Q^\beta \in \mathbb{C}_q\mathbb{X}^*(G)[x_1^{\pm1},\ldots,x_r^{\pm1}][\![Q^\vee_+]\!]$$

be the polynomial obtained from f by replacing $e^{-\varpi_i}$ with $x_{i'}$ (for each $i \in \mathtt{I}\setminus\mathtt{J}$ and $i' \in \mathtt{I}$ such that $\varpi_i = -w_0\varpi_{i'}$). For each $\lambda \in \Lambda$, we have

$$\begin{aligned}
&\left\langle A^\lambda \tilde{f}(e^\mu, A^{\varpi_i}, Q)[\mathcal{O}_\mathcal{B}], [\mathcal{O}_\mathcal{B}]\right\rangle^{\mathtt{GW}}_G \\
&\qquad= \sum_{\beta\in Q^\vee_+}\sum_{\vec{m},\nu,\gamma}\tilde{f}_{\vec{m},\nu,\gamma}Q^{\beta+\gamma}e^\nu\chi(\mathcal{X}_G(\beta), \mathcal{O}_{\mathcal{X}_G}(\lambda+\sum_{i\in\mathtt{I}}m_i\varpi_i)) \\
&\qquad= \sum_{\beta\in Q^\vee_+}\sum_{\vec{m},\nu,\gamma}\tilde{f}_{\vec{m},\nu,\gamma}Q^{\beta+\gamma}e^\nu\chi(\mathcal{Q}_G(\beta), \mathcal{O}_{\mathcal{Q}_G}(\lambda+\sum_{i\in\mathtt{I}}m_i\varpi_i)) \\
&\equiv \sum_{\beta\in Q^\vee_{\mathtt{J}',+}}\sum_{\vec{m},\nu,\gamma} e^\nu D_{w_0}(\tilde{f}_{\vec{m},\nu,\gamma}Q^{\beta+\gamma}\,\chi(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}}(\lambda+\sum_{i\in\mathtt{I}}m_i\varpi_i))) \\
&\qquad\qquad\mod (Q_i \mid i \in \mathtt{J}^\#),
\end{aligned}$$

where the first equality is the the definition, the second equality follows from Theorem 1.11, and the third equality follows from Lemma 4.2 and the fact that $\mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}}(\lambda)$ is the restriction of $\mathcal{O}_{\mathcal{Q}_G}(\lambda)$. Similarly, we have

$$\begin{aligned}
0 &= \langle A^\lambda f(e^\mu, A^{\varpi_i}, Q)[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}], [\mathcal{O}_{\mathcal{B}^{\mathtt{J}}}]\rangle^{\mathtt{GW}}_{L^{\mathtt{J}}} \\
&= \sum_{\beta\in Q^\vee_+}\sum_{\vec{m},\mu,\gamma} f_{\vec{m},\mu,\gamma}\,Q^{\beta+\gamma}e^\mu\chi(\mathcal{Q}_{L^{\mathtt{J}}}(\beta), \mathcal{O}_{\mathcal{Q}_{L^{\mathtt{J}}}}(\lambda+\sum_{i\in\mathtt{I}}m_i\varpi_i))
\end{aligned}$$

for $\lambda \in \Lambda$. By examining the relation between f and $\widetilde{f}$, we conclude

$$\left\langle A^\lambda \tilde{f}(e^\mu, A^{\varpi_i}, Q)[\mathcal{O}_\mathcal{B}], [\mathcal{O}_\mathcal{B}]\right\rangle^{\mathtt{GW}}_G \equiv 0 \mod (Q_i \mid i \in \mathtt{J}^\#)$$

for $\lambda \in \Lambda$. In view of Theorem 1.12, this is equivalent to

$$\tilde{f}(e^\mu, A^{\varpi_i}, Q)[\mathcal{O}_\mathcal{B}] \equiv 0 \mod (Q_i \mid i \in \mathtt{J}^\#).$$

This yields a map $qK_{\mathbf{G}}(\mathcal{B}) \to qK_{\mathbf{L}^{\mathtt{J}}}(\mathcal{B}^{\mathtt{J}})$ that intertwines A^{λ} ($\lambda \in \Lambda$), Q_i ($i \in \mathtt{I}$), and $\mathbb{C}_q\mathbb{X}^*(G)$-actions. The $Q_i \equiv 0$ ($i \in \mathtt{I}$) specialization of this map is the restriction map, that is an isomorphism (as a consequence of the bijection between equivariant line bundles through the restriction; cf. Corollary 1.7). Since the $\mathbb{C}Q^{\vee}_{\mathtt{J}',+}$-actions are free on the both of $qK_{\mathbf{G}}(\mathcal{B})/(Q_i \mid i \in \mathtt{J}^{\#})$ and $qK_{\mathbf{L}^{\mathtt{J}}}(\mathcal{B}^{\mathtt{J}})$, we conclude that

$$qK_{\mathbf{G}}(\mathcal{B})/(Q_i \mid i \in \mathtt{J}^{\#}) \xrightarrow{\cong} qK_{\mathbf{L}^{\mathtt{J}}}(\mathcal{B}^{\mathtt{J}})$$

as required.

5 Finkelberg-Tsymbaliuk's conjecture

We work in the settings of §1.1. The goal of this section is to prove the following main theorem of this paper, originally conjectured by Finkelberg-Tsymbaliuk [17]:

Theorem 5.1. *Let G be a connected reductive algebraic group over $\mathbb{C}$ such that $[G,G]$ is simply connected and $G \cong [G,G] \times H'$ for a subtorus $H' \subset H$. Let L be a reductive subgroup that contains H. The embedding of Corollary 3.11 induces algebra embeddings*

$$K_{\mathbf{G}}(\mathrm{Gr}_G) \hookrightarrow K_{\mathbf{L}}(\mathrm{Gr}_L) \hookrightarrow K_{\mathbf{H}}(\mathrm{Gr}_H).$$

Theorem 5.1 is proved in §5.2. From Theorem 5.1, we conclude the following enhancement:

Corollary 5.2. *Let G be a connected reductive algebraic group over $\mathbb{C}$ such that $[G,G]$ is simply connected and $[G,G] \times H'$ for a subtorus $H' \subset H$. Let L be a connected reductive subgroup of G that contains H. Let $Z \subset H \cap Z(G)$ be a finite subgroup. Theorem 5.1 induces embeddings*

$$K_{\mathbf{G}}(\mathrm{Gr}_{G/Z}) \hookrightarrow K_{\mathbf{L}}(\mathrm{Gr}_{L/Z}) \hookrightarrow K_{\mathbf{H}}(\mathrm{Gr}_{H/Z})$$

of algebras.

Proof. We set $G' := G/Z, L' := L/Z$. Note that the quotient $H \to H/Z$ induces an injective map

$$\mathbb{X}_* \cong \mathrm{Gr}_H \longrightarrow \mathrm{Gr}_{H/Z}$$

that identifies $\mathbb{X}_*$ with a subset of the group of cocharacters $\mathbb{X}'_*$ of H/Z via the quotient map. This gives rise to an isomorphism

$$K_{\mathbf{H}}(\mathrm{Gr}_{H/Z}) \cong \bigoplus_{\chi \in \mathsf{Irr}\, Z} K_{\mathbf{H}}(\mathrm{Gr}_H)$$

of algebras. In particular, the connected components of $\mathrm{Gr}_{H/Z}$ is the union of the contributions

$$\mathrm{Gr}_{H/Z} = \bigsqcup_{\chi \in \mathsf{Irr}\, Z} \mathrm{Gr}^{\chi}_{H/Z}.$$

The same is true for $\mathrm{Gr}_{G'}$ and $\mathrm{Gr}_{L'}$, that we denote by

$$\mathrm{Gr}_{G'} = \bigsqcup_{\chi \in \mathsf{Irr}\, Z} \mathrm{Gr}^{\chi}_{G'} \quad \text{and} \quad \mathrm{Gr}^{\chi}_{L'} = \bigsqcup_{\chi \in \mathsf{Irr}\, Z} \mathrm{Gr}^{\chi}_{L'}.$$

Note that the content of Theorem 5.1 under this setup is the algebra embeddings:

$$K_{\mathbf{G}}(\mathrm{Gr}^1_{G'}) \hookrightarrow K_{\mathbf{L}}(\mathrm{Gr}^1_{L'}) \hookrightarrow K_{\mathbf{H}}(\mathrm{Gr}^1_{H/Z}), \tag{5.1}$$

where $1 \in \mathsf{Irr}\, Z$ is the trivial representation.

The action of $\mathbb{X}'_*/\mathbb{X}_*$ induces outer automorphisms of the affine Dynkin diagram of G. This twists the embedding $K_{\mathbf{G}}(\mathrm{Gr}^\chi_{G'}) \subset K_{\mathbf{H}}(\mathrm{Gr}^\chi_{G'})$ into $K_{\mathbf{G}}(\mathrm{Gr}^1_{G'}) \subset K_{\mathbf{H}}(\mathrm{Gr}^1_{G'})$ by the Dynkin diagram automorphisms. These outer automorphisms induce automorphisms of $\mathcal{H}_q$, and hence gives rise to an algebra structure of $K_{\mathbf{G}}(\mathrm{Gr}_{G'})$ induced from $K_{\mathbf{H}}(\mathrm{Gr}_{G'})$. If we employ these twists of $R(\mathbf{H})$ also to the coefficients of $K_{\mathbf{H}}(\mathrm{Gr}^\chi_{H/Z})$, we obtain embeddings

$$K_{\mathbf{G}}(\mathrm{Gr}^\chi_{G'}) \longrightarrow K_{\mathbf{H}}(\mathrm{Gr}^\chi_{H/Z}) \qquad \chi \in \mathrm{Irr}\, Z. \tag{5.2}$$

Such twists, altogether along $\mathsf{Irr}\, Z$, give rise to a twist of the algebra structure of $K_{\mathbf{H}}(\mathrm{Gr}_{H/Z})$ (that prolongs $K_{\mathbf{H}}(\mathrm{Gr}^1_{H/Z}) \cong K_{\mathbf{H}}(\mathrm{Gr}_H)$). With these twisted algebra structures, we obtain a morphism

$$K_{\mathbf{G}}(\mathrm{Gr}_{G'}) \longrightarrow K_{\mathbf{H}}(\mathrm{Gr}_{H/Z})$$

of algebras that prolongs (5.1) and (5.2).

It remains to find that such a twisting can be taken to be compatible with the analogously defined embedding $K_{\mathbf{L}}(\mathrm{Gr}_{L'}) \subset K_{\mathbf{H}}(\mathrm{Gr}_{H/Z})$. To see this, it is enough to mind that the twisting by $\chi \in \mathsf{Irr}\, Z$ gives a twisting of $G'[\![z]\!] \subset G'(\!(z)\!)$ by a lift of χ in $\mathbb{X}'_*$ (up to internal automorphism), and it naturally induce a twisting of $L'[\![z]\!] \subset G'(\!(z)\!)$. □

Example 5.3. We assume that $G = SL(2)$ and $L = H$ is its maximal torus. We have $Q^\vee = \mathbb{X}_* = \mathbb{Z}\alpha$, where α is the positive simple root of G identified with its coroot. Let ϖ be the fundamental weight. We have

$$R(G) = \mathbb{C}[e^{\pm\varpi}]^{\mathfrak{S}_2} \subset \mathbb{C}[e^{\pm\varpi}] = R(H).$$

Theorem 5.1 yields an algebra map

$$R(G) \equiv R(G)[\mathcal{O}_{\mathrm{Gr}_G(0)}] \hookrightarrow K_G(\mathrm{Gr}_G) \longrightarrow K_H(\mathrm{Gr}_H) = \bigoplus_{\gamma \in Q^\vee} R(H)\mathtt{t}_\gamma,$$

where $\mathtt{t}_\gamma$ represents the class of the structure sheaf of $\mathrm{Gr}_H(\gamma)$, that is a point. In view of Proposition 3.8, we find that

$$[\mathcal{O}_{\mathrm{Gr}_G(0)}] \mapsto \mathtt{t}_0, \qquad (e^{\varpi} + e^{-\varpi})[\mathcal{O}_{\mathrm{Gr}_G(0)}] \mapsto e^{\varpi}(\mathtt{t}_0 - \mathtt{t}_\alpha) + e^{-\varpi}\mathtt{t}_0.$$

We remark that Example 5.3 is obtained from the $n = 2$ case of Example D by applying Theorem 5.1 (and Theorem 3.7).

5.1 Classes $E(\beta, \lambda)$ and $\mathcal{O}^\star(\lambda)$

We find $\mathtt{J} \subset \mathtt{I}$ such that L in Theorem 5.1 is written as $L^{\mathtt{J}}$. For $\beta \in \mathbb{X}_*^{\le}(\mathtt{J})$, we set $\mathtt{J}(\beta) = \{j \in \mathtt{J} \mid \langle \alpha_j^\vee, \beta \rangle = 0\} \subset \mathtt{J}$. We set $w(\mathtt{J}, \beta) := w_0^{\mathtt{J}} w_0^{\mathtt{J}(\beta)} w_0^{\mathtt{J}}$ and $\mathtt{J}(\beta)^\# := \{j \in \mathtt{J} \mid \exists j' \in \mathtt{J}(\beta) \text{ s.t. } \varpi_j = -w_0^{\mathtt{J}} \varpi_{j'}\}$ (i.e. $w(\mathtt{J}, \beta) = w_0^{\mathtt{J}(\beta)^\#}$). We set $\Lambda^{\mathtt{J}}_+(\beta) := \Lambda^{\mathtt{J} \setminus \mathtt{J}(\beta)} + \Lambda_+^{\mathtt{J}(\beta)}$. For each $\lambda \in \Lambda^{\mathtt{J}}_+(\beta)$, we define

$$E^{\mathtt{J}}[\beta; \lambda] := D_{w_0^{\mathtt{J}}}(e^{w_0^{\mathtt{J}}\lambda}[\mathcal{O}_{\mathrm{Gr}_L(u^{\mathtt{J}}_\beta)}]) \in K_{\mathbf{L}}(\mathrm{Gr}_{\mathbf{L}}),$$

where $u^{\mathtt{J}}_\beta \in W^{\mathtt{J}} t_\beta W^{\mathtt{J}}$ is the minimal length element inside the double coset.

Lemma 5.4. *The $\mathcal{H}_q(\mathtt{J})$-module $K_{\mathbf{L}}(\mathrm{Gr}_L)$ admits a direct sum decomposition whose associated graded pieces are parametrized by $\mathbb{X}_*^{\le}(\mathtt{J})$. The associated graded piece corresponding to β is isomorphic to $K_{\mathbf{L}}(\mathcal{B}^{\mathtt{J}}_{\mathtt{J}(\beta)^\#})$ and the correspondence is given by*

$$E^{\mathtt{J}}[\beta;\lambda] \mapsto D_{w_0^{\mathtt{J}}}(e^{w_0^{\mathtt{J}}\lambda} D_{w(\mathtt{J},\beta)}[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}(w_0^{\mathtt{J}})}]) \qquad \lambda \in \Lambda_+^{\mathtt{J}}(\beta).$$

In particular, the set $\{E^{\mathtt{J}}[\beta;\lambda]\}_{\beta\in\mathbb{X}_*^{\le}(\mathtt{J}),\lambda\in\Lambda_+^{\mathtt{J}}(\beta)}$ forms a $\mathbb{C}_q\mathbb{X}_0^*(\mathtt{J})$-basis of $K_{\mathbf{L}}(\mathrm{Gr}_{\mathbf{L}})$.

Proof. By definition, we have a $\mathbb{C}[\mathbf{H}]$-basis of $K_{\mathbf{H}}(\mathrm{Gr}_{\mathbf{L}})$ offered by $[\mathcal{O}_{\mathrm{Gr}_{\mathbf{L}}(wt_\beta)}]$ for $\beta \in \mathbb{X}_*^{\le}(\mathtt{J})$ and $w \in W^{\mathtt{J}}/W^{\mathtt{J}(\beta)}$. We have $K_{\mathbf{L}}(\mathrm{Gr}_{\mathbf{L}}) = D_{w_0^{\mathtt{J}}}(K_{\mathbf{H}}(\mathrm{Gr}_{\mathbf{L}}))$. By the Leibniz rule of D_i for each $i \in \mathtt{I}$ (Lemma 1.5), we conclude that the space of $D_{w_0^{\mathtt{J}}}$-invariants in $K_{\mathbf{H}}(\mathrm{Gr}_{\mathbf{L}})$ is the direct sum of the $D_{w_0^{\mathtt{J}}}$-invariants in

$$\bigoplus_{w\in W^{\mathtt{J}}/W^{\mathtt{J}(\beta)}} \mathbb{C}[\mathbf{H}][\mathcal{O}_{\mathrm{Gr}_{\mathbf{L}}(wt_\beta)}] \tag{5.3}$$

for all $\beta \in \mathbb{X}_*^{\le}(\mathtt{J})$. The space (5.3) is stable under the action of D_j ($j \in \mathtt{J}$) again by the Leibniz rule. In addition, it is generated from $[\mathcal{O}_{\mathrm{Gr}_L(u^{\mathtt{J}}_\beta)}]$, that is $D_{w(\mathtt{J},\beta)}$-invariant as $s_i\beta = \beta$ for $i \in \mathtt{J}(\beta)$. By Corollary 1.8 (and Theorem 1.6), we deduce that (5.3) is isomorphic to $K_{\mathbf{H}}(\mathcal{B}^{\mathtt{J}}_{\mathtt{J}(\beta)^\#})$ as $\mathcal{H}_q(\mathtt{J})$-module via the assignment

$$[\mathcal{O}_{\mathrm{Gr}_L(u^{\mathtt{J}}_\beta)}] \mapsto D_{w(\mathtt{J},\beta)}([\mathcal{O}_{\mathcal{B}^{\mathtt{J}}(w_0^{\mathtt{J}})}]).$$

This yields the desired correspondence between elements. Note that we have some $u \in W^{\mathtt{J}}$ such that $w_0^{\mathtt{J}} = uw(\mathtt{J},\beta)$ and $\ell(w_0^{\mathtt{J}}) = \ell(u) + \ell(w(\mathtt{J},\beta))$. It follows that

$$D_{w_0^{\mathtt{J}}}(e^{w_0^{\mathtt{J}}\lambda} D_{w(\mathtt{J},\beta)}[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}(w_0^{\mathtt{J}})}]) = D_u\left(D_{w(\mathtt{J},\beta)}(e^{w_0^{\mathtt{J}}\lambda} D_{w(\mathtt{J},\beta)}[\mathcal{O}_{\mathcal{B}^{\mathtt{J}}(w_0^{\mathtt{J}})}])\right),$$

represents a $\mathbf{L}$-equivariant vector bundle whose fiber is a $L^{\mathtt{J}(\beta)^\#}$-module with its character $D_{w(\mathtt{J},\beta)}(e^{w_0^{\mathtt{J}}\lambda})$. The latter is $\mathrm{ch}\, V^{\mathtt{J}(\beta)^\#}(w(\mathtt{J},\beta)w_0^{\mathtt{J}}\lambda)$ by the Weyl character formula. We have

$$K_{\mathbf{L}}(\mathcal{B}^{\mathtt{J}}_{\mathtt{J}(\beta)^\#}) \cong R(\mathbf{P}^{\mathtt{J}(\beta)^\#}) = R(\mathbf{L}^{\mathtt{J}(\beta)^\#}),$$

and the set of characters $\mathrm{ch}\, V^{\mathtt{J}(\beta)^\#}(w(\mathtt{J},\beta)w_0^{\mathtt{J}}\lambda)$ for $\lambda \in \Lambda_+^{\mathtt{J}}(\beta)$ is a $\mathbb{C}_q\mathbb{X}_0^*(\mathtt{J})$-basis of $R(\mathbf{L}^{\mathtt{J}(\beta)^\#})$. Therefore, we conclude that $\{E^{\mathtt{J}}[\beta;\lambda]\}_{\lambda\in\Lambda_+^{\mathtt{J}}(\beta)}$ is the $\mathbb{C}_q\mathbb{X}_0^*(\mathtt{J})$-basis of the $D_{w_0^{\mathtt{J}}}$-invariant part of (5.3). Since $K_{\mathbf{L}}(\mathrm{Gr}_{\mathbf{L}})$ is the direct sum of $D_{w_0^{\mathtt{J}}}$-invariant parts of (5.3), we conclude the result. ☐

We set $E^{\mathtt{J}}_{\mathrm{st}}[\gamma;\lambda] := E^{\mathtt{J}}[\gamma+\beta;\lambda] \odot_q \mathtt{t}_{-\beta}$ for $\lambda \in \Lambda^{\mathtt{J}}, \gamma \in \mathbb{X}_*, \beta, \beta+\gamma \in \mathbb{X}_*^-(\mathtt{J})$.

Corollary 5.5. *The element $E^{\mathtt{J}}_{\mathrm{st}}[\gamma;\lambda]$ does not depend on the choice (of β).*

Proof. The assertion follows from the fact that the right action of $\mathtt{t}_\beta$ commutes with the left action of D_i ($i \in \mathtt{J}$). ☐

By construction, we have $L \cong H'' \times [L,L]$ for a connected subtorus $H'' \subset H$. In particular, we have

$$L \cong H'' \times \prod_{k=1}^n L_k$$

where each L_k is a simply connected simple algebraic group. Let $Q_k^\vee \subset Q^\vee$ be the span of simple coroots corresponding to (co-)roots in L_k. We have

$$K_{\mathbf{L}}(\mathrm{Gr}_{\mathbf{L}}) \cong K_{\mathbf{H}''}(\mathrm{Gr}_{H''}) \otimes_{\mathbb{C}_q} \bigotimes_{k=1}^{n} K_{\mathbf{L}_k}(\mathrm{Gr}_{L_k}), \tag{5.4}$$

where the big tensor product is also taken over $\mathbb{C}_q$. On $K_{\mathbf{L}}(\mathrm{Gr}_{\mathbf{L}})$, we have the translation elements $\mathtt{t}_\beta$ for each $\beta \in \mathbb{X}_*$ obtained as the product of $\mathtt{t}_\gamma$'s that act on one of the tensor factors. This makes (5.4) into the isomorphism between their localized versions.

Using this, we consider the maps $\Psi_{\mathtt{J}}$ and $\Phi'_{\mathtt{J}}$ obtained from these of Theorem 3.1 and Theorem 2.17 by employing the following spaces:

$$K_{\mathbf{L}}(\mathbf{Q}_{\mathtt{J}}^{\mathrm{rat}}) := \bigotimes_{k=1}^{n} K_{\mathbf{L}_k}(\mathbf{Q}_{L_k}^{\mathrm{rat}}) \otimes K_{\mathbf{H}''}(\mathrm{Gr}_{H''}) \quad \text{and} \quad qK_{\mathbf{L}}(\mathscr{B}^{\mathtt{J}})_{\mathrm{loc}} \otimes K_{\mathbf{H}''}(\mathrm{Gr}_{H''}),$$

where all the tensor products are taken over $\mathbb{C}_q$, the $\Phi_{\mathtt{J}}$ is $K_{\mathbf{H}''}(\mathrm{Gr}_{H''})$-linear, and the map $\Psi'_{\mathtt{J}}$ is also $K_{\mathbf{H}''}(\mathrm{Gr}_{H''})$-linear, though the Novikov variables and line bundles (including the Heisenberg generators of $K_{\mathbf{H}''}(\mathrm{Gr}_{H''})$) are twisted by $-w_0$ from its naive definition. Note that the multiplication by $\mathtt{t}_\beta$ ($\beta \in \mathbb{X}_*$) corresponds to $Q^{-w_0\beta}$ only if $\beta \in Q_{\mathtt{J}}^\vee$, and the multiplication by Q^β for $\mathbb{X}_*$ is extended formally.

Lemma 5.6. *For $\beta \in \mathbb{X}_*$ and $\lambda \in \Lambda^{\mathtt{J}}$, we have*

$$E_{\mathrm{st}}^{\mathtt{J}}[\beta; \lambda] = \Phi_{\mathtt{J}}^{-1} \circ \Psi'_{\mathtt{J}}([\mathcal{O}_{\mathscr{B}^{\mathtt{J}}}(-w_0\lambda)]Q^{-w_0\beta}).$$

In particular, the set $\{E_{\mathrm{st}}^{\mathtt{J}}[\beta; \lambda]\}_{\beta \in \mathbb{X}_*, \lambda \in \Lambda^{\mathtt{J}}}$ is a $\mathbb{C}_q\mathbb{X}_0^*(\mathtt{J})$-basis of $K_{\mathbf{L}^{\mathtt{J}}}(\mathrm{Gr}_{\mathbf{L}^{\mathtt{J}}})_{\mathrm{loc}}$.

Proof. We have $[\mathcal{O}_{\mathscr{B}^{\mathtt{J}}}(\lambda)] = D_{w_0^{\mathtt{J}}}(e^{w_0^{\mathtt{J}}\lambda}[\mathcal{O}_{\mathscr{B}^{\mathtt{J}}(w_0^{\mathtt{J}})}]) \in K_{\mathbf{H}}(\mathscr{B}^{\mathtt{J}})$. In view of the correspondence between Schubert classes under the maps Ψ [26, Theorem 4.1 and its proof] and Φ [26, Proposition 2.13 and Remark 2.14], we deduce the first assertion. Taking into account of the first assertion and Theorem 3.1, the second assertion follows from Theorem 2.12 and Theorem 2.15. □

Lemma 5.7. *The embedding of Proposition 3.8 induces algebra embeddings*

$$K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} \hookrightarrow K_{\mathbf{L}}(\mathrm{Gr}_L)_{\mathrm{loc}} \hookrightarrow K_{\mathbf{H}}(\mathrm{Gr}_H).$$

Proof. In view of Corollary 3.10 and Proposition 3.8, we find that $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}$ and $K_{\mathbf{L}}(\mathrm{Gr}_L)_{\mathrm{loc}}$ are obtained by replacing the generator e^{ϖ_i} ($i \in \mathtt{I}$) in $K_{\mathbf{H}}(\mathrm{Gr}_H)$ to ξ_i for $i \in \mathtt{J}$ ($e^{-\varpi_i}$ and ϕ_i are the same for every $i \in \mathtt{I}$). The commutation relation in Proposition 3.8 implies $K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}} \subset K_{\mathbf{L}}(\mathrm{Gr}_L)_{\mathrm{loc}}$ inside $K_{\mathbf{H}}(\mathrm{Gr}_H)$. □

For $\lambda \in \Lambda$, we write $\lambda = \sum_{j \in \mathtt{I}} m_j \varpi_j$ for some $m_j \in \mathbb{Z}$. For each $\beta \in \mathbb{X}_*$, we define

$$[\mathcal{O}_\beta^\star(\lambda)] := \left(\prod_{j \in \mathtt{I}, m_j < 0} \phi_i^{-m_j} \right) \left(\prod_{j \in \mathtt{I}, m_j > 0} \xi_i^{m_j} \right) (\mathtt{t}_\beta) \in K_{\mathbf{G}}(\mathrm{Gr}_G)_{\mathrm{loc}}.$$

Similarly, for each $\lambda \in \Lambda$, we write $\lambda = \mu + \sum_{j \in \mathtt{J}} m_j \varpi_j$ for some $\mu \in \Lambda^{\mathtt{I} \setminus \mathtt{J}}$ and $m_j \in \mathbb{Z}$, and we define

$$[\mathcal{O}^{\star}_{\mathtt{J},\beta}(\lambda)] := e^{\mu} \left(\prod_{j \in \mathtt{J}, m_j < 0} \phi_i^{-m_j} \right) \left(\prod_{j \in \mathtt{J}, m_j > 0} \xi_i^{m_j} \right) (\mathtt{t}_\beta) \in K_{\mathbf{L}}(\mathrm{Gr}_L)_{\mathrm{loc}}.$$

Lemma 5.8. *For $\lambda \in \Lambda^{\mathtt{J}}$, we have*

$$[\mathcal{O}^{\star}_{\mathtt{J},0}(\lambda)] = E^{\mathtt{J}}_{\mathrm{st}}[0; \lambda] \mod (\mathtt{t}_{\alpha_j^\vee} \mid j \in \mathtt{J}).$$

Proof. In view of Theorem 1.1 and Theorem 3.1, the assertion follows from Theorem 2.9 4) and the definitions of ϕ_i's and ξ_i's. □

By the comparison of Lemma 5.4 and Lemma 5.8, we have a transition matrix (that is a finite sum in view of Corollary 3.10)

$$E^{\mathtt{J}}[\beta; \lambda] = \sum_{\gamma \in \mathbb{X}_*, \mu \in \Lambda^{\mathtt{J}}} a^{\gamma,\mu}_{\beta,\lambda}(\mathtt{J}) [\mathcal{O}^{\star}_{\mathtt{J},\gamma}(\mu)]$$

for $a^{\gamma,\mu}_{\beta,\lambda}(\mathtt{J}) \in \mathbb{C}_q \mathbb{X}^*_0(\mathtt{J})$. Moreover, we have:

Lemma 5.9. *We have $a^{\beta,\lambda}_{\beta,\lambda}(\mathtt{J}) = 1$, and*

$$a^{\gamma,\mu}_{\beta,\lambda}(\mathtt{J}) = 0 \quad \textit{for every} \quad \gamma \not\in \beta + Q^{\vee}_{\mathtt{J},+}.$$

Proof. The assertion follows by Lemma 5.8 and the fact that the effect of line bundle twists of $\mathbf{Q}_{L_k}$ raises the translation parts by $Q^{\vee}_{\mathtt{J},+}$. □

Proposition 5.10. *For each $\lambda \in \Lambda^{\mathtt{J}}$ and $\beta \in \mathbb{X}^-_*$, we have*

$$a^{\gamma,\mu}_{\beta,\lambda}(\mathtt{J}) = a^{\gamma,\mu}_{\beta,\lambda} \quad \gamma \in \beta + Q^{\vee}_{\mathtt{J},+}.$$

Proof. By assumption, we have $E[\beta; \lambda] = E_{\mathrm{st}}[\beta; \lambda]$ and $E^{\mathtt{J}}[\beta; \lambda] = E^{\mathtt{J}}_{\mathrm{st}}[\beta; \lambda]$. Thanks to Theorem 3.1 and Theorem 2.17, we transplant the problem to the quantum K-groups via $(\Psi'_{\mathtt{J}})^{-1} \circ \Phi_{\mathtt{J}}$. In view of Corollary 1.7, the assertion follows by Theorem 4.1 and Lemma 5.6. □

Proposition 5.11. *For each $\beta \in \mathbb{X}^{\leq}_*$ and $\lambda \in \Lambda_+(\beta)$, we have*

$$a^{\gamma,\mu}_{\beta,\lambda} = \sum_{\lambda'} c_{\lambda'} a^{\gamma,\mu}_{\beta,\lambda'}(\mathtt{J}) \quad \gamma \in \beta + Q^{\vee}_{\mathtt{J},+},$$

where $\lambda' \in \Lambda^{\mathtt{J}}_+(\beta)$ and $c_{\lambda'} \in \mathbb{C}_q \mathbb{X}^*_0(\mathtt{J})$.

Proof. We borrow the setting in the proof of Lemma 5.4. The element $E[\beta; \lambda]$ corresponds a G-equivariant vector bundle over $\mathscr{B}_{\mathtt{I}(\beta)\#}$ inflated from a $L^{\mathtt{I}(\beta)}$-module $V^{\mathtt{I}(\beta)}(\lambda)$, while the element $E^{\mathtt{J}}[\beta; \lambda']$ corresponding to a $L^{\mathtt{J}}$-equivariant vector bundle over $\mathscr{B}^{\mathtt{J}}_{\mathtt{J}(\beta)\#}$ inflated from a $L^{\mathtt{J}(\beta)}$-module $V^{\mathtt{J}(\beta)}(\lambda')$. These are parametrized by $\Lambda_+(\beta)$ and $\Lambda^{\mathtt{J}}_+(\beta)$, respectively. In particular, we have

$$V^{\mathtt{I}(\beta)}(\lambda) \cong \bigoplus_{\lambda' \in \Lambda^{\mathtt{J}}_+(\beta)} V^{\mathtt{J}(\beta)}(\lambda')^{\oplus c_{\lambda'}}, \tag{5.5}$$

where $c_{\lambda'} \in \mathbb{C}_q\mathbb{X}_0^*(\mathtt{J}) \subset \mathbb{C}_q\mathbb{X}^*$ is understood to be the multiplicity space that carries the information of character twists.

Consider the expansions

$$E^{\mathtt{J}}[\beta;\lambda] = \sum_{\mu} d_\mu^\lambda E_{\mathrm{st}}^{\mathtt{J}}[\beta;\mu] \;\; (\lambda \in \Lambda_+^{\mathtt{J}(\beta)}) \;\text{ and }\; E[\beta;\lambda] = \sum_{\mu} e_\mu^\lambda E_{\mathrm{st}}[\beta;\mu] \;\; (\lambda \in \Lambda_+^{\mathtt{I}(\beta)})$$

with $d_\mu^\lambda \in \mathbb{C}_q\mathbb{X}_0^*(\mathtt{J}), e_\mu^\lambda \in \mathbb{C}_q\mathbb{X}^*(G)$. These correspond to the expansions of the pullbacks of the class of vector bundles on $\mathscr{B}^{\mathtt{J}}_{\mathtt{J}(\beta)\#}$ and $\mathscr{B}_{\mathtt{I}(\beta)\#}$ to $\mathscr{B}^{\mathtt{J}}$ and $\mathscr{B}$ in terms of line bundles by Corollary 1.8, respectively. It respects the decomposition through the comparison given by Corollary 1.7, that sends $E_{\mathrm{st}}[\beta;\lambda]$ $(\lambda \in \Lambda)$ to $e^{\lambda-\lambda'}E_{\mathrm{st}}^{\mathtt{J}}[\beta;\lambda']$ for $\lambda' \in \Lambda^{\mathtt{J}}$ such that $\lambda - \lambda' \in \Lambda^{\mathtt{I}\setminus\mathtt{J}}$.

It follows that

$$d_\mu^\lambda = \sum_{\lambda'} c_{\lambda'} e_\mu^{\lambda'}.$$

Now the assertion follows by transplanting the problem to the quantum K-groups via $(\Psi'_{\mathtt{J}})^{-1} \circ \Phi_{\mathtt{J}}$ thanks to Proposition 4.1. $\Box$

5.2 Proof of Theorem 5.1

This subsection is totally devoted to the proof of Theorem 5.1. We consider elements of $K_{\mathbf{G}}(\mathrm{Gr}_G)$ and $K_{\mathbf{L}}(\mathrm{Gr}_L)$ as elements of $K_{\mathbf{H}}(\mathrm{Gr}_H)$ via Corollary 3.11. Since we have $\phi_i, \xi_i, \mathtt{t}_{\pm\alpha_i^\vee} \in K_{\mathbf{L}}(\mathrm{Gr}_L)$ for $i \notin \mathtt{J}$, we have

$$K_{\mathbf{G}}(\mathrm{Gr}_G) \subset K_{\mathbf{L}}(\mathrm{Gr}_L) \tag{5.6}$$

if and only if

$$K_{\mathbf{G}}(\mathrm{Gr}_G)[\phi_i, \xi_i, \mathtt{t}_{\pm\alpha_i^\vee} \mid i \notin \mathtt{J}] \subset K_{\mathbf{L}}(\mathrm{Gr}_L), \tag{5.7}$$

where the LHS exist as a subalgebra of $K_{\mathbf{H}}(\mathrm{Gr}_H)$. We consider the completions of the both sides of (5.7) using the variables $\{\mathtt{t}_\beta\}_{\beta\in\mathbb{X}_*}$ with respect to the direction $\langle \beta, \varpi_i \rangle \to \infty$ for $i \notin \mathtt{J}$. We denote the completion of the LHS of (5.7) by $\mathbf{K}_G^\wedge$ and the completion of the RHS of (5.7) by $\mathbf{K}_L^\wedge$. We have $(\sum_{k=0}^\infty \mathtt{t}_{k\alpha_i^\vee})\xi_i \in \mathbf{K}_G^\wedge$ for $i \notin \mathtt{J}$, that is an inverse of ϕ_i. We have (5.6) if and only if $\mathbf{K}_G^\wedge \subset \mathbf{K}_L^\wedge$.

For a collection $\vec{m} := \{m_i\}_{i\in(\mathtt{I}\setminus\mathtt{J})} \in \mathbb{Z}^{(\mathtt{I}\setminus\mathtt{J})}$, we set $\Lambda(\vec{m}) := \{\lambda \in \Lambda \mid \langle \alpha_i^\vee, \lambda \rangle = m_i, i \in (\mathtt{I} \setminus \mathtt{J})\}$. Assume that

$$\sum_{\lambda\in\Lambda,\beta\in\gamma+Q_+^\vee} c_{\lambda,\beta}[\mathcal{O}_\beta^\star(\lambda)] \in K_{\mathbf{G}}(\mathrm{Gr}_G) \qquad c_{\lambda,\beta} \in \mathbb{C}_q\mathbb{X}^*(G).$$

By taking the conjugations by $\mathtt{t}_{\alpha_i^\vee}$ for each $i \in (\mathtt{I} \setminus \mathtt{J})$ and separate out the eigenvectors, we conclude that

$$\sum_{\lambda\in\Lambda(\vec{m}),\beta\in\gamma+Q_+^\vee} c_{\lambda,\beta}[\mathcal{O}_\beta^\star(\lambda)] \in K_{\mathbf{G}}(\mathrm{Gr}_G)[\phi_i, \xi_i, \mathtt{t}_{\pm\alpha_i^\vee} \mid i \notin \mathtt{J}].$$

Inside $\mathbf{K}_G^\wedge$, we can take conjugation by ϕ_i for each $i \notin \mathtt{J}$. By examining their eigenvalues, we have

$$\sum_{\lambda\in\Lambda(\vec{m}),\beta\in\gamma+Q_{\mathtt{J},+}^\vee} c_{\lambda,\beta}[\mathcal{O}_\beta^\star(\lambda)] \in \mathbf{K}_G^\wedge.$$

Summing them up with respect to $\vec{m}$, we find that

$$\sum_{\lambda \in \Lambda, \beta \in \gamma + Q^\vee_{\mathtt{J},+}} c_{\lambda,\beta}[\mathcal{O}^\star_\beta(\lambda)] \in \mathbf{K}^\wedge_G.$$

Recall that we have $\mathbb{X}^{\leq}_* \subset \mathbb{X}^{\leq}_*(\mathtt{J})$ and $\Lambda_+(\beta) \subset \Lambda^{\mathtt{J}}_+(\beta) + \Lambda^{\mathtt{I} \setminus \mathtt{J}}$, and hence there is a natural inclusion between the (labels of the) $\mathbb{C}_q\mathbb{X}^*(G)$-basis

$$\{E(\beta, \lambda)\}_{\beta \in \mathbb{X}^{\leq}_*, \lambda \in \Lambda_+(\beta)} \subset K_{\mathbf{G}}(\mathrm{Gr}_G) \tag{5.8}$$

into the (labels of the) $\mathbb{C}_q\mathbb{X}^*(G)$-basis

$$\{E^{\mathtt{J}}(\beta, \lambda_1)e^{\lambda_2}\}_{\beta \in \mathbb{X}^{\leq}_*(\mathtt{J}), \lambda_1 \in \Lambda^{\mathtt{J}}_+(\beta), \lambda_2 \in \Lambda^{\mathtt{I} \setminus \mathtt{J}}} \subset K_{\mathbf{L}}(\mathrm{Gr}_L). \tag{5.9}$$

If a (formal) linear combination

$$\sum_{\lambda \in \Lambda, \beta \in \gamma + Q^\vee_+} c_{\lambda,\beta}[\mathcal{O}^\star_\beta(\lambda)] \quad c_{\lambda,\beta} \in \mathbb{C}_q\mathbb{X}^*(G) \tag{5.10}$$

belongs to $K_{\mathbf{G}}(\mathrm{Gr}_G)$, then it represents a $\mathbb{C}_q\mathbb{X}^*(G)$-linear combination of (5.8). In view of Proposition 5.11, the partial sum corresponding to $(\gamma + Q^\vee_{\mathtt{J},+}) \subset (\gamma + Q^\vee_+)$ yields the $\mathbb{C}_q\mathbb{X}^*(G)$-linear combination of (5.9) through $K_{\mathbf{H}}(\mathrm{Gr}_H)$. Therefore, (5.10) belongs to $K_{\mathbf{G}}(\mathrm{Gr}_G)$ only if

$$\sum_{\lambda \in \Lambda, \beta \in \gamma + Q^\vee_{\mathtt{J},+}} c_{\lambda,\beta}[\mathcal{O}^\star_{\mathtt{J},\beta}(\lambda)] \in K_{\mathbf{L}}(\mathrm{Gr}_L).$$

Since the corresponding leading term element belongs to $K_{\mathbf{G}}(\mathrm{Gr}_G) \subset \mathbf{K}^\wedge_G$ as a linear combination of (5.8) thanks to Lemma 5.4, we conclude that $\mathbf{K}^\wedge_G \subset \mathbf{K}^\wedge_L$ by removing the leading terms inductively. This forces $K_{\mathbf{G}}(\mathrm{Gr}_G) \subset K_{\mathbf{L}}(\mathrm{Gr}_L)$ as required. Thus, we conclude Theorem 5.1.

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Appendix A A quantum analogue of the induction equivalence

Let G be a connected reductive semi-simple group over $\mathbb{C}$, with a Borel subgroup B and a maximal torus H. Let $B \subset P \subset G$ be a parabolic subgroup. Let $Q^\vee_+$ denote the span of positive coroots (inside the coroot lattice of G) identified with the effective cone of G/B. Let $Q^\vee_{P,+} \subset Q^\vee_+$ be the span of positive coroots of G that does not belong to the standard Levi subgroup of P (cf. §1.1). Let $\{(\alpha^P_i)^\vee\}_i$ be the set of positive simple coroots in $Q^\vee_{P,+}$.

For a smooth projective variety $\mathfrak{X}$ over $\mathbb{C}$, we have a subset $H_2(\mathfrak{X})_+ \subset H_2(\mathfrak{X}, \mathbb{Z})$ of the effective classes (that is a submonoid). Let $\mathcal{M}_{g,n,\beta}(\mathfrak{X})$ be the moduli stack of genus g stable maps with n-marked points with degree $\beta \in H_2(\mathfrak{X})_+$ (see [2, 36]).

Theorem A.1. *Let X be a smooth projective algebraic variety over $\mathbb{C}$ equipped with the P-action. We assume $H_1(X, \mathbb{Z}) = \{0\}$. Then, we have a surjective map of algebras*

$$QK_G(G \times_P X) \longrightarrow QK_P(X),$$

where QK denotes the big quantum K-group defined in Lee [36].

Proof. Since X is projective with P-action, we can consider $X \subset \mathbb{P}(V)$ for a finite-dimensional P-module V. We can twist by P-character if necessary to assume that all the T-weights λ appearing in V satisfies $\langle (\alpha_i^P)^\vee, \lambda \rangle \geq 0$ for all i (with respect to the standard pairing, cf. §1.1). Then, we have an algebraic induction $V^\#$ of V, that is the maximal finite-dimensional G-module that is generated by V. We have $G \times_P X \subset \mathbb{P}(V^\#)$, and hence $G \times_P X$ is again projective. The variety $G \times_P X$ is evidently smooth as X is.

Since $H_1(X, \mathbb{Z}) = 0$, the Leray spectral sequence yields

$$H_2(G \times_P X, \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \oplus H_2(G/P, \mathbb{Z}).$$

The projection map yields

$$\pi : H_2(G \times_P X, \mathbb{Z})_+ \longrightarrow H_2(G/P, \mathbb{Z})_+ \cup \{0\},$$

and the preimage of 0 is $H_2(X, \mathbb{Z})_+$ by inspection. By the above identification of the effective classes, we find

$$\mathcal{M}_{g,n,\beta}(G \times_P X) \cong G \times_P \mathcal{M}_{g,n,\beta}(X) \tag{A.11}$$

whenever $\beta \in \pi^{-1}(0) \cong H_2(X, \mathbb{Z})_+$. In particular, we have an inflation map

$$\mathrm{infl} : K_P(\mathcal{M}_{g,n,\beta}(X)) \xrightarrow{\cong} K_G(G \times_P \mathcal{M}_{g,n,\beta}(X)).$$

By (A.11), the perfect obstruction theory of $G \times_P \mathcal{M}_{g,n,\beta}(X)$ ([36, §2.3 (3)]) can be taken as the inflation of that of $\mathcal{M}_{g,n,\beta}(X)$. It follows that

$$\mathrm{infl}([\mathcal{O}^{\mathrm{vir}}_{\mathcal{M}_{g,n,\beta}(X)}]) = [\mathcal{O}^{\mathrm{vir}}_{\mathcal{M}_{g,n,\beta}(G \times_P X)}].$$

Note that the quantum K-invariants of $\mathcal{M}_{g,n,\beta}(X)$ ([36, §4.2]) with respect to the classes from $K_P(X)$ are P-characters (corresponding to finite-dimensional virtual representations of P). If $\beta \in \pi^{-1}(0) \cong H_2(X, \mathbb{Z})_+$, then we find that the inflation isomorphisms $K_P(X) \cong K_G(G \times_P X)$ and $K_P(\mathcal{M}_{g,n,\beta}(X)) \cong K_G(\mathcal{M}_{g,n,\beta}(G \times_P X))$ send the P-equivariant Euler-Poincaré characteristic maps to G-equivariant Euler-Poincaré characteristic maps through the algebraic (virtual) induction of the P-characters to G-characters. In particular, the quantum K-potential ([36, (16)]) of $G \times_P X$ is the inflation of that of X (from P-characters to G-characters) modulo the Novikov monomial Q^β with $\pi(\beta) \neq 0$. This induces an algebra map

$$QK_G(G \times_P X)/(Q^\beta \mid \pi(\beta) \neq 0) \longrightarrow QK_P(X)$$

that is an isomorphism as being an isomorphism as vector spaces. □

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