Higher level BGG reciprocity for current algebras

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Abstract

We exhibit a higher-level analogue of the Bernstein-Gelfand-Gelfand (BGG) reciprocity for twisted current algebras for each positive integer, which recovers the original one (established by Bennett, Berenstein, Chari, Ion, Khoroshkin, Loktev, and Manning) as its level-one case.

This work brings theta functions and modular forms into the theory of symmetric polynomials. Furthermore, we establish branching properties for both versions of Demazure modules and provide a new interpretation of level-restricted generalized Kostka polynomials in terms of symmetric polynomials.

Introduction

An affine Kac-Moody algebra $\widetilde{\mathfrak{g}}$ over $\mathbb{C}$ has a non-negative part $\widetilde{\mathfrak{g}}_{\geq 0}$, which contains a simple Lie algebra $\mathfrak{g}$ whose Dynkin diagram is obtained by removing the zeroth node from the Dynkin diagram of $\widetilde{\mathfrak{g}}$ ([36]). Let P denote the set of weights of $\mathfrak{g}$, and let P^+ be the set of dominant weights of $\mathfrak{g}$. Chari-Pressley [11] defined the local and global Weyl modules of $\widetilde{\mathfrak{g}}_{\geq 0}$, denoted by $\mathbb{W}_\lambda$ and W_λ ($\lambda \in P^+$), respectively. Let V_λ ($\lambda \in P^+$) be the irreducible finite-dimensional $\mathfrak{g}$-module with its highest weight λ. Bennett, Berenstein, Chari, Ion, Khoroshkin, Loktev, and Manning [2, 1, 10] showed the following Ext-orthogonality:

$$\mathrm{Ext}^i(\mathbb{W}_\lambda, W_\mu^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } V_\lambda \cong V_\mu^* \\ 0 & \text{otherwise} \end{cases}, \qquad (0.1)$$

This relation can be understood as an analogue of the BGG reciprocity. The graded characters of $\mathbb{W}_\lambda$ and W_λ are proportional to each other, with the proportionality factor being an element of $\mathbb{Z}[\![q]\!]$. They are realized as specializations of a Macdonald polynomial ([47]) at $t = 0$. In addition, the numerical counterpart of (0.1) is precisely the orthogonality relation with respect to the Hall inner product.

Replacing the subalgebra $\widetilde{\mathfrak{g}}_{\geq 0}$ with the Iwahori subalgebra of $\widetilde{\mathfrak{g}}$ (i.e., the upper triangular part in the sense of Kac-Moody algebras) yields another class of modules, $\mathbb{D}_\lambda$ and D_λ, indexed by $\lambda \in P$. These modules possess properties

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analogous to (0.1), as established in [18, 15]. In addition, their graded characters are given by

$$\operatorname{gch} D_\lambda = E_\lambda(q, 0),$$
$$\operatorname{gch} \mathbb{D}_\lambda \in c_\lambda \cdot E_{-\lambda}^\dagger(q^{-1}, \infty),$$

where $E_\lambda(q, t)$ is a non-symmetric Macdonald polynomial ([13]), $E_\lambda^\dagger(q, t)$ is its character conjugate, and c_λ is an invertible element in $\mathbb{Z}[\![q]\!]$.

As pointed out by Sanderson [57] and Ion [32], the module D_λ can be obtained as the Demazure module of a level-one integrable representation of $\widetilde{\mathfrak{g}}$ whenever $\widetilde{\mathfrak{g}}$ is the twisted affinization of a simple Lie algebra. In fact, the module $\mathbb{D}_\lambda$ can also be understood as a version of a Demazure module in this case. Hence, the relation (0.1) can be viewed as an affine analogue of the duality between Demazure modules, which extends the classical result by van der Kallen [62] (cf. [50], [56]). This perspective suggests that it should be possible to generalize these results to an arbitrary Demazure module of an integrable highest weight module of a Kac-Moody algebra. However, the proof in the finite case does not straightforwardly generalize to the infinite-dimensional setting. The main reason is that there are two versions of Demazure modules for an arbitrary Kac-Moody algebra, and these two versions (essentially) coincide only for finite types. Outside of finite types, a proper formulation of (0.1) always involves both versions of Demazure modules: one finite-dimensional and the other infinite-dimensional.

The finite-dimensional Demazure modules, which we refer to as *thin Demazure modules*, are obtained by successive applications of Demazure functors starting from one-dimensional modules. (These are what are typically called "Demazure modules" in the literature; see, e.g., [45].) This fact is quite helpful in their analysis (see Remark 2.23). Infinite-dimensional Demazure modules, which we refer to as *thick Demazure modules*, are much more intractable. Unlike their thin counterparts, they are defined in terms of irreducible highest weight integrable representations and cannot be constructed from finite-dimensional objects via successive applications of Demazure functors. In fact, because Demazure functors lose information, there have been practically no effective means to analyze the precise module-theoretic structure of thick Demazure modules.

For the twisted affinization of a simple Lie algebra, the level-one case discussed above has connections to the theory of Macdonald polynomials. Moreover, this level-one framework is sufficient to derive the Rogers-Ramanujan identity and its generalizations ([14, 15]). In addition, their higher-level analogues have been investigated from a combinatorial viewpoint as level-restricted generalized Kostka polynomials, and some of their explicit descriptions are known ([28, 29, 58, 59, 44]). Thus, at least since the appearance of the foundational works [14, 10, 18, 15], extending the reciprocity in (0.1) to the higher-level case has been a highly desirable goal.

In this paper, we build upon the framework developed in [15, 43] to address the aforementioned difficulties. We begin by considering the category $\mathcal{C}$, which consists of $[\widetilde{\mathfrak{g}}_{\geq 0}, \widetilde{\mathfrak{g}}_{\geq 0}]$-modules that are semi-simple when viewed as $\mathfrak{g}$-modules. In this setting, we disregard the central charge and the grading.

Theorem A ($\doteq$ Theorems 5.1, 5.4, and 9.5). *For each $k \in \mathbb{Z}_{>0}$ and $\lambda \in P^+$, there exist two modules, $W_\lambda^{(k)}$ and $\mathbb{W}_\lambda^{(k)}$, satisfying the following properties:*

1. The modules $W_\lambda^{(k)}$ and $\mathbb{W}_\lambda^{(k)}$ are $\widetilde{\mathfrak{g}}_{\geq 0}$-modules whose heads are isomorphic to V_λ.

2. The module $W_\lambda^{(k)}$ is the thin Demazure module of an irreducible integrable highest weight module of $\widetilde{\mathfrak{g}}$ of level k.

3. The following Ext-orthogonality holds in the category $\mathcal{C}$:

$$\mathrm{Ext}_\mathcal{C}^i(\mathbb{W}_\lambda^{(k)}, (W_\mu^{(k)})^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } V_\lambda \cong V_\mu^* \\ 0 & \text{otherwise} \end{cases}. \qquad (0.2)$$

4. The module $W_\lambda^{(k)}$ has a filtration whose factors are of the form $W_\mu^{(k+1)}$ for $\mu \in P^+$, and the module $\mathbb{W}_\lambda^{(k+1)}$ has a filtration whose factors are of the form $\mathbb{W}_\mu^{(k)}$ for $\mu \in P^+$.

5. For each $\mu \in P^+$, the following multiplicity formulas hold:

$$(P_\lambda : \mathbb{W}_\mu^{(k)}) = [W_\mu^{(k)} : V_\lambda] \quad and \quad (\mathbb{W}_\lambda^{(k+1)} : \mathbb{W}_\mu^{(k)}) = (W_\mu^{(k)} : W_\lambda^{(k+1)}),$$

where $(M : N)$ denotes the multiplicity of N in a given filtration of M, and P_λ is the projective cover of V_λ in $\mathcal{C}$.

We also present the non-symmetric counterparts of Theorem A in §5.

We remark that the $k = 1$ case of (0.2) is (0.1), and their non-symmetric counterparts are contained in [15]. Theorem A asserts that $\mathcal{C}$ possesses a structure similar to that of highest weight categories [17, 7]. In fact, if we also incorporate the grading into the structure, then we deduce that $W_\lambda^{(k)}$ and $\left(\mathbb{W}_\lambda^{(k)}\right)^\vee$ can be naturally understood as standard/costandard modules in a certain graded category (Corollary 11.5 and Remark 11.6). From this perspective, our relation (0.2) exactly matches the Ext-vanishing expected from the framework of highest weight theory. Hence, Theorem A(3) may be regarded as the higher-level analogue of (0.1), and Theorem A(5) may be considered as the higher-level analogue of the BGG reciprocity ([10]).

Except for the $k = 1$ case and the first two items, the only previously known case of Theorem A is the existence of a filtration of $W_\lambda^{(k)}$ by $\{W_\mu^{(k+1)}\}_{\mu \in P^+}$ when $\mathfrak{g}$ is of type ADE (Joseph-Naoi [35, 51]; see also [12]). In particular, Theorem A is new even for $\mathfrak{g} = \mathfrak{sl}(2)$ in its full generality. Let us add that our proof here (and consequently our proof of Corollary B below) is independent of that of Joseph-Naoi.

Note that we can compute $(W_\mu^{(k)} : W_\lambda^{(k+1)})$ by counting highest weight elements in suitable tensor products of crystals (Joseph [34]). There is also another combinatorial formula for $\mathfrak{g} = \mathfrak{sl}(2)$ by Biswal-Kus [3]. In this sense, Theorem A yields the first effective means to analyze the structure of the infinite-dimensional Demazure modules studied in [41, 14, 43, 15] for an arbitrary level, since the description in [34, §3.3] *cannot* capture $(\mathbb{W}_\lambda^{(k+1)} : \mathbb{W}_\mu^{(k)})$ (as these modules do not contain highest weight vectors).

Let us point out that for a fixed $\lambda \in P^+$ and for $k \gg 0$ we have $W_\lambda^{(k)} = V_\lambda$, and $\mathbb{W}_\lambda^{(k)}$ is isomorphic to a level-$(1 - k)$ integrable lowest weight module of $\widetilde{\mathfrak{g}}$ (Lemma 10.1). Thus, Theorem A enlarges the scope of the theory of symmetric polynomials ([46]) to include theta functions and modular forms (Remark 10.2).

As an important application, our framework resolves a long-standing speculation, which generalizes a result from [35] for simply-laced types:

Theorem B ($\doteq$ Corollary 9.3). *Every finite-dimensional Demazure module of $\widetilde{\mathfrak{g}}$ admits a filtration whose successive quotients are isomorphic to Demazure modules of $\mathfrak{g}$ by restriction.*

As a further consequence of our analysis, we identify the level-k-restricted Kostka polynomials (in the sense of [44]) as a subfamily of the branching polynomials arising from Theorem A (Corollary 10.4 and Remark 10.6). It follows that the graded characters of our modules provide a clear interpretation of level-restricted generalized Kostka polynomials in the language of symmetric polynomials:

Corollary C ($\doteq$ Corollary 10.5 and Remark 10.6). *Let $k \in \mathbb{Z}_{>0}$. Let B be a tensor product of Kirillov-Reshetikhin crystals of level at most k (see [30, §3.3]), and suppose that its character is given by*

$$\sum_{\lambda \in P^+} X_\lambda(q) \cdot \operatorname{ch} V_\lambda, \quad where \ X_\lambda(q) \in \mathbb{Z}[q].$$

We expand this character in terms of the graded characters of $W_\lambda^{(k+1)}$ as follows:

$$\sum_{\lambda \in P^+} X_\lambda(q) \cdot \operatorname{ch} V_\lambda = \sum_{\lambda \in P^+} X_\lambda^{(k)}(q) \cdot \operatorname{gch} W_\lambda^{(k+1)}, \quad where \ X_\lambda^{(k)}(q) \in \mathbb{Z}[q].$$

Here, gch denotes the graded character, which reduces to the ordinary character ch by setting $q = 1$. Then, the level-k-restricted generalized Kostka polynomials of B are precisely the coefficients $X_\lambda^{(k)}(q)$ for all $\lambda \in P^+$ that can be understood as level-k integrable highest weights.

The level shift observed previously may be explained by the following branching formula:

Corollary D ($\doteq$ Corollary 10.7). *Let $k \in \mathbb{Z}_{>0}$. For each $\lambda, \mu \in P^+$ such that λ can be understood as a level-k integrable highest weight, we have*

$$(W_\mu^{(k)} : W_\lambda^{(k+1)}) = \begin{cases} 1 & if \ \mu \uparrow_k \lambda, \\ 0 & otherwise, \end{cases}$$

where $\mu \uparrow_k \lambda$ means that μ is an extremal weight of the level-k integrable highest weight module of $\widetilde{\mathfrak{g}}$ whose highest weight is λ.

Example E. Assume that $\mathfrak{g} = \mathfrak{sl}(2)$, whose fundamental weight is denoted by ϖ. The first few branching rules are:

$$\operatorname{gch} W_{2\varpi}^{(1)} = \operatorname{gch} W_{2\varpi}^{(2)} + q \operatorname{gch} W_0^{(2)}, \quad \operatorname{gch} W_{3\varpi}^{(1)} = \operatorname{gch} W_{3\varpi}^{(2)} + q^2 \operatorname{gch} W_\varpi^{(2)},$$

$$\operatorname{gch} W_{3\varpi}^{(2)} = \operatorname{gch} W_{3\varpi}^{(3)} + q \operatorname{gch} W_\varpi^{(3)}, \quad \operatorname{gch} W_{4\varpi}^{(2)} = \operatorname{gch} W_{4\varpi}^{(3)} + q^2 \operatorname{gch} W_0^{(3)},$$

$$\operatorname{gch} W_{4\varpi}^{(3)} = \operatorname{gch} W_{4\varpi}^{(4)} + q \operatorname{gch} W_{2\varpi}^{(4)}, \qquad \cdots .$$

By iterating these rules, we obtain expansions such as

$$\operatorname{gch} W_{4\varpi}^{(1)} = \operatorname{gch} W_{4\varpi}^{(2)} + (q^2 + q^3) \operatorname{gch} W_{2\varpi}^{(2)} + q^4 \operatorname{gch} W_0^{(2)},$$

$$\operatorname{gch} W_{6\varpi}^{(1)} = \operatorname{gch} W_{6\varpi}^{(2)} + (q^3 + q^4 + q^5) \operatorname{gch} W_{4\varpi}^{(2)} + (q^6 + q^7 + q^8) \operatorname{gch} W_{2\varpi}^{(2)} + q^9 \operatorname{gch} W_0^{(2)}.$$

The appearance of the highest degree terms q^4 and q^9 in these expressions implies that the module $W_0^{(2)}$ appears in the filtrations of $W_{4\varpi}^{(1)}$ and $W_{6\varpi}^{(1)}$ only at their socles. Item (5) of Theorem A then asserts the dual relations:

$$\operatorname{gch} \mathbb{W}_0^{(2)} = \operatorname{gch} \mathbb{W}_0^{(1)} + q \operatorname{gch} \mathbb{W}_{2\varpi}^{(1)} + q^4 \operatorname{gch} \mathbb{W}_{4\varpi}^{(1)} + q^9 \operatorname{gch} \mathbb{W}_{6\varpi}^{(1)} + \cdots,$$
$$\operatorname{gch} \mathbb{W}_{2\varpi}^{(2)} = \operatorname{gch} \mathbb{W}_{2\varpi}^{(1)} + (q^2 + q^3) \operatorname{gch} \mathbb{W}_{4\varpi}^{(1)} + (q^6 + q^7 + q^8) \operatorname{gch} \mathbb{W}_{6\varpi}^{(1)} + \cdots.$$

An overview of our proof of Theorem A.

Since the central achievement in this paper is Theorem A, we discuss the structure of its proof in more detail. The base of the induction, the $k = 1$ case, was established for $\mathfrak{g}$ of certain types in [18], and for general $\mathfrak{g}$ in [15]. The main argument proceeds by induction on k, relying on the following two statements, which form the technical core of Theorem A:

($\clubsuit$)$_k$ The modules $W_\lambda^{(k)}$ and $\mathbb{W}_\mu^{(k)}$ and their variants satisfy the Ext-orthogonality condition (0.2).

($\spadesuit$)$_k$ An Ext-criterion for a module to admit a filtration by $\{W_\lambda^{(k)}\}_{\lambda \in P^+}$ or $\{\mathbb{W}_\mu^{(k)}\}_{\mu \in P^+}$, and their variants.

The precise statement of ($\clubsuit$)$_k$ is given by Theorem 5.1 at level k, and that of ($\spadesuit$)$_k$ is given by Theorem 5.4 at level k. The key idea behind the implication ($\clubsuit$)$_k \Rightarrow$ ($\spadesuit$)$_k$ is a rather formal observation, stated as Proposition 5.2. It therefore remains to prove ($\clubsuit$)$_k$ using the induction hypothesis ($\clubsuit$)$_{k-1}$ and ($\spadesuit$)$_{k-1}$.

This is achieved in several steps, and the crucial ingredients of the proof include the following:

Reduction. A structural property of special Demazure modules that holds particularly in the setting of twisted affinizations (Proposition 3.3).

Lifting. Morphisms of certain types between modules admit a lifting to a morphism from a larger module obtained by applying a Demazure functor (Theorem 4.1 and Corollary 4.3).

Vanishing. Ext^1-vanishing between modules arising from level k and level $(k-1)$ (Theorem 6.4).

Here, *Reduction* is a property that fails to hold for a general affine Lie algebra $\widetilde{\mathfrak{g}}$. Consequently, if one could find a proper replacement for this property, it might be possible to generalize our results to the remaining types of affine Lie algebras. The proof of *Lifting* consists of a careful examination of so-called "string modules", borrowed from [33]. The proof of *Reduction* is a case-by-case analysis using the structure of affine root systems. The proof of *Vanishing* requires a characterization of (global) Weyl modules ([15, 42]), a rough estimate of the size of $W_\lambda^{(k)}$ (Lemma 2.22), and the adjoint property of Demazure functors ([18, 50]), in addition to the *Lifting* and *Reduction*. The proofs of *Reduction* and *Lifting* do not, by themselves, require the induction hypotheses ($\clubsuit$) or ($\spadesuit$).

With these three kinds of auxiliary results, we obtain a filtration of $\mathbb{W}_\lambda^{(k)}$ by modules of the form $\{\mathbb{W}_\mu^{(k-1)}\}_\mu$ (Proposition 7.1) by assuming ($\spadesuit$)$_{k-1}$. This,

in turn, leads to a necessary vanishing theorem (Theorem 7.2) to carry out the proof of $(\clubsuit)_k$.

Organization of the paper.

The remainder of the paper is organized as follows. First, in Sections 1 and 2, we provide the necessary background material, including the notation used throughout the paper and foundational module-theoretic results.

Sections 3, 4 establish two of the three crucial ingredients mentioned in the overview. Section 3 establishes a structural property of special Demazure modules, namely the *Reduction* property. Section 4 proves our key theorem on the lifting of morphisms, referred to as the *Lifting* property.

Section 5 provides the precise formulation of our main theorems (Theorems 5.1 and 5.4) which includes a refined version of Theorem A(3).

Sections 6, 7 are in the induction loop of the proofs of our main results. Section 6 utilizes the results on Lifting and Reduction to derive the key Ext^1-vanishing theorems, which is the *Vanishing* property. Section 7 then combines the induction hypothesis with the Vanishing results from Section 6 to establish the crucial filtration properties of our modules, a decisive step for advancing the induction.

Finally, Section 8 assembles all these ingredients to complete the inductive argument and finalize the proofs of the main theorems.

The subsequent sections are devoted to applications. Section 9 proves the branching rules (Theorem A(4)(5)); Section 10 connects our results with level-restricted Kostka polynomials; and Section 11 relates our modules to standard and costandard modules in the sense of highest weight theory.

Appendix A presents several numerical examples to help illustrate the theory developed in the paper.

We end this introduction with a heuristic remark. While we have expressed our results by using ungraded Homs and Exts in Introduction, we mainly employ the graded ones, denoted by homs and exts (§1.3.3). They are isomorphic as vector spaces in the arguments of this paper, but we prefer to keep the gradings in mind, since the latter retain crucial information.

1   Preliminaries

We work over the field $\mathbb{C}$ of complex numbers. In this paper, a graded vector space V will always mean a $\mathbb{Z}$-graded vector space, whose degree-m component is denoted by V_m. For a graded vector space, we set

$$\mathrm{gdim}\, V := \sum_{m \in \mathbb{Z}} q^m \dim V_m.$$

The general references for this section are Kac [36] and Kumar [45].

1.1   Lie algebras and their root systems

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, with a fixed Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. We set $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$. We have the root space decompositions:

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \quad \text{and} \quad \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n},$$

where $\Delta^+ \subset \mathfrak{h}^*$ is the set of positive roots, $\Delta := \Delta^+ \sqcup (-\Delta^+)$, and $\mathfrak{g}_\alpha$ is the root space for a root $\alpha \in \Delta$. Let r be the lacing number of $\mathfrak{g}$, defined as the ratio of the squared length of a long root to that of a short root. Namely, $r = 1$ for types $\mathsf{A}, \mathsf{D}, \mathsf{E}$; $r = 2$ for types $\mathsf{B}, \mathsf{C}, \mathsf{F}$; and $r = 3$ for type G_2.

1.1.1 Finite root systems and weights

Let $\mathtt{I}$ be the set of vertices of the Dynkin diagram of $\mathfrak{g}$. Let $\Pi = \{\alpha_i\}_{i \in \mathtt{I}} \subset \Delta^+$ be the set of simple roots, and let $\vartheta \in \Delta^+$ be the highest short root. We set $\Delta_s := W\vartheta \subset \Delta$, and $\Delta_l := \Delta \setminus \Delta_s$. For each $\alpha \in \Delta$, we have its coroot $\alpha^\vee \in \mathfrak{h}$. We set $\Pi^\vee := \{\alpha_i^\vee\}_{i \in \mathtt{I}}$. The Weyl group W of $\mathfrak{g}$ is generated by the simple reflections $\{s_i\}_{i \in \mathtt{I}}$, where $s_i := s_{\alpha_i}$. The length function with respect to these generators is denoted by ℓ. Let $w_0 \in W$ be the longest element. We have $-w_0\vartheta = \vartheta$, as ϑ is the lowest non-zero dominant weight.

Let $Q := \sum_{i \in \mathtt{I}} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ be the root lattice, and let $Q_+ := \sum_{i \in \mathtt{I}} \mathbb{Z}_{\geq 0}\alpha_i \subset Q$ be its positive submonoid. We set $Q^\vee := \sum_{i \in \mathtt{I}} \mathbb{Z}\alpha_i^\vee \subset \mathfrak{h}$ to be the coroot lattice. The set of integral weights P and the set of dominant integral weights P^+ are defined as follows:

$$P := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i \in \mathtt{I}\},$$
$$P^+ := \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i \in \mathtt{I}\}.$$

Let $\{\varpi_i\}_{i \in \mathtt{I}} \subset P^+$ be the set of fundamental weights of $\mathfrak{g}$, defined by the condition $\langle \varpi_j, \alpha_i^\vee \rangle = \delta_{ij}$ for all $i, j \in \mathtt{I}$. Each $\lambda \in P^+$ defines an irreducible finite-dimensional $\mathfrak{g}$-module V_λ with a unique (up to a scalar multiple) highest-weight vector $\mathbf{v}_\lambda$ that is $\mathfrak{n}$-invariant and has $\mathfrak{h}$-eigenvalue λ.

1.1.2 Twisted Affine Root Systems and Their Weyl Groups

The twisted affine root system associated with $\mathfrak{g}$ is given by

$$\Delta_{\mathrm{af}} := (\Delta_s \times \mathbb{Z}\delta) \sqcup (\Delta_l \times r\mathbb{Z}\delta)$$

We consider the standard set of positive roots $\Delta_{\mathrm{af}}^+ \subset \Delta_{\mathrm{af}}$, which contains the set of finite positive roots Δ^+. We define the simple affine roots by $\Pi_{\mathrm{af}} := \Pi \cup \{\alpha_0\}$, where $\alpha_0 := -\vartheta + \delta$, and index them by $\mathtt{I}_{\mathrm{af}} := \mathtt{I} \cup \{0\}$.

The affine Weyl group is the semidirect product $W_{\mathrm{af}} := W \ltimes Q$. It is a reflection group generated by $\{s_i \mid i \in \mathtt{I}_{\mathrm{af}}\}$, where s_0 is the reflection with respect to α_0. This equips W_{af} with a length function $\ell: W_{\mathrm{af}} \to \mathbb{Z}_{\geq 0}$ that extends the one on W. For an element $w \in W_{\mathrm{af}}$, a sequence of indices $i_1, \dots, i_k \in \mathtt{I}_{\mathrm{af}}$ is a *reduced expression* for w if $w = s_{i_1} \cdots s_{i_k}$ and $k = \ell(w)$.

The Bruhat order on W_{af} is defined by setting $w \leq v$ if a reduced expression for w can be obtained as a subexpression of a reduced expression for v. The embedding $Q \hookrightarrow W_{\mathrm{af}}$ associates each $\gamma \in Q$ with a translation element t_γ. A key relation involving these elements is $t_{-\vartheta} = s_\vartheta s_0$.

1.2 Twisted affine Lie algebras

Let $\widetilde{\mathfrak{g}}$ be the twisted affinization of $\mathfrak{g}$. This is the affine Kac-Moody algebra associated with the root system Δ_{af}. The types of algebras under consideration

are given in [36] and include:

$$\mathsf{A}_\ell^{(1)}, \mathsf{D}_\ell^{(1)}, \mathsf{E}_6^{(1)}, \mathsf{E}_7^{(1)}, \mathsf{E}_8^{(1)} \quad \text{and} \quad \mathsf{A}_{2\ell}^{(2)}, \mathsf{D}_{\ell+1}^{(2)}, \mathsf{E}_6^{(2)}, \mathsf{D}_4^{(3)}.$$

The set $\mathtt{I}$ of the vertices of the Dynkin diagram of the finite-dimensional algebra $\mathfrak{g}$ is obtained from the set $\mathtt{I}_{\mathrm{af}}$ of the vertices of the Dynkin diagram of $\widetilde{\mathfrak{g}}$ by removing the node 0 [36, Chap. 4, Table Aff].

1.2.1 Triangular decomposition

We have the triangular decomposition:

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{n}}^- \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}},$$

which contains the decomposition of the finite part, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, component-wise. The components are given by

$$\widetilde{\mathfrak{n}} = \bigoplus_{\alpha \in \Delta_{\mathrm{af}}^+} \widetilde{\mathfrak{g}}_\alpha, \quad \widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d, \quad \text{and} \quad \widetilde{\mathfrak{n}}^- = \bigoplus_{\alpha \in \Delta_{\mathrm{af}}^+} \widetilde{\mathfrak{g}}_{-\alpha}.$$

Here, K is the central element ($[K, \widetilde{\mathfrak{g}}] = 0$), d is the derivation, and the enlarged Cartan subalgebra $\widetilde{\mathfrak{h}}$ is abelian. The root space $\widetilde{\mathfrak{g}}_\alpha$ is the eigenspace for the adjoint action of $\widetilde{\mathfrak{h}}$, where the roots $\alpha \in \Delta_{\mathrm{af}}$ are viewed as functionals in $\widetilde{\mathfrak{h}}^*$ via the inclusion $\Delta_{\mathrm{af}} \subset (\mathfrak{h} \oplus \mathbb{C}d)^* \subset \widetilde{\mathfrak{h}}^*$.

We extend the definition of simple coroots from the finite case by setting $\alpha_0^\vee = K - \vartheta^\vee$. We also define the derived subalgebra $\widehat{\mathfrak{g}} := [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ and the Borel subalgebras $\widetilde{\mathfrak{b}} := \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}$ and $\widetilde{\mathfrak{b}}^- := \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}}^-$.

Finally, let $\widetilde{\mathfrak{g}}_{\geq 0}$ and $\widetilde{\mathfrak{g}}_{\leq 0}$ denote the direct sums of the non-negative and non-positive d-eigenspaces of $\widetilde{\mathfrak{g}}$, respectively. These are Lie subalgebras of $\widetilde{\mathfrak{g}}$.

1.2.2 Simple roots and fundamental weights

For each $i \in \mathtt{I}_{\mathrm{af}}$, let $E_i \in \widetilde{\mathfrak{g}}_{\alpha_i}$ and $F_i \in \widetilde{\mathfrak{g}}_{-\alpha_i}$ be the standard Kac-Moody generators, satisfying $[E_i, F_i] = \alpha_i^\vee \in \widetilde{\mathfrak{h}}$ (see [36, §1.2]). The Chevalley involution θ on $\widetilde{\mathfrak{g}}$ is an automorphism that satisfies

$$\theta(E_i) = F_i, \quad \theta(F_i) = E_i \quad (i \in \mathtt{I}_{\mathrm{af}}) \quad \text{and} \quad \theta(d) = -d.$$

Let $\{\Lambda_i\}_{i \in \mathtt{I}_{\mathrm{af}}} \subset \widetilde{\mathfrak{h}}^*$ be the set of fundamental weights of $\widetilde{\mathfrak{g}}$, and let $\rho \in \widetilde{\mathfrak{h}}^*$ be the Weyl vector. These are defined for all $i, j \in \mathtt{I}_{\mathrm{af}}$ by the conditions:

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \langle \rho, \alpha_i^\vee \rangle = 1, \quad \text{and} \quad \langle \Lambda_i, d \rangle = \langle \rho, d \rangle = 0.$$

We set $P_{\mathrm{af}} := \bigoplus_{i \in \mathtt{I}_{\mathrm{af}}} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ and $P_{\mathrm{af}}^+ := \bigoplus_{i \in \mathtt{I}_{\mathrm{af}}} \mathbb{Z}_{\geq 0}\Lambda_i \oplus \mathbb{Z}\delta$. The inclusion $\mathfrak{h} \hookrightarrow \widetilde{\mathfrak{h}}$ induces a classical projection map $\lambda \mapsto \overline{\lambda}$ from P_{af} onto P, which satisfies

$$\overline{\Lambda}_i = \varpi_i \quad (i \in \mathtt{I}) \quad \text{and} \quad \overline{\Lambda}_0 = \overline{\delta} = 0.$$

Conversely, we define an inclusion $P \hookrightarrow P_{\mathrm{af}}$ by extending the map $\varpi_i \mapsto \Lambda_i - \langle \varpi_i, \vartheta^\vee \rangle \Lambda_0$ for $i \in \mathtt{I}$ linearly. This map is a section of the classical projection.

The affine Weyl group W_{af} acts on P_{af} such that each simple reflection s_i ($i \in \mathtt{I}_{\mathrm{af}}$) acts as the reflection with respect to the coroot $\alpha_i^\vee$.

1.2.3 Affine weights of level k

For each $k \in \mathbb{Z}$, we define the set of *level k weights* as

$$P_{\mathrm{af},k} := \{\Lambda \in P_{\mathrm{af}} \mid \langle \Lambda, K \rangle = k\}.$$

The set of *level k dominant weights*, denoted $P_{\mathrm{af},k}^+$, consists of weights $\Lambda \in P_{\mathrm{af},k}$ that also satisfy $\langle \Lambda, \alpha_i^\vee \rangle \geq 0$ for all $i \in \mathtt{I}_{\mathrm{af}}$.

For a positive level $k \in \mathbb{Z}_{>0}$, the set $P_{\mathrm{af},k}^+$ is finite up to translation by integer multiples of δ. For each dominant weight $\Lambda \in P_{\mathrm{af},k}^+$, there exists a unique (up to isomorphism) integrable highest-weight representation $L(\Lambda)$ of level k. This module contains a highest-weight vector $\mathbf{v}_\Lambda$, and for any $w \in W_{\mathrm{af}}$, the $w\Lambda$-weight space of $L(\Lambda)$ is one-dimensional. We denote a vector spanning this space by $\mathbf{v}_{w\Lambda}$, which is unique up to a scalar multiple.

Since $P_{\mathrm{af},k}^+$ is a fundamental domain for the action of W_{af} on $P_{\mathrm{af},k}$ [36, Corollary 10.1], any weight in $P_{\mathrm{af},k}$ is conjugate to a unique weight in $P_{\mathrm{af},k}^+$. In particular, for any finite weight $\lambda \in P$, the corresponding level-k weight $\lambda + k\Lambda_0$ satisfies

$$\lambda + k\Lambda_0 = w\Lambda \quad \text{for a unique } \Lambda \in P_{\mathrm{af},k}^+ \text{ and some } w \in W_{\mathrm{af}}. \tag{1.1}$$

1.2.4 Definition of the modules $D_\lambda^{(k)}$ and $\widehat{\mathbb{D}}_\lambda^{(k)}$

Using the correspondence from (1.1), we define the *level k thin Demazure module* as

$$D_\lambda^{(k)} := U(\widetilde{\mathfrak{b}})\mathbf{v}_{w\Lambda} \subset L(\Lambda).$$

We remark that $D_\lambda^{(k)}$ is finite-dimensional, since its character is obtained by finitely many applications of Demazure operators [45, Proposition 8.1.17]. Similarly, we define the *level k thick Demazure module* as

$$L(\Lambda)^w := U(\widetilde{\mathfrak{b}}^-)\mathbf{v}_{w\Lambda} \subset L(\Lambda).$$

We then twist the $\widetilde{\mathfrak{b}}^-$-module structure on $L(\Lambda)^w$ by the Chevalley involution θ to obtain a $\widetilde{\mathfrak{b}}$-module, which we denote by $\widehat{\mathbb{D}}_{-\lambda}^{(k)}$. When convenient, we use the notation $^\theta M$ for a $\widetilde{\mathfrak{g}}$-module M with the twisted action; thus, $\widehat{\mathbb{D}}_{-\lambda}^{(k)}$ is the module $^\theta(L(\Lambda)^w)$, that we frequently refer as $^\theta L(\Lambda)^w$.

Both modules $D_\lambda^{(k)}$ and $\widehat{\mathbb{D}}_\lambda^{(k)}$, when regarded as $\widetilde{\mathfrak{b}}$-modules, are $\widetilde{\mathfrak{h}}$-semisimple. Each is generated by a cyclic vector whose $\mathfrak{h}$-weight is λ.

As a $\widetilde{\mathfrak{g}}$-module, $^\theta L(\Lambda)$ has level $-k$. Consequently, $\widehat{\mathbb{D}}_{-\lambda}^{(k)}$ can be viewed as a submodule of the irreducible lowest-weight integrable representation of $\widetilde{\mathfrak{g}}$ with lowest weight $-\Lambda$.

1.2.5 Order Relations on P and W_{af}

Definition 1.1. For each $\lambda \in P$, let λ_+ and λ_- denote the unique elements in $W\lambda \cap P^+$ and $W\lambda \cap (-P^+)$, respectively. Let $\preceq$ be the partial order on P defined by the following conditions:

$$\begin{aligned}
\lambda \preceq \mu \quad \Longleftrightarrow \quad & (\mu_+ - \lambda_+ \in Q_+ \text{ and } \lambda_+ \neq \mu_+), \quad \text{or} \\
& (\lambda_+ = \mu_+ \text{ and } \mu - \lambda \in Q_+).
\end{aligned}$$

We write $\lambda \prec \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$. We set

$$\Sigma(\lambda) := \{\mu \in P \mid \mu \preceq \lambda\} \quad \text{and} \quad \Sigma_*(\lambda) := \{\mu \in P \mid \mu \prec \lambda\}.$$

The partial order $\preceq$ equips P_{af} with a preorder induced by the projection map $\lambda \mapsto \overline{\lambda}$. We define the shifted action of W_{af} on $\mathbb{R} \otimes_{\mathbb{Z}} P$ as

$$w(\!(\lambda)\!) := \overline{w(\lambda + \Lambda_0) - \Lambda_0} \quad \text{for} \quad \lambda \in \mathbb{R} \otimes_{\mathbb{Z}} P, w \in W_{\mathrm{af}}.$$

Note that we have the relation $\lambda_+ = w_0 \lambda_-$.

Remark 1.2. The order $\preceq$ originates in the work of Heckman [31], and is also recorded in Macdonald's Bourbaki lecture [47, §4.10].

Lemma 1.3 ([15] §2.2.1). *For each $\lambda \in P$, we have*

$$\Sigma(\lambda) = (\mathrm{Conv}\, \Sigma(\lambda)) \cap (\lambda + Q),$$

where Conv *denotes the convex hull in* $\mathbb{R} \otimes_{\mathbb{Z}} P$. $\qquad\qquad\square$

Proposition 1.4 ([15] §4.1.1). *Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_{\mathrm{af},k}$ and $w, v \in W_{\mathrm{af}}$, if $w \leq v$ in the Bruhat order, then $\overline{w\Lambda} \preceq \overline{v\Lambda}$.* $\qquad\square$

1.2.6 Definition of the module $\mathbb{D}^{(k)}_{\lambda}$

Theorem 1.5 ([39]; see also [43, Theorem C]). *Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_{\mathrm{af},k}$ and $w, v \in W_{\mathrm{af}}$ such that $w \leq v$, we have an inclusion of $\widetilde{\mathfrak{b}}^-$-modules $L(\Lambda)^w \supset L(\Lambda)^v$. This inclusion is an equality if and only if $w\Lambda = v\Lambda$.* $\qquad\square$

Corollary 1.6 ([43, Theorem C]; cf. [39]). *Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_{\mathrm{af},k}$ and $w, v, \in W_{\mathrm{af}}$, there exists a subset $S(w,v) \subset W_{\mathrm{af}}$ such that*

$$L(\Lambda)^w \cap L(\Lambda)^v = \sum_{u \in S(w,v)} L(\Lambda)^u \subset L(\Lambda). \qquad\qquad\square$$

Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_{\mathrm{af},k}$ and $w \in W_{\mathrm{af}}$, we define

$$\mathrm{gr}^w L(\Lambda) := \frac{L(\Lambda)^w}{\sum_{v > w,\, w\Lambda \neq v\Lambda} L(\Lambda)^v}.$$

Assuming the correspondence $\lambda + k\Lambda_0 = w\Lambda$ from (1.1), we denote by $\mathbb{D}^{(k)}_{-\lambda}$ the $\widetilde{\mathfrak{b}}$-module obtained by twisting the action on $\mathrm{gr}^w L(\Lambda)$ by the Chevalley involution. Since $\mathrm{gr}^w L(\Lambda)$ is a quotient of $L(\Lambda)^w$ as a $\widetilde{\mathfrak{b}}^-$-module, it follows that $\mathbb{D}^{(k)}_{-\lambda}$ is a quotient of $\widehat{\mathbb{D}}^{(k)}_{-\lambda}$ as a $\widetilde{\mathfrak{b}}$-module. In particular, $\mathbb{D}^{(k)}_{-\lambda}$ is a cyclic $\widetilde{\mathfrak{b}}$-module generated by a vector of $\mathfrak{h}$-weight $-\lambda$.

Corollary 1.7. *Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P^+_{\mathrm{af},k}$ and $w \in W_{\mathrm{af}}$, the module $^{\theta}L(\Lambda)^w$ admits a decreasing, separable filtration whose associated graded components are direct sums of modules from the set $\{\mathbb{D}^{(k)}_{\mu} \mid \mu \in P\}$.*

Proof. Applying the involution θ, the assertion is equivalent to the claim that $L(\Lambda)^w$ has a decreasing, separable filtration whose successive quotients are isomorphic to modules of the form $\mathrm{gr}^v L(\Lambda)$. This is a direct consequence of Theorem 1.5 and Corollary 1.6. $\qquad\square$

1.3 Categories of representations

We set $Q_{\mathrm{af}}^+ := \sum_{\alpha \in \Delta_{\mathrm{af}}^+} \mathbb{Z}_{\geq 0}\alpha \subset P_{\mathrm{af}}$. For an inclusion of Lie algebras $\mathfrak{q} \subset \mathfrak{p}$ where $\mathfrak{q}$ is abelian, we denote by $(\mathfrak{p}, \mathfrak{q})$ the category of $\mathfrak{p}$-modules equipped with a semisimple $\mathfrak{q}$-action.

A $\widetilde{\mathfrak{g}}_{\geq 0}$-module M is said to be $\mathfrak{g}$-*integrable* if it decomposes into a direct sum of finite-dimensional $\mathfrak{g}$-modules upon restriction.

Definition 1.8 (Category $\mathfrak{B}$). A $\widetilde{\mathfrak{b}}$-module M is called *graded* if it is $\widetilde{\mathfrak{h}}$-semisimple, all of its $\widetilde{\mathfrak{h}}$-eigenvalues belong to P_{af}, and each of its $\widetilde{\mathfrak{h}}$-weight spaces has at most countable dimension. Let

$$\Psi(M) := \{\Lambda \in P_{\mathrm{af}} \mid \mathrm{Hom}_{\widetilde{\mathfrak{h}}}(\mathbb{C}_\Lambda, M) \neq 0\}$$

denote the set of $\widetilde{\mathfrak{h}}$-weights of M. The module M is said to be *bounded* if its d-degrees are bounded from below and each d-weight space is finite-dimensional. Let $\mathfrak{B}$ and $\mathfrak{B}_{\mathrm{bdd}}$ be the full subcategories of the category of $\widetilde{\mathfrak{b}}$-modules consisting of graded modules and bounded graded modules, respectively.

Definition 1.9 (Category $\mathfrak{C}$). A $\widetilde{\mathfrak{g}}_{\geq 0}$-module M is called *graded* if its restriction to $\widetilde{\mathfrak{b}}$ is a graded module. Let $\mathfrak{C}$ be the full subcategory of graded $\widetilde{\mathfrak{g}}_{\geq 0}$-modules whose objects are also $\mathfrak{g}$-integrable. Let $\mathfrak{C}_{\mathrm{bdd}}$ denote the full subcategory of $\mathfrak{C}$ consisting of modules whose restrictions to $\widetilde{\mathfrak{b}}$ belong to $\mathfrak{B}_{\mathrm{bdd}}$.

Note that any module M in $\mathfrak{B}_{\mathrm{bdd}}$ or $\mathfrak{C}_{\mathrm{bdd}}$ is generated by (any $\widetilde{\mathfrak{h}}$-splitting of) its head, $\mathsf{hd}\, M$. This can be verified by collecting vectors that do not arise from lower d-degrees, using the boundedness of the weights.

1.3.1 Simple and projective objects in $\mathfrak{B}$

A simple object in $\mathfrak{B}$ is a one-dimensional representation corresponding to a character of $\widetilde{\mathfrak{h}}$. A simple object in $\mathfrak{C}$ is isomorphic to V_λ for some $\lambda \in P^+$, up to a twist by a one-dimensional character.

For each $\Lambda \in P_{\mathrm{af}}$, we define the standard module

$$Q_\Lambda := U(\widetilde{\mathfrak{b}}) \otimes_{U(\widetilde{\mathfrak{h}})} \mathbb{C}_\Lambda.$$

The module Q_Λ is the projective cover of the simple module $\mathbb{C}_\Lambda$ in the category $\mathfrak{B}$. A projective cover in $\mathfrak{B}$ of an object in $\mathfrak{B}_{\mathrm{bdd}}$ is the direct sum of $\{Q_\Lambda\}_\Lambda$ since each module in $\mathfrak{B}_{\mathrm{bdd}}$ is generated by (any $\widetilde{\mathfrak{h}}$-splitting of) its head.

1.3.2 Simple and projective objects in $\mathfrak{C}$

For each $\lambda \in P^+$, the simple $\mathfrak{g}$-module V_λ can be regarded as a graded $\widetilde{\mathfrak{g}}_{\geq 0}$-module via the natural surjection $\widetilde{\mathfrak{g}}_{\geq 0} \to \mathfrak{g}$ that annihilates summands with positive d-degree. The parabolic Verma module

$$P_\lambda := U(\widetilde{\mathfrak{g}}_{\geq 0}) \otimes_{U(\mathfrak{g}+\widetilde{\mathfrak{h}})} V_\lambda$$

is the projective cover of V_λ in the category $\mathfrak{C}$ [9].

For any affine weight $\Lambda \in P_{\mathrm{af}}$, we define the corresponding projective module P_Λ and simple module V_Λ by twisting their finite counterparts. Let $\lambda := \overline{\Lambda}$ be

the classical projection and let λ_+ be the dominant weight in its W-orbit (the same one as in Definition 1.1). We then define

$$P_\Lambda := P_{\lambda_+} \otimes \mathbb{C}_{\Lambda-\lambda} \quad \text{and} \quad V_\Lambda := V_{\lambda_+} \otimes \mathbb{C}_{\Lambda-\lambda}.$$

Here, we identify λ with its canonical image in P_{af} to form the difference $\Lambda - \lambda$. The module V_Λ is regarded as a $(\mathfrak{g} + \widetilde{\mathfrak{h}})$-module. A projective object in $\mathfrak{C}_{\mathrm{bdd}}$ is the direct sum of $\{P_\Lambda\}_\Lambda$ since each module in $\mathfrak{C}_{\mathrm{bdd}}$ is generated by (any $\widetilde{\mathfrak{h}}$-splitting of) its head.

1.3.3 Graded Homs and Exts

The four categories defined above are all abelian. They are equipped with shift functors q^m that tensors a module with the one-dimensional representation $\mathbb{C}_{m\delta}$ for each $m \in \mathbb{Z}$. Namely, if we are given a module M equipped with a graded decomposition $M = \bigoplus_{s \in \mathbb{Z}} M_s$, then for each $m, s \in \mathbb{Z}$, we define

$$(\mathsf{q}^m M)_s := M_{s-m}.$$

This functor commutes with natural functors among our categories. Using these, we define the graded Hom-space between objects M and N in one of these categories $\mathcal{C}$ as

$$\hom_{\mathcal{C}}(M, N) := \bigoplus_{m, l \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(\mathsf{q}^m M, N \otimes \mathbb{C}_{l\Lambda_0}). \tag{1.2}$$

We similarly define the graded Ext-groups using a projective resolution of M:

$$\mathrm{ext}_{\mathcal{C}}^i(M, N) := \bigoplus_{m, l \in \mathbb{Z}} \mathrm{Ext}_{\mathcal{C}}^i(M, \mathsf{q}^{-m} N \otimes \mathbb{C}_{l\Lambda_0}). \tag{1.3}$$

We consider these as graded vector spaces, where the grading corresponds to the d-degree (i.e., the q-shifts).

Despite the conventional isomorphism of graded vector spaces

$$\mathrm{ext}_{\mathcal{C}}^\bullet(M, N) \cong \mathrm{ext}_{\mathcal{C}}^\bullet(M, N \otimes \mathbb{C}_{\Lambda_0}), \tag{1.4}$$

we generally keep track of the grading by Λ_0-twists. This is because the corresponding action of the central element K is crucial for understanding the underlying structure in many cases.

1.3.4 Restricted duals

For a module M in $\mathfrak{B}_{\mathrm{bdd}}$ or $\mathfrak{C}_{\mathrm{bdd}}$, its *restricted dual* is defined as

$$M^\vee := \bigoplus_{\Lambda \in P_{\mathrm{af}}} \mathrm{Hom}_{\widetilde{\mathfrak{h}}}(\mathbb{C}_\Lambda, M)^*.$$

The dual module $M^\vee$ is an object in $\mathfrak{B}$ or $\mathfrak{C}$, respectively.

In addition, there is a natural isomorphism for Hom-spaces. For any $M, N \in \mathfrak{B}_{\mathrm{bdd}}$, we have:

$$\hom_{\mathfrak{B}}(M, N^\vee) \cong \hom_{\mathfrak{B}}(N, M^\vee) \tag{1.5}$$

This isomorphism is functorial. Standard arguments in homological algebra show that it extends to the higher Ext-groups as well ([27, Chap. 2 §1]).

Remark 1.10. The extension of (1.5) to higher Ext-groups is obtained by appealing to the formalism of δ-functors, in particular through the universal property of derived functors regarded as δ-functors [27, Proposition 2.2.1].

If (1.5) were available functorially for all $\widetilde{\mathfrak{h}}$-semisimple $\widetilde{\mathfrak{b}}$-modules, then the passage to higher Ext-groups would follow directly from the formalism of derived categories (cf. [63]). However, since hom does not in general commute with direct sums, such a functorial extension is not valid in complete generality. To address this issue, we restrict to situations where (1.5) holds under certain boundedness assumptions, and we promote it to a natural transformation by fixing one of the arguments.

This technical constraint explains the asymmetric role of $\mathfrak{B}_{\mathrm{bdd}}$ in our setup. Indeed, objects in $\mathfrak{B}_{\mathrm{bdd}}$ are compact in the derived category of $\mathfrak{B}$ (cf. (1.8)), and this compactness ensures the desired control over direct sums.

In the sequel, we shall refrain from further use of the δ-functor formalism. All arguments will instead be carried out within the more concrete setting of standard homological algebra (e.g., [8]).

1.3.5 Filtrations

Consider a decreasing, separable filtration of an object M in one of the categories defined above:

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots, \qquad \text{with} \quad \bigcap_{r \geq 0} M^r = \{0\}.$$

Assume that this filtration is exhaustive with respect to the d-grading; that is, for any degree bound $m \in \mathbb{Z}$, there exists an integer r such that all weights in the support $\Psi(M^r)$ have d-degree at least m.

Under this assumption, the object M is complete with respect to the filtration, i.e.,

$$\varprojlim_r M/M^r \cong M$$

as an object in the respective category.

In particular, this implies that M is complete with respect to the topology induced by its d-grading. It is important to note, however, that this notion of completion—taken within specific categories such as $\mathfrak{B}$ or $\mathfrak{C}$—may differ from the standard completion in the larger category of all modules.

1.3.6 Yoneda interpretation of ext^1

Proposition 1.11 (See e.g., [64, Theorem 3.4.3]). *Let $\mathcal{A}$ be an abelian category, and let $M, N \in \mathcal{A}$. Then there is a natural bijection:*

$$\mathrm{Ext}^1_{\mathcal{A}}(M, N) \;\cong\; \left\{ \begin{array}{c} \textit{equivalence classes of short exact sequences of the form} \\ 0 \to N \to E \to M \to 0 \end{array} \right\}.$$

Two extensions are considered equivalent if there exists a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle \sim} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & E' & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

in which the middle vertical map is an isomorphism.

In particular, there exists a nontrivial extension of M by N in $\mathcal{A}$ if and only if $\mathrm{Ext}^1_{\mathcal{A}}(M, N) \neq 0$.

We apply Proposition 1.11 below for $\mathcal{A} = \mathfrak{B}_{\mathrm{bdd}}$ or $\mathfrak{C}_{\mathrm{bdd}}$, replacing $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ with $\mathrm{ext}^1_{\mathcal{A}}(M, N)$. This amounts to allowing nontrivial extensions of M by $\mathsf{q}^m N$ for some $m \in \mathbb{Z}$, in accordance with our convention (1.3).

1.3.7 ext and inverse limits

The following result is a variant of [64, Theorem 3.5.8].

Lemma 1.12. *Let $M, N \in \mathfrak{B}_{\mathrm{bdd}}$, and suppose that*

$$M \cong \varprojlim_{r} M(r)$$

as a projective system $\{M(r)\}_{r \in \mathbb{Z}_{\geq 0}}$ with surjective transition maps. Then, for each $i \in \mathbb{Z}$,

$$\varinjlim_{r} \mathrm{ext}^i_{\mathfrak{B}}\big(M(r), N^{\vee}\big) \cong \mathrm{ext}^i_{\mathfrak{B}}\big(\varprojlim_{r} M(r), N^{\vee}\big). \tag{1.6}$$

Proof. Let $X(r) := M(r) \otimes N$ and

$$X := \varprojlim_{r} X(r) \cong \big(\varprojlim_{r} M(r)\big) \otimes N \qquad (\text{since } M, N \in \mathfrak{B}_{\mathrm{bdd}}).$$

By the duality from §1.3.4, we have

$$\mathrm{ext}^{\bullet}_{\mathfrak{B}}\big(M(r), N^{\vee}\big) \cong \mathrm{ext}^{\bullet}_{\mathfrak{B}}\big(\mathbb{C}, X(r)^{\vee}\big), \quad \mathrm{ext}^{\bullet}_{\mathfrak{B}}\big(M, N^{\vee}\big) \cong \mathrm{ext}^{\bullet}_{\mathfrak{B}}\big(\mathbb{C}, X^{\vee}\big). \tag{1.7}$$

Since all $\widetilde{\mathfrak{h}}$-weights of $\widetilde{\mathfrak{n}}$ are (strictly) positive, the Koszul resolution of $\mathbb{C}$ has finite-dimensional $\widetilde{\mathfrak{h}}$-weight components (and therefore, for each fixed weight, the corresponding component complex is bounded). In particular, we have

$$\mathrm{ext}^{\bullet}_{\mathfrak{B}}\big(\mathbb{C}, \bigoplus_{r} X(r)^{\vee}\big) \cong \bigoplus_{r} \mathrm{ext}^{\bullet}_{\mathfrak{B}}\big(\mathbb{C}, X(r)^{\vee}\big). \tag{1.8}$$

The transition maps $\iota_r : X(r)^{\vee} \hookrightarrow X(r+1)^{\vee}$ are injective. Hence we have a standard short exact sequence

$$0 \longrightarrow \bigoplus_{r} X(r)^{\vee} \xrightarrow{f} \bigoplus_{r} X(r)^{\vee} \longrightarrow \varinjlim_{r} X(r)^{\vee} \longrightarrow 0,$$

where $f(\{a_r\}_r) = \{\iota_{r-1}(a_{r-1}) - a_r\}_r$ with $\iota_{-1}(a_{-1}) := 0$, and $\varinjlim_r X(r)^{\vee} \cong \big(\varprojlim_r X(r)\big)^{\vee}$.

Applying $\mathrm{ext}^{\bullet}_{\mathfrak{B}}(\mathbb{C}, -)$ yields the long exact sequence

$$0 \to \hom_{\mathfrak{B}}(\mathbb{C}, \bigoplus_{r} X(r)^{\vee}) \to \hom_{\mathfrak{B}}(\mathbb{C}, \bigoplus_{r} X(r)^{\vee}) \to \hom_{\mathfrak{B}}(\mathbb{C}, \varinjlim_{r} X(r)^{\vee})$$

$$\to \mathrm{ext}^1_{\mathfrak{B}}(\mathbb{C}, \bigoplus_{r} X(r)^{\vee}) \to \cdots.$$

On the other hand, taking inductive limits after applying $\mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{C}, -)$ to the system $\{X(r)^\vee\}_r$ gives another long exact sequence

$$0 \to \bigoplus_r \hom_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) \to \bigoplus_r \hom_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) \to \varinjlim_r \hom_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee)$$

$$\to \bigoplus_r \mathrm{ext}^1_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) \to \cdots .$$

By (1.8), these two long exact sequences fit into a commutative diagram

$$\begin{CD} \bigoplus_r \mathrm{ext}^i_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) @>>> \bigoplus_r \mathrm{ext}^i_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) \\ @VVV @VVV \\ \mathrm{ext}^i_{\mathfrak{B}}(\mathbb{C}, \bigoplus_r X(r)^\vee) @>>> \mathrm{ext}^i_{\mathfrak{B}}(\mathbb{C}, \bigoplus_r X(r)^\vee) \end{CD} \qquad (1.9)$$

for each $i \in \mathbb{Z}$. In addition, we have natural maps

$$\varinjlim_r \mathrm{ext}^i_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) \longrightarrow \mathrm{ext}^i_{\mathfrak{B}}(\mathbb{C}, X^\vee) \qquad (i \in \mathbb{Z})$$

which commute with (1.9). By the five lemma, we obtain

$$\varinjlim_r \mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{C}, X(r)^\vee) \; \cong \; \mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{C}, \varinjlim_r X(r)^\vee).$$

Combining with (1.7) yields (1.6). $\qquad\square$

Corollary 1.13. *Let $M, N \in \mathfrak{B}_{\mathrm{bdd}}$, and suppose that N is given as a projective system*

$$N \; \cong \; \varprojlim_r N(r),$$

where $\{N(r)\}_r$ has surjective transition maps. Then there are natural isomorphisms

$$\varinjlim_r \mathrm{ext}^\bullet_{\mathfrak{B}}\big(M, N(r)^\vee\big) \; \cong \; \mathrm{ext}^\bullet_{\mathfrak{B}}\Big(M, (\varprojlim_r N(r))^\vee\Big) \; \cong \; \mathrm{ext}^\bullet_{\mathfrak{B}}\Big(M, \varinjlim_r N(r)^\vee\Big).$$

Proof. By the duality from §1.3.4, we have

$$\mathrm{ext}^\bullet_{\mathfrak{B}}\big(M, N(r)^\vee\big) \cong \mathrm{ext}^\bullet_{\mathfrak{B}}\big(N(r), M^\vee\big), \qquad \mathrm{ext}^\bullet_{\mathfrak{B}}\big(M, N^\vee\big) \cong \mathrm{ext}^\bullet_{\mathfrak{B}}\big(N, M^\vee\big).$$

Applying Corollary 1.13, we obtain

$$\varinjlim_r \mathrm{ext}^\bullet_{\mathfrak{B}}\big(M, N(r)^\vee\big) \; \cong \; \mathrm{ext}^\bullet_{\mathfrak{B}}\Big(M, (\varprojlim_r N(r))^\vee\Big).$$

Finally, in $\mathfrak{B}_{\mathrm{bdd}}$ the dual interchanges inductive limits and inverse limits (since weight spaces are finite-dimensional); hence $\varinjlim_r N(r)^\vee \cong (\varprojlim_r N(r))^\vee$, and this yields the last isomorphism. $\qquad\square$

1.4 BGG resolution and Demazure functors

1.4.1 BGG resolutions

For each $\Lambda \in P_{\mathrm{af}}$, we define the *Verma module* $M(\Lambda)$ and its dual counterpart $^{\theta}M(\Lambda)$ as

$$M(\Lambda) := U(\widetilde{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{b}})} \mathbb{C}_{\Lambda}, \quad \text{and} \quad ^{\theta}M(\Lambda) := U(\widetilde{\mathfrak{g}}) \otimes_{U(\widetilde{\mathfrak{b}}^{-})} \mathbb{C}_{-\Lambda}.$$

Since $^{\theta}M(\Lambda)$ is isomorphic to the standard projective module $Q_{-\Lambda}$, it is a projective object in the category $\mathfrak{B}$.

For any dominant integral weight $\Lambda \in P_{\mathrm{af}}^{+}$, the simple module $L(\Lambda)$ admits a *BGG resolution*. This is the following exact sequence [45, Theorem 9.1.3]:

$$\cdots \to \bigoplus_{\ell(w)=2} M(w \cdot \Lambda) \to \bigoplus_{\ell(w)=1} M(w \cdot \Lambda) \to M(\Lambda) \to L(\Lambda) \to 0, \qquad (1.10)$$

where $w \cdot \Lambda := w(\Lambda + \rho) - \rho$ denotes the shifted "dot" action of the Weyl group.

Thus, applying the Chevalley involution θ to the BGG resolution (1.10) yields a projective resolution of the module $^{\theta}L(\Lambda)$ in $\mathfrak{B}$.

1.4.2 Demazure functors

For each $i \in \mathtt{I}_{\mathrm{af}}$, we define the parabolic subalgebra $\widetilde{\mathfrak{p}}_i := \widetilde{\mathfrak{b}} \oplus \widetilde{\mathfrak{g}}_{-\alpha_i}$. This contains the Lie subalgebra

$$\mathfrak{sl}(2, i) := \mathbb{C}E_i \oplus \mathbb{C}\alpha_i^{\vee} \oplus \mathbb{C}F_i.$$

A $\widetilde{\mathfrak{p}}_i$-module is called $\mathfrak{sl}(2, i)$-*integrable* if, upon restriction, it decomposes into a direct sum of finite-dimensional $\mathfrak{sl}(2, i)$-modules.

The *Demazure functor* $\mathcal{D}_i$ is an endofunctor on $\mathfrak{B}$ that sends a module M to the maximal $\mathfrak{sl}(2, i)$-integrable quotient of the induced module $U(\widetilde{\mathfrak{p}}_i) \otimes_{U(\widetilde{\mathfrak{b}})} M$.

Theorem 1.14 (Joseph [33])**.** *Let $i, j \in \mathtt{I}_{\mathrm{af}}$. The Demazure functors have the following properties:*

1. *There is a natural transformation of functors* $\mathrm{Id} \to \mathcal{D}_i$.

2. *There is a natural isomorphism of functors* $\mathcal{D}_i \cong \mathcal{D}_i \circ \mathcal{D}_i$.

3. *The functor $\mathcal{D}_i$ is right exact, and its left derived functors satisfy* $\mathbb{L}^m \mathcal{D}_i = 0$ *for* $m \neq 0, -1$.

4. *If $M \in \mathfrak{B}$ is the restriction of an $\mathfrak{sl}(2, i)$-integrable $\widetilde{\mathfrak{p}}_i$-module, then there is a natural isomorphism of functors:*

$$M \otimes \mathcal{D}_i(\bullet) \cong \mathcal{D}_i(M \otimes \bullet).$$

5. *(Braid Relations) If s_i and s_j satisfy* $\overbrace{s_i s_j s_i \cdots}^{m} = \overbrace{s_j s_i s_j \cdots}^{m}$, *then*

$$\overbrace{\mathcal{D}_i \mathcal{D}_j \mathcal{D}_i \cdots}^{m} \cong \overbrace{\mathcal{D}_j \mathcal{D}_i \mathcal{D}_j \cdots}^{m}. \qquad \square$$

Any element $w \in W_{\mathrm{af}}$ has a *reduced expression* $w = s_{i_1} \cdots s_{i_\ell}$, where the sequence of indices $i_1, \ldots, i_\ell \in \mathtt{I}_{\mathrm{af}}$ has minimal length $\ell = \ell(w)$. As a consequence of the braid relations in Theorem 1.14(5), the composite functor

$$\mathcal{D}_w := \mathcal{D}_{i_1} \circ \mathcal{D}_{i_2} \circ \cdots \circ \mathcal{D}_{i_\ell} \tag{1.11}$$

is well-defined and does not depend on the choice of reduced expression for w.

Corollary 1.15. *Let $i \in \mathtt{I}_{\mathrm{af}}$, and consider a sequence of inclusions of $\widetilde{\mathfrak{h}}$-stable Lie algebras*

$$\mathfrak{sl}(2, i) \subset \mathfrak{q} = \mathfrak{sl}(2, i) \oplus \mathfrak{q}_{>0} \subset \widetilde{\mathfrak{p}}_i, \tag{1.12}$$

with $\widetilde{\mathfrak{h}} \subset \mathfrak{q}$.

Then, for any $\widetilde{\mathfrak{h}}$-semisimple $(\mathfrak{q} \cap \widetilde{\mathfrak{b}})$-module M, the Demazure module $\mathcal{D}_i(M)$ computed using $(\mathfrak{sl}(2, i) \cap \widetilde{\mathfrak{b}})$-structure coincides with that computed using $(\mathfrak{q} \cap \widetilde{\mathfrak{b}})$-structure.

Proof. Let $\mathfrak{b}_i := \mathfrak{sl}(2, i) \cap \widetilde{\mathfrak{b}} = \mathbb{C}\alpha_i^\vee \oplus \mathbb{C}E_i$, and set $\mathfrak{q}_{\geq 0} := \mathfrak{q} \cap \widetilde{\mathfrak{b}}$.

The decomposition $\mathfrak{q} = \mathfrak{sl}(2, i) \oplus \mathfrak{q}_{>0}$ is compatible with the $\widetilde{\mathfrak{h}}$-action, and $\mathfrak{q}_{>0}$ decomposes into finite-dimensional $\mathfrak{sl}(2, i)$-modules. We may write

$$\mathfrak{q}_{>0} = \widetilde{\mathfrak{h}}' \oplus \bigoplus_{\beta \in \Delta'} \mathfrak{g}_\beta, \tag{1.13}$$

where $\widetilde{\mathfrak{h}}' \subset \widetilde{\mathfrak{h}}$ and $\Delta' \subset \Delta_{\mathrm{af}}^+ \setminus \{\alpha_i\}$ is s_i-stable, again using the assumptions in (1.12).

Let $\widetilde{M} := U(\mathfrak{q}) \otimes_{U(\mathfrak{q}_{\geq 0})} M \cong U(\mathfrak{sl}(2, i)) \otimes_{U(\mathfrak{b}_i)} M$. Let $\mathcal{D}_i(M)$ and $\mathcal{D}'_i(M)$ denote the Demazure modules obtained by imposing maximality with respect to $\mathfrak{q}_{\geq 0}$ and $\mathfrak{b}_i$, respectively. That is,

$$\mathcal{D}_i(M) = \widetilde{M}/R, \qquad \mathcal{D}'_i(M) = \widetilde{M}/R',$$

where R (resp. R') is the largest submodule of $\widetilde{M}$ such that all simple $\mathfrak{q}$-module (resp. $\mathfrak{sl}(2, i)$-module) quotients contain an infinite-dimensional irreducible constituent as a $\mathfrak{sl}(2, i)$-module.

Here, we have $R' \subset R$ as the latter carries an action of a strictly larger Lie algebra. Suppose $R' \subsetneq R$. Then the quotient R/R' has a simple $\mathfrak{sl}(2, i)$-module quotient which, by construction, must be finite-dimensional. Since the $\widetilde{\mathfrak{h}}$-action on R/R' is semisimple, it follows that R/R' admits a nonzero simple finite-dimensional $(\mathfrak{sl}(2, i) + \widetilde{\mathfrak{h}})$-module quotient.

However, any such simple $(\mathfrak{sl}(2, i) + \widetilde{\mathfrak{h}})$-module extends to a $\mathfrak{q}$-module via (1.13). Indeed, each $\mathfrak{g}_\beta$ (for $\beta \in \Delta'$) strictly raises $\widetilde{\mathfrak{h}}$-weights, so every nonzero action by $\mathfrak{g}_\beta$ maps into a proper submodule. This contradicts the defining property of R, whose simple $\mathfrak{q}$-module quotients contain an infinite-dimensional irreducible $(\mathfrak{sl}(2, i) + \widetilde{\mathfrak{h}})$-constituent.

We conclude that $R = R'$, and hence $\mathcal{D}_i(M) = \mathcal{D}'_i(M)$ as claimed. $\qquad\square$

Thanks to Corollary 1.15, the Demazure functors $\mathcal{D}_i$ and $\mathcal{D}_w$ are independent of the ambient Lie algebra module structures, whenever they are well-defined.

A module $M \in \mathfrak{B}$ admits an integrable $\mathfrak{g}$-action if and only if $M \cong \mathcal{D}_{w_0}(M)$, which is equivalent to the condition $\mathcal{D}_i(M) \cong M$ for all $i \in \mathtt{I}$.

1.4.3 The adjoint property

The following result is stated under the assumption $\dim N < \infty$ in [18], but holds whenever (1.5) holds:

Theorem 1.16 ([18, Proposition 5.7]; cf. [50, Lemma 8]). *For any $M, N \in \mathfrak{B}_{\mathrm{bdd}}$, there is a functorial isomorphism between hypercohomology groups:*

$$\mathrm{ext}^i_{\mathfrak{B}}(\mathbb{L}^\bullet \mathcal{D}_i(M), N^\vee) \cong \mathrm{ext}^i_{\mathfrak{B}}(M, \mathbb{R}^\bullet(\mathcal{D}_i(N))^\vee) \quad (i \in \mathbb{Z}).$$

2 Basic module-theoretic results

We keep the setting of the previous section.

2.1 Comparison of Exts

Proposition 2.1 (Cartan–Eilenberg [8]). *For each $M, N \in \mathfrak{C}$ such that $\dim N < \infty$, we have*

$$\mathrm{Ext}^\bullet_{\mathfrak{C}}(M, N) = \mathrm{Ext}^\bullet_{\mathfrak{B}}(M, N).$$

Proof. Let $\widetilde{\mathfrak{g}}_{>0} := \ker(\widetilde{\mathfrak{g}}_{\geq 0} \to \mathfrak{g} + \widetilde{\mathfrak{h}})$. The inclusions $\widetilde{\mathfrak{g}}_{>0} \subset \widetilde{\mathfrak{b}}$ and $\widetilde{\mathfrak{g}}_{>0} \subset \widetilde{\mathfrak{g}}_{\geq 0}$ form Lie algebra ideals with their quotients $\mathfrak{b} + \widetilde{\mathfrak{h}}$ and $\mathfrak{g} + \widetilde{\mathfrak{h}}$ admit Lie algebra splittings.

Thus the Hochschild–Serre spectral sequence [8, Chapter XVI, §6] provides, for $(\widetilde{\mathfrak{b}}, \widetilde{\mathfrak{h}})$-modules M and N, a spectral sequence that calculates the extension groups by two-steps

$$E_2^{q,p} := \mathrm{Ext}^q_{(\mathfrak{b}+\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}})}(\mathbb{C}, \mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) \Rightarrow \mathrm{Ext}^{q+p}_{(\widetilde{\mathfrak{b}}, \widetilde{\mathfrak{h}})}(M, N).$$

Since $M, N \in \mathfrak{C}$, we also have the Hochschild–Serre spectral sequence

$${}'E_2^{q,p} := \mathrm{Ext}^q_{(\mathfrak{g}+\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}})}(\mathbb{C}, \mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) \Rightarrow \mathrm{Ext}^{q+p}_{(\widetilde{\mathfrak{g}}_{\geq 0}, \widetilde{\mathfrak{h}})}(M, N).$$

Hence, the assertion reduces to the analysis of $\mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)$. The module $\mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)$ inherits a $\mathfrak{g}$-action, which is $\mathfrak{g}$-integrable because $U(\widetilde{\mathfrak{g}}_{>0})$ is itself $\mathfrak{g}$-integrable under the adjoint action. Owing to the complete reducibility of integrable $\mathfrak{g}$-modules in $(\mathfrak{b}, \mathfrak{h})$, we have

$$\mathrm{Ext}^{>0}_{(\mathfrak{b}+\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}})}(\mathbb{C}, \mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) = 0,$$
$$\mathrm{Hom}_{(\mathfrak{b}+\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}})}(\mathbb{C}, \mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)) = \mathrm{Hom}_{(\mathfrak{g}+\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}})}(\mathbb{C}, \mathrm{Ext}^p_{\widetilde{\mathfrak{g}}_{>0}}(M, N)).$$

The vanishing of $E_2^{>0,\bullet}$ and ${}'E_2^{>0,\bullet}$ implies that the spectral sequences collapse, and the second equality identifies the E_2-pages of the Ext-groups. This yields the desired isomorphism

$$\mathrm{Ext}^\bullet_{\mathfrak{C}}(M, N) = \mathrm{Ext}^\bullet_{\mathfrak{B}}(M, N),$$

as claimed. $\qquad\square$

2.2 Properties of affine Demazure modules

Theorem 2.2 (see Kumar [45, Chapter VIII], [15, Proposition 4.16]). *Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P$ and $i \in \mathtt{I}_{\mathrm{af}}$, we have*

$$\mathbb{L}^{\bullet}\mathcal{D}_i(D_{\lambda}^{(k)}) = \begin{cases} \mathsf{q}^{\delta_{i0}(\langle \vartheta^{\vee}, \lambda \rangle - k)} D_{s_i(\lambda+k\Lambda_0)}^{(k)} & \text{if } \overline{s_i(\lambda + k\Lambda_0)} \succ \lambda, \\ D_{\lambda}^{(k)} & \text{if } \overline{s_i(\lambda + k\Lambda_0)} \preceq \lambda. \end{cases} \qquad \square$$

Theorem 2.3 (Kashiwara [39], see also [43, Theorem C]). *Fix $k \in \mathbb{Z}_{>0}$, and let $\Lambda \in P_{\mathrm{af},k}^+$, $w \in W_{\mathrm{af}}$, and $i \in \mathtt{I}_{\mathrm{af}}$. Then we have*

$$\mathbb{L}^{\bullet}\mathcal{D}_i({}^{\theta}L(\Lambda)^w) = \begin{cases} {}^{\theta}L(\Lambda)^{s_i w} & \text{if } s_i w < w, \\ {}^{\theta}L(\Lambda)^w & \text{if } s_i w > w. \end{cases} \qquad \square$$

Corollary 2.4. *Fix $k \in \mathbb{Z}_{>0}$, and let $\Lambda \in P_{\mathrm{af},k}^+$ and $w \in W_{\mathrm{af}}$. For a subset $S \subset W_{\mathrm{af}}$, define*

$$M := \sum_{w \in S} {}^{\theta}L(\Lambda)^w, \quad N := \bigcap_{w \in S} {}^{\theta}L(\Lambda)^w.$$

Then for each $i \in \mathtt{I}_{\mathrm{af}}$, the lower derived functors vanish:

$$\mathbb{L}^{<0}\mathcal{D}_i(M) = \mathbb{L}^{<0}\mathcal{D}_i(N) = 0,$$

and both M and N are preserved under $\mathcal{D}_i$:

$$M \subset \mathcal{D}_i(M), \quad N \subset \mathcal{D}_i(N).$$

Proof. By [39, Proposition 4.1] (specialized to $q = 1$), the module $L(\Lambda)$ admits a basis such that, for each $w \in W_{\mathrm{af}}$, a subset of this basis yields a basis of $L(\Lambda)^w$. These bases consist of $\widetilde{\mathfrak{h}}$-eigenvectors and satisfy the so-called *string property* of Demazure crystals [39, (0.4)(iii), §4], which is preserved under both sums and intersections of these subspaces.

This property determines decreasing separable filtrations $F_i M$ and $F_i N$ with respect to E_i-stable subspaces, such that their associated graded modules decompose as direct sums of indecomposable $\widetilde{\mathfrak{h}}$-semisimple $\mathbb{C}[E_i]$-modules (cf. [43, Proof of Corollary 4.8]).

By [39, (0.5)], we have

$$\mathbb{L}^{<0}\mathcal{D}_i(F_j M / F_{j+1} M) = 0 = \mathbb{L}^{<0}\mathcal{D}_i(F_j N / F_{j+1} N) \qquad (j \geq 0).$$

Therefore, taking inverse limits, we obtain

$$\mathbb{L}^{<0}\mathcal{D}_i(M) \cong \varprojlim_j \mathbb{L}^{<0}\mathcal{D}_i(M/F_j M) = 0 = \varprojlim_j \mathbb{L}^{<0}\mathcal{D}_i(N/F_j N) \cong \mathbb{L}^{<0}\mathcal{D}_i(N).$$

Here, the functor $\mathcal{D}_i$ commutes with $\varprojlim$ since M and N are direct sums of finitely generated $\widetilde{\mathfrak{h}}$-semisimple $\mathbb{C}[E_i]$-modules, and the derived functors $\mathbb{L}^{<0}\mathcal{D}_i$ vanish on each direct summand as well as at each stage of the induced filtrations (cf. [43, Proof of Lemma 4.4]). This proves the assertion. $\square$

Corollary 2.5. *The setting is as in Theorem 2.3. The module ${}^{\theta}L(\Lambda)^w$ is $\mathfrak{g}$-stable if and only if $\overline{w\Lambda}$ is dominant.*

Proof. For the "only if" direction, note that ${}^\theta L(\Lambda)^w$ being a $\mathfrak{g}$-module requires the equality $U(\mathfrak{b}^-)\mathbf{v}_{w\Lambda} = U(\mathfrak{g})\mathbf{v}_{w\Lambda}$, which can occur only if $\overline{w\Lambda} \in P^+$.

For the "if" direction, assume $\overline{w\Lambda} \in P^+$. By combining Proposition 1.4, Theorem 1.5, and Theorem 2.3, we see that $\mathcal{D}_i({}^\theta L(\Lambda)^w) = {}^\theta L(\Lambda)^w$ for each $i \in \mathtt{I}$. This implies that ${}^\theta L(\Lambda)^w$ is $\mathfrak{g}$-stable. $\qquad\square$

Let $\lambda \in P^+$ be such that $-\lambda + k\Lambda_0 = w\Lambda$ for some $w \in W_{\mathrm{af}}$ and $\Lambda \in P_{\mathrm{af},k}^+$. We define
$$W_\lambda^{(k)} := D_{\lambda_-}^{(k)}.$$

We also set
$$\mathbb{W}_\lambda^{(k)} := \frac{{}^\theta L(\Lambda)^{w_0 w}}{\sum_{v \in W_{\mathrm{af}},\ \lambda_- \prec (-\overline{v\Lambda})_-} {}^\theta L(\Lambda)^v}. \qquad (2.1)$$

Lemma 2.6. *Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P^+$, $\Lambda \in P_{\mathrm{af},k}^+$, and $w \in W_{\mathrm{af}}$ such that $-\lambda + k\Lambda_0 = w\Lambda$, the module ${}^\theta L(\Lambda)^{w_0 w}$ admits a decreasing separable filtration by grading shifts of the modules $\{\mathbb{W}_\mu^{(k)}\}_\mu$, where μ runs over the set*
$$\left\{ \mu \in \overline{W_{\mathrm{af}}\Lambda} \cap P^+ \mid -\mu = \overline{v\Lambda} \text{ for some } v \geq w \right\}.$$

Proof. By Theorem 1.5 and Corollary 1.6, the assertion follows immediately from the definition of $\mathbb{W}_\lambda^{(k)}$ (see also Corollary 1.7). $\qquad\square$

Lemma 2.7. *Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P^+$, the modules $W_\lambda^{(k)}$ and $\mathbb{W}_\lambda^{(k)}$ are $\widetilde{\mathfrak{g}}_{\geq 0}$-modules generated by V_λ.*

Proof. Both $W_\lambda^{(k)}$ and $\mathbb{W}_\lambda^{(k)}$ are quotients of Demazure modules, and hence each admits a unique simple quotient as a $\widetilde{\mathfrak{h}}$-semisimple $\widetilde{\mathfrak{b}}$-module. Moreover, they are $\mathfrak{g}$-stable by construction. The assertion follows from the fact that both $W_\lambda^{(k)}$ and $\mathbb{W}_\lambda^{(k)}$ are cyclic $\widetilde{\mathfrak{b}}$-modules generated by a vector of $\mathfrak{h}$-weight λ_-. In either case, this vector generates an irreducible $\mathfrak{g}$-submodule isomorphic to V_λ, which is the unique simple quotient of the corresponding $\widetilde{\mathfrak{g}}_{\geq 0}$-module. $\qquad\square$

2.3 The module $\mathbb{W}_\lambda$ and its properties

By definition, $W_\lambda^{(1)}$ and $\mathbb{W}_\lambda^{(1)}$ (for $\lambda \in P^+$) are level-one modules on which the central element $K \in \widetilde{\mathfrak{g}}$ acts by $+1$ and -1, respectively.

Theorem 2.8 (Fourier–Littelmann–Manning–Senesi [23, 24])**.** *For each $\lambda \in P^+$, the module $\mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0}$ is a nontrivial self-extension of $W_\lambda^{(1)} \otimes \mathbb{C}_{-\Lambda_0}$ as a $\widetilde{\mathfrak{g}}_{\geq 0}$-module. Moreover, its endomorphism ring*

$$\mathrm{End}_{\mathfrak{C}}(\mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0})$$

is isomorphic to a graded polynomial ring generated by elements of strictly positive degree. In addition, $\mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0}$ is a projective object in the full subcategory of $\mathfrak{C}$ consisting of those modules M with

$$\overline{\Psi(M)} \leq \lambda. \qquad\square$$

The following result is a reformulation of [15, Theorem 5.2], adapted to the setting of [42, Theorem 4.13]. Note that there exists a W-equivariant inclusion $Q \cap P^+ \subset Q^\vee$, which identifies short roots with long coroots. This identification is specific to the twisted affinizations.

Theorem 2.9 ([15, 42]). *Let $\lambda \in P^+$ and $\beta \in Q \cap P^+ \subset Q^\vee$. Then there is an isomorphism of graded $\widetilde{\mathfrak{b}}$-modules:*

$$\mathbb{L}^\bullet \mathcal{D}_{t_{-\beta}} \left(\mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0} \right) \cong \mathsf{q}^{-\langle \beta, \lambda \rangle} \, \mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0}. \qquad \Box$$

Let us extend the definition of $\mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0}$ to arbitrary $\lambda \in P$ in a slightly twisted manner:

Proposition 2.10 ([15, §4.2.2]). *For each $\lambda \in P$, there exists a $\widetilde{\mathfrak{h}}$-semisimple $\widetilde{\mathfrak{b}}$-module $\mathbb{W}_\lambda$ that carries a $\widetilde{\mathfrak{b}}$-cyclic vector of $\widetilde{\mathfrak{h}}$-weight λ with the following properties:*

1. *We have an isomorphism $\mathbb{W}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0} \cong \mathbb{W}_{\lambda_-}$ as $\widetilde{\mathfrak{b}}$-modules;*

2. *We have an isomorphism $\mathbb{D}_{\lambda_+}^{(1)} \otimes \mathbb{C}_{\Lambda_0} \cong \mathbb{W}_{\lambda_+}$ as $\widetilde{\mathfrak{b}}$-modules;*

3. *The module $\mathbb{W}_\lambda$ admits a finite filtration with successive quotients isomorphic to $\mathbb{D}_\mu^{(1)} \otimes \mathbb{C}_{\Lambda_0}$, where μ ranges over*

$$\{ \mu \in W\lambda \mid \lambda \le \mu \le \lambda_+ \},$$

   each occurring exactly once;

4. *For each $i \in \mathtt{I}_{\mathrm{af}}$, we have*

$$\mathbb{L}^\bullet \mathcal{D}_i (\mathbb{W}_\lambda) \cong \begin{cases} \mathbb{W}_\lambda & \text{if } s_i(\!(\lambda)\!) \preceq \lambda, \\ \mathsf{q}^{-\langle \vartheta^\vee, \lambda \rangle \delta_{i,0}} \mathbb{W}_{\overline{s_0 \lambda}} & \text{if } s_i(\!(\lambda)\!) \succ \lambda. \end{cases}$$

5. *For each $i \in \mathtt{I}_{\mathrm{af}}$ such that $s_i(\!(\lambda)\!) \preceq \lambda$, we have $\mathbb{L}^\bullet \mathcal{D}_i(\mathbb{D}_\lambda^{(1)}) \equiv 0$. For each $i \in \mathtt{I}$ such that $s_i(\!(\lambda)\!) \succ \lambda$, we have a short exact sequence*

$$0 \to \mathbb{D}_\lambda^{(1)} \longrightarrow \mathcal{D}_i(\mathbb{D}_\lambda^{(1)}) \longrightarrow \mathbb{D}_{s_i(\!(\lambda)\!)}^{(1)} \longrightarrow 0.$$

Proof. The cyclic vector is provided by the construction at the beginning of [15, §4.2.2]; the first item appears at the beginning of [15, §4.2.2]; the second is [15, Corollary 4.15]; the third is [15, Lemma 4.13]; the fourth [15, Proposition 4.14], while the case $i = 0$ follows from [15, Theorem 5.2] and its proof; the fifth is [15, Corollary 4.15]. $\qquad \Box$

Remark 2.11. The family of modules $\{\mathbb{W}_\lambda\}_{\lambda \in P}$ was first realized both as the duals of the global sections of line bundles on semi-infinite flag varieties [42] and, simultaneously, as the nonsymmetric counterpart of q-Whittaker functions for untwisted affinizations [21]. Extensions to general twisted affinizations are given in [15], and the case of type $\mathsf{A}_{2\ell}^{(2)}$ is treated in [16].

Corollary 2.12. *Let $\mathbb{W}$ be the module obtained by applying an affine Dynkin diagram automorphism to $\mathbb{W}_\lambda$ for $\lambda \in P$. Then there exist $\lambda' \in P$ such that*

$$\mathbb{W} \cong \mathbb{W}_{\lambda'}.$$

Proof. Combining Theorems 2.8 and 2.9 and Proposition 2.10, we observe that the module

$$\bigcup_{w \in W_{\mathrm{af}}} \mathcal{D}_w(\mathbb{W}_\lambda)$$

satisfies the universal property of a level-zero extremal weight module of $\widetilde{\mathfrak{g}}$ (see [39, §8]). It follows that $\mathbb{W}_\lambda$ must be a Demazure module of such a level-zero extremal weight module (see [40]). Thus, up to grading shifts, we obtain a bijection between the set $\{\mathbb{W}_\lambda\}_{\lambda \in P}$ and the set of Demazure submodules of level-zero extremal weight modules.

Since the structure of level-zero extremal weight modules and their Demazure submodules is preserved under diagram automorphisms, the same conclusion holds for the transformed module $\mathbb{W}$. □

Remark 2.13. Let θ be the automorphism of the affine Dynkin diagram that permutes the vertices $\mathtt{I}_{\mathrm{af}}$. Then the weights Λ appearing in Corollary 2.12 all satisfy the level zero condition $\Lambda(K) = 0$. The automorphism θ induces linear automorphisms of the spaces

$$\{\Lambda \in \widetilde{\mathfrak{h}}^* \mid \Lambda(K) = 0\} \subset \widetilde{\mathfrak{h}}^*$$

satisfying

$$\theta(\alpha_i) = \alpha_{\theta(i)}, \quad \theta(\Lambda_i) = \Lambda_{\theta(i)}, \quad \theta(\delta) = \delta \qquad (i \in \mathtt{I}_{\mathrm{af}}).$$

In §1.2.2, each fundamental weight $\varpi_i \in P$ $(i \in \mathtt{I})$ is identified with

$$\varpi_i = \Lambda_i - \langle \varpi_i, \vartheta^\vee \rangle \Lambda_0 = \Lambda_i - \Lambda_i(K)\Lambda_0 \in P_{\mathrm{af}}.$$

By convention, we formally set $\varpi_0 := \Lambda_0 - \Lambda_0(K)\Lambda_0 = 0$, corresponding to the node $0 \in \mathtt{I}_{\mathrm{af}} \setminus \mathtt{I}$.

It then follows that

$$\theta(\varpi_i) = \Lambda_{\theta(i)} - \Lambda_i(K)\Lambda_{\theta(i)} \qquad (i \in \mathtt{I}_{\mathrm{af}}),$$

which satisfies the consistency condition

$$\theta(\varpi_0) = \Lambda_{\theta(0)} - \Lambda_0(K)\Lambda_{\theta(0)} = \Lambda_{\theta(0)} - \Lambda_{\theta(0)} = 0.$$

Hence, applying θ linearly yields $\lambda' = \theta(\lambda)$ in Corollary 2.12.

2.4   Spaces of characters B and C

We define the $\mathbb{Q}((q))$-vector space

$$\mathbf{B} := \mathbb{Q}((q)) \otimes_{\mathbb{Z}[\![q]\!]} \varprojlim_m \bigoplus_{\lambda \in P} \left(\mathbb{Z}[q]/(q^{m+1}) \right) e^\lambda.$$

We define $\mathbf{C}$ to be the subspace of W-invariants in $\mathbf{B}$, where the W-action is given by $w \cdot (q^m e^\lambda) := q^m e^{w\lambda}$ for $w \in W$, $\lambda \in P$, and $m \in \mathbb{Z}$. Both vector spaces are equipped with the inverse limit topology induced by the q-adic filtration.

For each module M in $\mathfrak{B}_{\mathrm{bdd}}$ or $\mathfrak{C}_{\mathrm{bdd}}$, we define its graded character as

$$\operatorname{gch} M := \sum_{\lambda \in P, \ m \in \mathbb{Z}} q^m e^\lambda \cdot \dim \operatorname{Hom}_{(\mathfrak{h} + \mathbb{C}d)}(\mathbb{C}_{\lambda + m\delta}, \ M).$$

In particular, this character is invariant under level shifts:

$$\operatorname{gch} M = \operatorname{gch}(M \otimes \mathbb{C}_{k\Lambda_0}) \quad \text{for any } k \in \mathbb{Z}.$$

Lemma 2.14. *For each M in $\mathfrak{B}_{\mathrm{bdd}}$ or $\mathfrak{C}_{\mathrm{bdd}}$, we have $\operatorname{gch} M \in \mathbf{B}$ or $\operatorname{gch} M \in \mathbf{C}$, respectively.*

Proof. This follows directly from the definitions of the categories and the grading structure of M. $\square$

Proposition 2.15 ([18, Appendix]). *For each $M, N \in \mathfrak{B}_{\mathrm{bdd}}$, define the Euler–Poincaré pairing by*

$$\langle M, N \rangle_{\mathrm{EP}} := \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{gdim} \operatorname{Ext}^i_{\mathfrak{B}}(M, N^\vee).$$

This pairing extends uniquely to a $\mathbb{Z}[q^{\pm 1}]$-antibilinear map

$$[\mathfrak{B}_{\mathrm{bdd}}] \times [\mathfrak{B}_{\mathrm{bdd}}] \longrightarrow \mathbb{Q}((q^{-1})),$$

where $[\mathcal{C}]$ denotes the Grothendieck group of the abelian category $\mathcal{C}$. $\square$

Lemma 2.16. *Let $k \in \mathbb{Z}_{>0}$. We have the following:*

1. *The sets of characters $\{\operatorname{gch} D_\lambda^{(k)}\}_{\lambda \in P}$ and $\{\operatorname{gch} \mathbb{D}_\lambda^{(k)}\}_{\lambda \in P}$ are topological $\mathbb{Q}((q))$-bases of $\mathbf{B}$.*

2. *The sets of characters $\{\operatorname{gch} W_\lambda^{(k)}\}_{\lambda \in P^+}$ and $\{\operatorname{gch} \mathbb{W}_\lambda^{(k)}\}_{\lambda \in P^+}$ are topological $\mathbb{Q}((q))$-bases of $\mathbf{C}$.*

Proof. Any integrable lowest-weight module of $\widetilde{\mathfrak{g}}$ is a quotient of a parabolic Verma module P_λ for some $\lambda \in P^+$ (see [36, §9.2]). The graded character of such a module belongs to $\mathbf{C}$, as we have

$$\operatorname{gch} P_\lambda = \operatorname{gch}\big(U(\widetilde{\mathfrak{n}}/\mathfrak{n})\big) \cdot \operatorname{ch} V_\lambda \in \mathbf{C}.$$

The modules $\mathbb{D}_\mu^{(k)}$ (for $\mu \in W\lambda$) and $\mathbb{W}_\lambda^{(k)}$ are subquotients of such modules. It follows that their graded characters are termwise bounded above by that of a parabolic Verma module, and hence belong to $\mathbf{B}$ or $\mathbf{C}$, respectively.

By Lemma 2.7, both $W_\lambda^{(k)}$ and $\mathbb{W}_\lambda^{(k)}$ contain V_λ in degree zero and are generated by it as $\widetilde{\mathfrak{g}}_{\geq 0}$-modules. Therefore, we have the congruences

$$\operatorname{gch} W_\lambda^{(k)} \equiv \operatorname{gch} \mathbb{W}_\lambda^{(k)} \equiv \operatorname{gch} V_\lambda \pmod{q} \quad \text{for each } \lambda \in P^+.$$

Furthermore, for each $\mu \in P$, the module $D_\mu^{(k)}$ is contained in $W_{\mu_+}^{(k)}$, and $\mathbb{D}_{u\mu_+}^{(k)}$ appears in a filtration of $\mathbb{W}_{\mu_+}^{(k)}$. Thus, we have

$$\mathrm{gch}\, D_\mu^{(k)}\Big|_{q=0} \in \sum_{\lambda \in \Sigma(\mu)} \mathbb{Z}e^\lambda \ni \mathrm{gch}\, \mathbb{D}_\mu^{(k)}\Big|_{q=0}.$$

Here, the specialization $\mathrm{gch}\, M|_{q=0}$ denotes the image of $\mathrm{gch}\, M$ under the reduction map $\mathbb{Z}[q] \to \mathbb{Z}[q]/(q) \cong \mathbb{Z}$ applied coefficientwise.

These specializations belong to $\bigoplus_{\lambda \in P} \mathbb{Z}e^\lambda$, and the resulting sets

$$\big\{\mathrm{gch}\, D_\mu^{(k)}\Big|_{q=0}\big\}_{\mu \in P} \qquad \text{and} \qquad \big\{\mathrm{gch}\, \mathbb{D}_\mu^{(k)}\Big|_{q=0}\big\}_{\mu \in P}$$

form $\mathbb{Z}$-bases of this space. The same argument applies to $\{\mathrm{gch}\, W_\lambda^{(k)}\}_{\lambda \in P+}$ and $\{\mathrm{gch}\, \mathbb{W}_\lambda^{(k)}\}_{\lambda \in P+}$.

Since all the characters involved are q-series with integer coefficients and bounded below degrees, it follows that any element of $\mathbf{B}$ or $\mathbf{C}$ can be uniquely expressed as a convergent q-series in terms of these characters. This proves that the families listed in the statement form topological $\mathbb{Q}(\!(q)\!)$-bases. $\qquad \square$

2.5 Ext-orthogonality in the level one case

Theorem 2.17 ([15, Theorem 4.19]). *For each $\lambda, \mu \in P$, we have*

$$\mathrm{ext}_\mathfrak{B}^i\left(\mathbb{D}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0},\ (D_\mu^{(1)} \otimes \mathbb{C}_{-\Lambda_0})^*\right) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \lambda + \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the set $\Psi(D_\lambda^{(1)})$ of weights of $D_\lambda^{(1)}$ satisfies

$$\overline{\Psi(D_\lambda^{(1)})} \subset \Sigma(\lambda). \qquad \square$$

Lemma 2.18. *For $\lambda, \mu \in P$, we have*

$$\mathrm{ext}_\mathfrak{B}^{>0}\left(\mathbb{D}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0},\ (\mathbb{C}_{\mu-\Lambda_0})^*\right) = 0 \quad \text{for all } \mu \preceq \lambda. \tag{2.2}$$

Proof. We prove the assertion by induction on μ with respect to $\prec$, starting from minimal elements. In case μ is minimal, we have $\dim D_\mu^{(1)} = 1$ since $\mu + \Lambda_0$ must be a level one dominant weight.

Assume (2.2) holds for all $\nu \prec \mu$. Consider the short exact sequence

$$0 \to \ker f_\mu \longrightarrow D_\mu^{(1)} \xrightarrow{\ f_\mu\ } \mathbb{C}_\mu \to 0.$$

By the inductive hypothesis, all weights γ occurring in $\ker f_\mu$ satisfy $\gamma \prec \mu$, so we have

$$\mathrm{ext}_\mathfrak{B}^{>0}\left(\mathbb{D}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0}, (\ker f_\mu \otimes \mathbb{C}_{-\Lambda_0})^*\right) = 0.$$

Moreover, Theorem 2.17 gives

$$\mathrm{ext}_\mathfrak{B}^{>0}\left(\mathbb{D}_\lambda^{(1)} \otimes \mathbb{C}_{\Lambda_0}, (D_\mu^{(1)} \otimes \mathbb{C}_{-\Lambda_0})^*\right) = 0.$$

Applying the long exact sequence for $\mathrm{ext}_{\mathfrak{B}}^{\bullet}$ to the dual of the above short exact sequence (after twisting), we conclude that

$$\mathrm{ext}_{\mathfrak{B}}^{>0}\left(\mathbb{D}_{\lambda}^{(1)}\otimes\mathbb{C}_{\Lambda_0},(\mathbb{C}_{\mu-\Lambda_0})^*\right)=0,$$

which completes the induction step and proves (2.2). $\qquad\square$

Corollary 2.19. *Let $\lambda,\mu\in P$ with $-\mu\not\succeq-\lambda$. Then we have*

$$\mathrm{ext}_{\mathfrak{B}}^{>0}\left(\mathbb{D}_{\lambda}^{(1)},\mathbb{D}_{\mu}^{(1)}\right)=0.$$

Moreover, the set $\Psi(\mathbb{D}_{\lambda}^{(1)})$ of weights of $\mathbb{D}_{\lambda}^{(1)}$ satisfies

$$-\overline{\Psi(\mathbb{D}_{\lambda}^{(1)})}\subset\Sigma(-\lambda). \qquad\square$$

Proof. By [15, Proposition 4.17] and [15, Corollary 4.21], we have

$$\mathrm{gch}\, D_{\lambda}^{(1)}=E_{\lambda}(q,0),\qquad\text{and}\qquad(\mathrm{gch}\,\mathbb{D}_{\lambda}^{(1)})^{\dagger}=c_{\lambda}\cdot E_{-\lambda}(q^{-1},\infty)$$

for some unit $c_{\lambda}\in\mathbb{Z}[\![q]\!]^{\times}$, where the involution $\dagger$ sends e^{γ} to $e^{-\gamma}$ for all $\gamma\in P$.

In particular, the set of weights of $\mathbb{D}_{\lambda}^{(1)}$ satisfies

$$\overline{\Psi(\mathbb{D}_{\lambda}^{(1)})}=-\overline{\Psi(D_{-\lambda}^{(1)})}\subset-\Sigma(-\lambda).$$

Now fix $\mu\in P$ such that $-\mu\not\succeq-\lambda$, and consider the module $D_{\mu}^{(1)}$. By construction, $D_{\mu}^{(1)}$ admits a finite filtration with successive quotients of the form $\mathsf{q}^m\mathbb{C}_{\gamma}$ for some $\gamma\preceq\mu$. Moreover, only such $\mathbb{C}_{\gamma}$ can appear in the filtrations of $D_{\nu}^{(1)}$ for $\nu\prec\mu$.

Since negating weights does not affect the order relation $\succeq$ when the two weights lie in different W-orbits, and reverses the order when they lie in the same W-orbit, we deduce that $-\nu\not\succeq-\lambda$.

Now observe that $\mathbb{D}_{\mu}^{(1)}$ admits a decreasing, separated filtration whose associated graded components are of the form $\mathsf{q}^m\mathbb{C}_{\gamma}$ with $-\gamma\not\succeq-\lambda$. Since $\mathrm{ext}_{\mathfrak{B}}^{\bullet}$ commutes with limits in the second variable, it follows that

$$\mathrm{ext}_{\mathfrak{B}}^{>0}\left(\mathbb{D}_{\lambda}^{(1)},\mathbb{D}_{\mu}^{(1)}\right)=0,$$

as desired. $\qquad\square$

2.6 Two technical Lemmas

Proposition 2.20 ([42, Lemma 4.4]; see also [33, 49])**.** *Assume that $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$. Let $\mathcal{D}$ denote the Demazure functor with respect to $\mathfrak{b}$, and let M be a $\mathfrak{b}$-module with semisimple $\mathfrak{h}$-action. Suppose that:*

(i) $\mathbb{L}^{-1}\mathcal{D}(M)=0$, and

(ii) there exists a $\mathfrak{b}$-module embedding $M\hookrightarrow\mathcal{D}(M)$.

Then M admits a finite filtration whose associated graded is a direct sum of modules of the following two types:

1. *irreducible finite-dimensional* $\mathfrak{sl}(2,\mathbb{C})$*-modules;*

2. *one-dimensional* $\mathfrak{b}$*-modules* $\mathbb{C}_\mu$ *with* $\langle \alpha^\vee, \mu \rangle \geq 0$*, where* $\alpha^\vee$ *denote the unique positive coroot of* $\mathfrak{sl}(2,\mathbb{C})$*.*

Proof. Since $\mathbb{L}^{-1}\mathcal{D}(M) = 0$, the derived Demazure functor coincides with its underived version in degree zero. In particular, we have

$$\mathrm{gch}\,\mathcal{D}(M) = \partial\,\mathrm{gch}\,M,$$

where ∂ denotes the Demazure operator associated with $\mathfrak{sl}(2,\mathbb{C})$. Therefore, the assumptions of [42, Lemma 4.4] are satisfied, and the desired conclusion follows. $\qquad\square$

Lemma 2.21. *Let* $i \in \mathtt{I}_{\mathrm{af}}$*, and let* B *be an indecomposable* $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$*-submodule of* $\mathcal{D}_i(M)$*. Suppose* $v \in B$ *is an* $\alpha_i^\vee$*-eigenvector with eigenvalue* t*. Then*

$$v = \sum_{|t| \leq t'} v_{t'},$$

where each $v_{t'}$ *is a nonzero* $\alpha_i^\vee$*-eigenvector with eigenvalue* t *lying in the irreducible* $\mathfrak{sl}(2,i)$*-submodule of* $\mathcal{D}_i(M)$ *with highest weight* $t'\varpi$*, where* ϖ *denotes the fundamental weight of* $\mathfrak{sl}(2,i)$*.*

Proof. Since B is an indecomposable $(\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)$-module, it suffices to assume that v is a cyclic vector with $\alpha_i^\vee$-eigenvalue t.

Each indecomposable $\mathfrak{sl}(2,i)$-summand of $\mathcal{D}_i(M)$ contains an E_i-fixed vector whose $\alpha_i^\vee$-eigenvalue is among the values $t' = |t|, |t|+2, \ldots, |t|+2m$ for some $m \geq 0$, as one sees by inspecting the action of $\mathfrak{sl}(2,i)$.

If some of these irreducible $\mathfrak{sl}(2,i)$-modules do not contribute to v, then the $\mathfrak{sl}(2,i)$-span of B would be strictly smaller than $\mathcal{D}_i(M)$, contradicting the defining property of the Demazure functor. Hence all possible contributions must be nonzero, as claimed. $\qquad\square$

2.7 An estimate of weights in $D_\lambda^{(k)}$

Lemma 2.22. *For each* $\lambda \in P$ *and* $k \in \mathbb{Z}_{\geq 2}$*, the module* $D_\lambda^{(k)}$ *is a quotient of* $D_\lambda^{(k-1)} \otimes \mathbb{C}_{\Lambda_0}$ *in* $\mathfrak{B}$*. Moreover, we have*

$$\overline{\Psi(D_\lambda^{(k)})} \subset \overline{\Psi(D_\lambda^{(1)})} \subset \Sigma(\lambda).$$

Proof. By [33, 3.4 Theorem], the defining relations of $D_\lambda^{(k)}$ are given by

$$e_\alpha^{\max\{-\langle \alpha^\vee, \lambda \rangle, 0\}+1}\mathbf{v} = 0 \qquad\qquad \text{for } \alpha \in \Delta_+,$$
$$e_{\alpha+m\delta}^{\max\{-\langle \alpha^\vee, \lambda \rangle - mk, 0\}+1}\mathbf{v} = 0 \qquad\qquad \text{for } \alpha \in \Delta,\ m \in \mathbb{Z}_{>0}, \qquad (2.3)$$

where $\mathbf{v}$ is the cyclic vector of $\mathfrak{h}$-weight λ.

From this description, it is clear that $D_\lambda^{(k)}$ is a quotient of $D_\lambda^{(k-1)}$. In particular, we obtain the inclusion

$$\Psi(D_\lambda^{(k)}) \subset \Psi(D_\lambda^{(1)}).$$

The further inclusion $\Psi(D_\lambda^{(1)}) \subset \Sigma(\lambda)$ follows from the identification of the graded character of $D_\lambda^{(1)}$ with the non-symmetric Macdonald polynomial $E_\lambda(q, 0)$ (see [32]), together with the weight support estimate in [13, (4.4)], which is part of the characterization of these polynomials. $\square$

Remark 2.23. Joseph's result [33] is stated only for semisimple Lie algebras $\mathfrak{g}$, but the construction and arguments extend without modification to thin Demazure modules for general symmetrizable Kac–Moody algebras. This is because the proof in [33, §4] proceeds by induction on the length of the sequence of simple reflections, starting from the one-dimensional case corresponding to the empty sequence. In contrast, the argument does not generalize to thick Demazure modules.

3 On the structure of affine Demazure modules

We retain the setting of the previous section.

In this section, we establish a technical but crucial structural result concerning affine Demazure modules, stated as Proposition 3.3, referred to as *Reduction* in the Introduction. The proof proceeds by a case-by-case analysis and hinges on features specific to twisted affinizations. While somewhat ad hoc in appearance, this result plays an essential role in the inductive proof of the ext[1]-vanishing theorem (Theorem 6.4) by enabling a reduction from level k to level $k-1$.

More precisely, in the argument for Theorem 6.4, the ext[1]-vanishing for weights $\Lambda \in P_{\mathrm{af},k}^+$ is formally reduced to that for weights of the form $\Lambda - \Lambda_0 \in P_{\mathrm{af},(k-1)}$. Proposition 3.3 enables a further key reduction by replacing this with a vanishing condition for another dominant weight $\Lambda' \in P_{\mathrm{af},(k-1)}^+$, in a way that is compatible with the Demazure module structure.

For $k \in \mathbb{Z}_{\geq 1}$ and $\Lambda \in P_{\mathrm{af},k}^+$, we set

$$S^+(\Lambda) := \{j \in \mathtt{I}_{\mathrm{af}} \mid \langle \alpha_j^\vee, \Lambda \rangle = 0\},$$

and let $S(\Lambda) \subset S^+(\Lambda)$ be the maximal subset that forms a connected subdiagram of the Dynkin diagram of $\widetilde{\mathfrak{g}}$ containing 0. Since $k > 0$, we have $S(\Lambda) \subsetneq \mathtt{I}_{\mathrm{af}}$. Let $W_{S(\Lambda)} := \langle s_j \mid j \in S(\Lambda) \rangle$, and let $\mathfrak{g}_{S(\Lambda)}$ be the simple Lie subalgebra of $\widetilde{\mathfrak{g}}$ generated by $\{E_j, \alpha_j^\vee, F_j\}_{j \in S(\Lambda)}$.

Lemma 3.1. *Let $k \in \mathbb{Z}_{\geq 2}$ and $\Lambda \in P_{\mathrm{af},k}^+$. Then the weight $\Lambda - \Lambda_0$ is a minuscule antidominant weight of $\mathfrak{g}_{S(\Lambda)}$. In particular, $\mathfrak{g}_{S(\Lambda)}$ is not of types E_8, F_4, or G_2.*

Proof. The weight Λ_0, viewed as a weight of $\mathfrak{g}_{S(\Lambda)}$, is the fundamental weight corresponding to the Dynkin vertex 0. Since $S(\Lambda) \subsetneq \mathtt{I}_{\mathrm{af}}$ and $\widetilde{\mathfrak{g}}$ is a twisted affinization, a case-by-case inspection using the classification ([36, Chap. 4, Table Aff] and [6, VIII, §7.2]) shows that Λ_0 is a minuscule dominant weight of $\mathfrak{g}_{S(\Lambda)}$. It follows that $\Lambda - \Lambda_0$ is also a minuscule weight of $\mathfrak{g}_{S(\Lambda)}$, and it is antidominant. This establishes the first assertion.

The second assertion follows from the fact that the simple Lie algebras of types E_8, F_4, and G_2 admit no nonzero minuscule weights (see the above classification). $\square$

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Lemma 3.2. *Let $k \in \mathbb{Z}_{\geq 2}$. Suppose that $\Lambda \in P_{\mathrm{af},k}^+$ and $w \in W_{S(\Lambda)}$ satisfy $\langle \vartheta^\vee, \overline{\Lambda} \rangle = k$ and $w^{-1}(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$. In addition, assume that w is the minimal element with this property. Then $\ell(s_0 w) < \ell(w)$, and we have*

$$\mathcal{D}_w(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)}) \big/ \mathcal{D}_{s_0 w}(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)}) \cong \mathbb{C}_{\Lambda - \Lambda_0}.$$

Proof. Since $w^{-1}(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$ and

$$s_0 w w^{-1}(\Lambda - \Lambda_0) = (\Lambda - \Lambda_0) + \alpha_0 = w w^{-1}(\Lambda - \Lambda_0) + \alpha_0,$$

we deduce that $s_0 w < w$, which implies $\ell(s_0 w) < \ell(w)$. This proves the first assertion.

Let $S := S(\Lambda)$. By Lemma 3.1, every weight space of $\mathcal{D}_w(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)})$ is one-dimensional and of the form $v(\Lambda - \Lambda_0)$ for some $v \in W_S$. Thus, the second assertion holds if every such $v(\Lambda - \Lambda_0) \neq \Lambda - \Lambda_0$ yields a nonzero weight space in $\mathcal{D}_{s_0 w}(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)})$.

Since $\Lambda - \Lambda_0$ is antidominant for $\mathfrak{g}_S$, there exists a minimal sequence

$$i_1, i_2, \ldots, i_t \in S$$

such that

$$(s_{i_1} s_{i_2} \cdots s_{i_t}) v(\Lambda - \Lambda_0) = \Lambda - \Lambda_0, \quad \text{provided } v(\Lambda - \Lambda_0) \neq \Lambda - \Lambda_0.$$

By the minimality of t, we must have $i_1 = 0$, so that

$$(s_{i_2} \cdots s_{i_t}) v(\Lambda - \Lambda_0) = s_0(\Lambda - \Lambda_0).$$

Moreover, we may replace v with $(s_{i_1} s_{i_2} \cdots s_{i_t})^{-1}$ without loss of generality. Then, by the minimality assumption on w, we have $\ell(vw) = \ell(w) - \ell(v)$, and

$$s_{i_2} \cdots s_{i_t} \cdot vw = s_0 w.$$

Therefore,

$$\mathbb{C}_{v(\Lambda - \Lambda_0)} \subset (\mathcal{D}_{i_2} \circ \cdots \circ \mathcal{D}_{i_t})\big(\mathcal{D}_{vw}(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)})\big) = \mathcal{D}_{s_0 w}(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)}).$$

Hence, each such $v(\Lambda - \Lambda_0)$ appears as a weight of $\mathcal{D}_{s_0 w}(\mathbb{C}_{w^{-1}(\Lambda - \Lambda_0)})$, as desired. $\square$

Proposition 3.3. *Let $k \in \mathbb{Z}_{\geq 2}$. For each $\Lambda \in P_{\mathrm{af},k}^+$ satisfying $\langle \vartheta^\vee, \overline{\Lambda} \rangle = k$, there exist $w \in W_{\mathrm{af}}$ and $\Lambda' \in P_{\mathrm{af},(k-1)}^+$ such that*

$$\mathcal{D}_{s_0 w}(\mathbb{C}_{\Lambda'}) \big/ \mathcal{D}_w(\mathbb{C}_{\Lambda'}) \cong \mathbb{C}_{\Lambda - \Lambda_0}. \tag{3.1}$$

Proof. We have

$$s_0(\Lambda - \Lambda_0) = \Lambda - \Lambda_0 + \alpha_0,$$

since $\langle \alpha_0^\vee, \Lambda - \Lambda_0 \rangle = 0 - 1 = -1$. If $s_0(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$, then

$$\mathcal{D}_0(\mathbb{C}_{s_0(\Lambda - \Lambda_0)}) \cong \mathbb{C}_{\Lambda - \Lambda_0} \oplus \mathbb{C}_{s_0(\Lambda - \Lambda_0)}.$$

Thus, the assertion holds in this case.

This includes, in particular, the case when $\mathfrak{g}$ is of type A_1. In the remainder of the proof, we assume that $\mathfrak{g}$ is not of type A_1.

We now assume that $s_0(\Lambda - \Lambda_0) \notin P^+_{\text{af},(k-1)}$. Thus, the possible values of $j \in \mathtt{I}_{\text{af}}$ satisfying $\langle \alpha_j^\vee, s_0(\Lambda - \Lambda_0) \rangle < 0$ are as follows.

If $\mathfrak{g}$ is of type A_n, then $j = 1$ or $j = n$ (cf. [6, p. 206 (VI)], with $n = \ell$). In all other types, $j = i$ for a uniquely determined i, by inspection (cf. [36, Table Aff]).

Suppose that $\langle \alpha_j^\vee, \Lambda \rangle > 0$ for all such possible j. Then necessarily $s_0(\Lambda - \Lambda_0) \in P^+_{\text{af},(k-1)}$, contradicting our assumption.

Hence, we must have $\langle \alpha_i^\vee, \Lambda \rangle = 0$ for at least one such i: specifically, $i = 1$ or $i = n$ in type A_n, and a uniquely determined i in all other types.

We write S for $S(\Lambda)$ for the remainder of the proof.

By Lemma 3.1, the weight $\Lambda - \Lambda_0$ is an antidominant minuscule weight (corresponding to the vertex 0) of $\mathfrak{g}_S$, and the subdiagram that appears in this construction is of type A, B, C, D, E_6, or E_7.

Hence, there exists a minimal length element $w \in W_S$ such that $w^{-1}(\Lambda - \Lambda_0)$ is a dominant minuscule (fundamental) weight of $\mathfrak{g}_S$.

If $w^{-1}(\Lambda - \Lambda_0) \in P^+_{\text{af},(k-1)}$, then the assertion follows from Lemma 3.2.

Thus, we must determine whether $w^{-1}(\Lambda - \Lambda_0) \in P^+_{\text{af},(k-1)}$, which we examine on a case-by-case basis on the shape of the Dynkin subdiagram determined by S.

We first remark that if $j \in \mathtt{I}_{\text{af}} \setminus S$ is adjacent to a vertex of S in the extended Dynkin diagram, then $\langle \alpha_j^\vee, \Lambda \rangle \geq 1$. Moreover, if $|\mathtt{I}_{\text{af}}| - |S| = 1$ and

$$\langle \vartheta^\vee, \varpi_j \rangle = 1 \quad \Leftrightarrow \quad \vartheta^\vee \in \alpha_j^\vee + \sum_{r \in \mathtt{I}, \, r \neq j} \mathbb{Z}_{\geq 0} \alpha_r^\vee, \tag{3.2}$$

then we obtain

$$\langle \alpha_j^\vee, \Lambda \rangle = \langle \vartheta^\vee, \Lambda \rangle = k \geq 2.$$

The condition (3.2) holds automatically when $\mathfrak{g}$ is of type A, or when $\mathfrak{g}$ is of type B, C, or D and $j \in \mathtt{I}$ corresponds to a short root connected to a unique other vertex in the (non-extended) Dynkin diagram, by inspection.

To distinguish between the original Dynkin indices and their rearrangement to match the conventions of [6], we may denote the key original Dynkin indices—namely 0 (the affine node) and i (a node adjacent to 0)—by $\underline{0}$ and $\underline{i}$, respectively.

Type A_l: $\underline{0}$ has degree one in S We then label the vertices as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad 1 = \underline{0}, \quad 2 = \underline{i},$$

so that j and $j-1$ are adjacent in S.

In this case, the weight $\Lambda - \Lambda_0$ is the lowest weight of the vector representation of type A_l. We then have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \sum_{j=1}^l \alpha_j.$$

For each $j \in S$ and $t \in I_{\mathrm{af}} \setminus S$, we have $\langle \alpha_t^\vee, \alpha_j \rangle \in \{0, -1\}$. If $j \in I_{\mathrm{af}} \setminus S$ is adjacent to two simple roots in S (which is the maximal possible), then necessarily $\mathfrak{g}$ is of type A_n and $|I_{\mathrm{af}}| - |S| = 1$.

Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$.

Type A_l: $\underline{0}$ has degree two in S We then label the vertices as

$$S = \{-r_1, -1, 0, 1, 2, \ldots, r_2\},$$

so that j and $j-1$ are adjacent for each $j \in S$.

In this case, the weight $\Lambda - \Lambda_0$ corresponds to the minuscule representation of type $A_{r_1+r_2+1}$, and $\mathfrak{g}$ is necessarily of type A_n. We have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \alpha_{-r_1} + 2\alpha_{-r_1+1} + \cdots + \alpha_{r_2}.$$

Each $j \in I_{\mathrm{af}} \setminus S$ that is adjacent to S in the extended Dynkin diagram is adjacent either to $-r_1$ or to r_2. Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$.

Type D_l: $\underline{i}$ has degree three in S We label the vertices as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad l = \underline{0}, \quad l-2 = \underline{i},$$

so that j and $j-1$ are adjacent for $j < l$, and l is adjacent to $l-2$.

This situation occurs only when $\mathfrak{g}$ is of type C or D. In this case, the weight $\Lambda - \Lambda_0$ is the lowest weight of a half-spin representation. According to [6, p. 209 (VI)], we have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \varpi_l + \varpi_{l-1} = \alpha_1 + 2\alpha_2 + \cdots. \qquad (3.3)$$

A vertex in $I_{\mathrm{af}} \setminus S$ can be adjacent only to 1 or 2 in the extended Dynkin diagram. The latter case occurs only when $|I_{\mathrm{af}}| - |S| = 1$ and $\mathfrak{g}$ is of type D. Moreover, α_1 must be a short root.

Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$.

Type D_l: $\underline{i}$ has degree two in S We then label the vertices as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad 1 = \underline{0}, \quad 2 = \underline{i},$$

so that j and $j-1$ are adjacent for $j < l$, and $(l-2)$ is adjacent to l.

This situation occurs only when $\mathfrak{g}$ is of type D or E. In this case, the weight $\Lambda - \Lambda_0$ is the lowest weight of a minuscule representation. According to [6, p. 209 (VI)], we have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = 2\varpi_1 = 2\alpha_1 + 2\alpha_2 + \cdots + \alpha_{l-1} + \alpha_l.$$

If $\mathfrak{g}$ is of type D_n, then $|I_{\mathrm{af}}| - |S| = 1$, and the unique vertex in $I_{\mathrm{af}} \setminus S$ is connected to vertex 2 in the extended Dynkin diagram; the corresponding root lengths are equal. If $\mathfrak{g}$ is of type E, then all external vertices are connected only to $l-1$ or l.

In either case, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P_{\mathrm{af},(k-1)}^+$.

Type B_l case We label the vertices as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad l = \underline{0}, \quad l-1 = \underline{i},$$

so that the vertices j and $j-1$ are adjacent in S for each $j > 1$, in accordance with [6, p. 202 (IV)].

In this case, the node l corresponds to the short root, and this situation occurs only when $\mathfrak{g}$ is of type B_n. Moreover, the weight $\Lambda - \Lambda_0$ is the lowest weight of a minuscule representation of $\mathfrak{g}_S$. Thus, we have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = 2\varpi_l = \alpha_1 + 2\alpha_2 + \cdots.$$

A vertex in $\mathtt{I}_{\mathrm{af}} \setminus S$ can be connected only to vertex 1 in the extended Dynkin diagram. The corresponding roots have different lengths only when $|\mathtt{I}_{\mathrm{af}}| - |S| = 1$, in which case α_1 is the long root.

Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P^+_{\mathrm{af}, (k-1)}$.

Type C_l case We label the vertices as

$$S = \{1, 2, \ldots, l\} \quad \text{with} \quad 1 = \underline{0}, \quad 2 = \underline{i},$$

so that the vertices j and $j-1$ are adjacent in S for all $j > 1$, and l corresponds to the long root. (This occurs only when $\mathfrak{g}$ is of type C or F.)

In this case, the weight $\Lambda - \Lambda_0$ is the lowest weight of the vector representation of $\mathfrak{g}_S$. Thus, we have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \alpha_l + \sum_{j=1}^{l-1} 2\alpha_j.$$

If l is connected to some $t \in \mathtt{I}_{\mathrm{af}} \setminus S$ in the extended Dynkin diagram, then α_l and α_t have the same length. If another vertex $j \in S$, with $j \neq l$, is connected to $t \in \mathtt{I}_{\mathrm{af}} \setminus S$, then $\mathfrak{g}$ must be of type C_n, $|\mathtt{I}_{\mathrm{af}}| - |S| = 1$, and α_j and α_t again have the same length.

Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P^+_{\mathrm{af}, (k-1)}$.

Type E_6 case Then $\mathfrak{g}$ must also be of type E_6, and in particular we have $|\mathtt{I}_{\mathrm{af}}| - |S| = 1$.

We label the vertices as

$$S = \{1, 2, \ldots, 6\} \quad \text{with} \quad 1 = \underline{0}, \quad 3 = \underline{i},$$

in accordance with [6, p. 218 (IV)], where vertex 4 is the trivalent node.

In this case, the weight $\Lambda - \Lambda_0$ is the lowest weight of the minuscule representation. Thus, we have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = \varpi_1 + \varpi_6 = 2\alpha_1 + 2\alpha_2 + \cdots.$$

Moreover, vertex 2 is the only node in S that is adjacent to the unique vertex $j \in \mathtt{I}_{\mathrm{af}} \setminus S$ in the extended Dynkin diagram. We have

$$\langle \vartheta^\vee, \varpi_j \rangle = 1$$

by [6, p. 218 (IV)].

Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P^+_{\mathrm{af}, (k-1)}$.

Type E_7 case Then $\mathfrak{g}$ must also be of type E_7, and in particular we have $|\mathtt{I}_{\mathrm{af}}| - |S| = 1$.

We label the vertices as

$$S = \{1, 2, \ldots, 7\} \quad \text{with} \quad 7 = \underline{0}, \quad 6 = \underline{i},$$

in accordance with [6, p. 216 (IV)]. Here 4 is again the trivalent vertex.

In this case, the weight $\Lambda - \Lambda_0$ is the lowest weight of the minuscule representation. We have

$$w^{-1}(\Lambda - \Lambda_0) - (\Lambda - \Lambda_0) = 2\varpi_7 = 2\alpha_1 + 3\alpha_2 + \cdots.$$

Moreover, vertex 1 is the only node in S that is adjacent to the unique vertex in $\mathtt{I}_{\mathrm{af}} \setminus S$ in the extended Dynkin diagram. We have

$$\langle \vartheta^\vee, \varpi_j \rangle = 1,$$

as recorded in [6, p. 216 (IV)].

Therefore, since $k \geq 2$, we conclude that $w^{-1}(\Lambda - \Lambda_0) \in P^+_{\mathrm{af},(k-1)}$.

This completes the proof. $\qquad\square$

4 A lifting theorem

We continue to work in the setting of the previous section.

Theorem 4.1. *Let* $M, N \in \mathfrak{B}_{\mathrm{bdd}}$ *be modules on which* K *acts by* $-k$ *and* k, *respectively. Suppose that for some* $i \in \mathtt{I}_{\mathrm{af}}$, *we have*

$$M \subset \mathcal{D}_i(M), \qquad N \subset \mathcal{D}_i(N), \qquad \text{and} \qquad \mathbb{L}^{<0}\mathcal{D}_i(M) = \mathbb{L}^{<0}\mathcal{D}_i(N) = 0.$$

Assume further that N *admits a* $\widetilde{\mathfrak{b}}$-*cyclic* $\widetilde{\mathfrak{h}}$-*eigenvector* $\mathbf{v}$. *Then the natural map*

$$\hom_{\mathfrak{B}}(\mathcal{D}_i(M), N^\vee) \longrightarrow \hom_{\mathfrak{B}}(M, N^\vee)$$

is surjective.

Remark 4.2. Theorem 4.1 extends naturally to the setting of an arbitrary symmetrizable Kac–Moody algebra.

Proof of Theorem 4.1. If either $\mathcal{D}_i(M) \cong M$ or $\mathcal{D}_i(N) \cong N$, then the assertion follows from Theorem 1.16 together with the idempotence $\mathcal{D}_i^2 \cong \mathcal{D}_i$.

Using the identification (1.5), it suffices to prove the surjectivity of the dual map

$$\hom_{\mathfrak{B}}(N, \mathcal{D}_i(M)^\vee) \longrightarrow \hom_{\mathfrak{B}}(N, M^\vee). \tag{4.1}$$

By Proposition 2.20 and the isomorphism $U(\mathfrak{g}_{\alpha_i}) \cong \mathbb{C}[E_i]$, the module N decomposes as a direct sum of indecomposable $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$-modules, each of which is isomorphic to an irreducible $\mathfrak{sl}(2, i)$-module tensored with a one-dimensional $\widetilde{\mathfrak{h}}$-module of weight λ satisfying $\langle \alpha_i^\vee, \lambda \rangle \geq 0$. Similarly, M admits a decomposition into a direct sum of modules of the same type.

Consider a nonzero morphism of $\widetilde{\mathfrak{b}}$-modules

$$f : N \longrightarrow M^\vee.$$

Let B be an indecomposable direct summand of N as a $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$-module, viewed as a module over $\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee$. Then B has head $\mathbb{C}_{r\varpi}$ and socle $\mathbb{C}_{s\varpi}$ for some integers $r \leq s$.

Let L be an indecomposable direct summand of M as a $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$-module, regarded also as a $(\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)$-module. Then L is isomorphic to an irreducible $\mathfrak{sl}(2, i)$-module tensored with a one-dimensional $(\widetilde{\mathfrak{h}} + \mathbb{C}E_i)$-module $\mathbb{C}_\lambda$, on which $\widetilde{\mathfrak{h}}$ acts by λ with $\langle \alpha_i^\vee, \lambda \rangle \geq 0$. Consequently, $L^* \subset M^\vee$ is obtained by tensoring its dual $\mathbb{C}_{-\lambda}$ with an irreducible $\mathfrak{sl}(2, i)$-module.

Let f_L denote the composition

$$N \longrightarrow M^\vee \longrightarrow L^*.$$

As the image of B under f_L, the module $f_L(B)$ is either zero or a quotient of B as a $(\mathbb{C}E_i \oplus \mathbb{C}\alpha_i^\vee)$-module. In the latter case, it has head $\mathbb{C}_{r\varpi}$ and socle $\mathbb{C}_{s'\varpi}$ for some $s' \leq s$.

Using the natural embedding $N \subset \mathcal{D}_i(N)$, we consider the $\mathfrak{sl}(2, i)$-submodule $\widetilde{B} \subset \mathcal{D}_i(N)$ generated by B. By the defining property of $\mathcal{D}_i$, we have $\mathcal{D}_i(B) = \widetilde{B}$. It follows that $\widetilde{B}$ decomposes as a direct sum of irreducible $\mathfrak{sl}(2, i)$-modules with highest weights

$$s\varpi, \ (s-2)\varpi, \ \ldots, \ (|r|+2)\varpi, \ |r|\varpi. \tag{4.2}$$

We now examine whether the map f_L extends from B to $\widetilde{B}$.

If $s' > |r|$, then there exists a unique irreducible $\mathfrak{sl}(2, i)$-direct summand of $\widetilde{B}$ with highest weight $s'\varpi$ that admits an injection into L^*. This summand contains the image $f_L(B)$, since $s'\varpi$ (and hence $-s'\varpi$) is a weight of L^*.

If $s' \leq |r|$, then there exists a unique irreducible $\mathfrak{sl}(2, i)$-direct summand of $\widetilde{B}$ with highest weight $|r|\varpi$ that surjects onto $f_L(B) \subset L^*$.

In either case, we obtain a nonzero map $\widetilde{f}_L$ from $\widetilde{B}$ to L^*.

By Lemma 2.21, the map $\widetilde{f}_L$ can be rearranged uniquely to satisfy $\widetilde{f}_L|_B = f_L$.

Now set $B_0 := U(\mathbb{C}E_i)\,\mathbf{v}$. This is a direct summand of N as a $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$-module, by $\widetilde{\mathfrak{h}}$-weight considerations. In particular, its $\mathfrak{sl}(2, i)$-span $\widetilde{B}_0 \subset \mathcal{D}_i(N)$ admits a lift

$$\widetilde{f} : \widetilde{B}_0 \longrightarrow M^\vee$$

of $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$-modules such that its restriction to B_0 coincides with $f|_{B_0}$.

Let

$$\mathfrak{u} := \bigoplus_{\beta \in \Delta_{\mathrm{af}}^+ \setminus \{\alpha_i\}} \widetilde{\mathfrak{g}}_\beta \subset \widetilde{\mathfrak{n}}.$$

Since $U(\widetilde{\mathfrak{n}}) \cong U(\mathfrak{u}) \otimes U(\mathbb{C}E_i)$, we can regard

$$U(\mathfrak{u}) \otimes_\mathbb{C} B_0 \qquad \text{and} \qquad U(\mathfrak{u}) \otimes_\mathbb{C} \widetilde{B}_0$$

as $\widetilde{\mathfrak{b}}$-modules, which are projective as $U(\mathfrak{u})$-modules. Moreover, $\mathfrak{u}$ is a $\mathfrak{sl}(2, i)$-integrable $\widetilde{\mathfrak{p}}_i$-module by inspection.

Therefore, we have

$$U(\mathfrak{u}) \otimes_\mathbb{C} \widetilde{B}_0 \ \cong \ U(\mathfrak{u}) \otimes_\mathbb{C} \mathcal{D}_i(B_0) \ \cong \ \mathcal{D}_i(U(\mathfrak{u}) \otimes_\mathbb{C} B_0).$$

In particular, we obtain surjections of $\widetilde{\mathfrak{b}}$-modules

$$U(\mathfrak{u}) \otimes_{\mathbb{C}} B_0 \longrightarrow N, \qquad \text{and} \qquad U(\mathfrak{u}) \otimes_{\mathbb{C}} \widetilde{B}_0 \longrightarrow \mathcal{D}_i(N).$$

Set

$$I := \ker\big(U(\mathfrak{u}) \otimes_{\mathbb{C}} B_0 \longrightarrow N\big).$$

By assumption, we have $\mathbb{L}^{-1}\mathcal{D}_i(N) = 0$. It follows that the exact sequence

$$0 = \mathbb{L}^{-1}\mathcal{D}_i(N) \longrightarrow \mathcal{D}_i(I) \longrightarrow U(\mathfrak{u}) \otimes_{\mathbb{C}} \widetilde{B}_0 \longrightarrow \mathcal{D}_i(N) \longrightarrow 0$$

is short exact.

By Theorem 1.16, giving a $\widetilde{\mathfrak{b}}$-module map

$$\widetilde{f} : \mathcal{D}_i(N) \longrightarrow M^{\vee}$$

is equivalent to giving a $\widetilde{\mathfrak{b}}$-module map

$$\widetilde{f}' : \mathcal{D}_i(N) \longrightarrow \mathcal{D}_i(M)^{\vee}.$$

Since $\mathcal{D}_i(M)^{\vee} \to M^{\vee}$ is surjective by assumption, it suffices to construct a map $\widetilde{f}'$ lifting f, that is, a map

$$\widetilde{\psi} : U(\mathfrak{u}) \otimes_{\mathbb{C}} \widetilde{B}_0 \longrightarrow \mathcal{D}_i(M)^{\vee}$$

such that $\widetilde{\psi}\big(\mathcal{D}_i(I)\big) = 0$.

Now, the map f lifts to a map

$$\psi : U(\mathfrak{u}) \otimes_{\mathbb{C}} B_0 \longrightarrow M^{\vee}$$

satisfying $\psi(I) = 0$. Let

$$\widetilde{\psi}' : U(\mathfrak{u}) \otimes_{\mathbb{C}} \widetilde{B}_0 \longrightarrow M^{\vee}$$

be the unique lift of ψ induced by the universal property of the $(\mathbb{C}E_i + \widetilde{\mathfrak{h}})$-module lift $B_0 \to \widetilde{B}_0$. Then $\widetilde{\psi}'(I) = 0$ holds.

By applying the correspondence given by Theorem 1.16, we obtain a map

$$\widetilde{\psi}'' : U(\mathfrak{u}) \otimes_{\mathbb{C}} \widetilde{B}_0 \longrightarrow \mathcal{D}_i(M)^{\vee}$$

corresponding to $\widetilde{\psi}'$, which satisfies $\widetilde{\psi}''\big(\mathcal{D}_i(I)\big) = 0$, since $\mathcal{D}_i$ sends zero maps to zero maps.

Therefore, we may take $\widetilde{\psi} := \widetilde{\psi}''$ as the desired lift. This defines the required map $\widetilde{f}'$, and hence the original map $\widetilde{f}$.

This completes the proof. $\qquad\qquad\square$

Corollary 4.3. *Let $k, l \in \mathbb{Z}_{\geq 0}$ with $k \geq l$, and let $M \in \mathfrak{B}_{\mathrm{bdd}}$ be a module on which K acts by $-k$. Suppose that for some $i \in \mathtt{I}_{\mathrm{af}}$, we have*

$$M \subset \mathcal{D}_i(M), \qquad \text{and} \qquad \mathbb{L}^{<0}\mathcal{D}_i(M) = 0.$$

Then, for each $\mu \in P$, there is a surjection:

$$\hom_{\mathfrak{B}}\big(\mathcal{D}_i(M), (D_{\mu}^{(l)} \otimes \mathbb{C}_{(k-l)\Lambda_0})^*\big) \longrightarrow \hom_{\mathfrak{B}}\big(M, (D_{\mu}^{(l)} \otimes \mathbb{C}_{(k-l)\Lambda_0})^*\big).$$

Proof. For each $i \in \mathtt{I}_{\mathrm{af}}$, we have

$$D_{\mu}^{(l)} \subset \mathcal{D}_i(D_{\mu}^{(l)}) \equiv \mathbb{L}^{\bullet}\mathcal{D}_i(D_{\mu}^{(l)})$$

by Theorem 2.2. For $i = 0$, we apply Proposition 2.20 to verify

$$D_{\mu}^{(l)} \otimes \mathbb{C}_{(k-l)\Lambda_0} \subset \mathcal{D}_i(D_{\mu}^{(l)} \otimes \mathbb{C}_{(k-l)\Lambda_0}) \equiv \mathbb{L}^{\bullet}\mathcal{D}_i(D_{\mu}^{(l)} \otimes \mathbb{C}_{(k-l)\Lambda_0}).$$

Hence the assertion follows from Theorem 4.1. $\qquad\qquad\square$

5 Statement of the Main Theorems

Keep the setting of the previous section. This section presents the main results of the paper and discusses some of their consequences. The proofs will be given in the subsequent sections.

5.1 ext-orthogonality theorem

Theorem 5.1 (Level k-duality). *Let $k \in \mathbb{Z}_{>0}$. Then the following statements hold:*

1. *For each $\lambda, \mu \in P$, we have*

$$\mathrm{ext}_{\mathfrak{B}}^i(\mathbb{D}_\lambda^{(k)}, \ (D_\mu^{(k)})^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \lambda + \mu = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (5.1)$$

2. *For each $\lambda, \mu \in P^+$, we have*

$$\mathrm{ext}_{\mathfrak{C}}^i(\mathbb{W}_\lambda^{(k)}, \ (W_\mu^{(k)})^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ and } \lambda + \mu_- = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (5.2)$$

The case $k = 1$ of Theorem 5.1(1) corresponds to [15, Theorem 4.19] (Theorem 2.17). The case $k = 1$ of Theorem 5.1(2) follows from Theorem 5.1(1), as explained in the final paragraph of its proof.

Let us record a consequence of Theorem 5.1.

Proposition 5.2. *Let $k \in \mathbb{Z}_{>0}$, and assume that Theorem 5.1 holds for level k when $k \geq 2$. For each $\lambda \in P$, any proper quotient M of either $\mathbb{D}_\lambda^{(k)}$ or $D_\lambda^{(k)}$ in $\mathfrak{B}$ satisfies*

$$\bigoplus_{\mu \in P} \mathrm{ext}_{\mathfrak{B}}^1 \left(M, \ (D_\mu^{(k)})^* \right) \neq 0 \quad or \quad \bigoplus_{\mu \in P} \mathrm{ext}_{\mathfrak{B}}^1 \left(\mathbb{D}_\mu^{(k)}, \ M^* \right) \neq 0, \qquad (5.3)$$

respectively. Moreover, in each case, one can find an extension of M by the socle of $(D_\mu^{(k)})^*$ or $(\mathbb{D}_\mu^{(k)})^\vee$ that realizes a nonzero class in (5.3). Similarly, for each $\lambda \in P^+$, any proper quotient M of either $\mathbb{W}_\lambda^{(k)}$ or $W_\lambda^{(k)}$ in $\mathfrak{C}$ satisfies

$$\bigoplus_{\mu \in P^+} \mathrm{ext}_{\mathfrak{B}}^1 \left(M, \ (W_\mu^{(k)})^* \right) \neq 0 \quad or \quad \bigoplus_{\mu \in P^+} \mathrm{ext}_{\mathfrak{B}}^1 \left(\mathbb{W}_\mu^{(k)}, \ M^* \right) \neq 0,$$

respectively.

Proof. Suppose that M is a proper quotient of $\mathbb{D}_\lambda^{(k)}$ with its quotient map f. The module $\ker f$ is generated by (any $\widetilde{\mathfrak{h}}$-splitting of) its head, which contains a ($\widetilde{\mathfrak{b}}$-module direct) summand of the form $\mathsf{q}^m \mathbb{C}_{\gamma - k\Lambda_0}$ for some $m \in \mathbb{Z}$ and $\gamma \in P$. This gives rise to an extension M^+ of M by $\mathsf{q}^m \mathbb{C}_{\gamma - k\Lambda_0}$, which remains a quotient of $\mathbb{D}_\lambda^{(k)}$ since $\ker f \neq 0$.

Note that $\mathsf{q}^m \mathbb{C}_{\gamma - k\Lambda_0}$ appears in the socle of $\mathsf{q}^m (D_{-\gamma}^{(k)})^*$. Consider the module $M^\sharp$ defined by the following short exact sequence:

$$0 \to \mathsf{q}^m \mathbb{C}_{\gamma - k\Lambda_0} \to M^+ \oplus \mathsf{q}^m (D_{-\gamma}^{(k)})^* \xrightarrow{g} M^\sharp \to 0,$$

where the map from $\mathsf{q}^m \mathbb{C}_{\gamma - k\Lambda_0}$ is given by the anti-diagonal embedding. This sequence induces a short exact sequence

$$0 \to \mathsf{q}^m (D_{-\gamma}^{(k)})^* \to M^\sharp \to M \to 0. \tag{5.4}$$

To show that $\mathrm{ext}^1_{\mathfrak{B}}(M, (D_{-\gamma}^{(k)})^*) \neq 0$, it suffices to prove that the sequence (5.4) does not split by the Yoneda interpretation (§1.3.6).

Assume, for contradiction, that (5.4) does split. Then there exists a nonzero map $h : M^\sharp \to \mathsf{q}^m (D_{-\gamma}^{(k)})^*$, which lifts to a map from $M^+ \oplus \mathsf{q}^m (D_{-\gamma}^{(k)})^*$ by precomposition of h with g. By examining the behavior of

$$\mathsf{q}^m \mathbb{C}_{\gamma - k\Lambda_0} \cong \ker(M^+ \to M) \subset M^+$$

under $h \circ g$, we find that $M^+ \to \mathsf{q}^m (D_{-\gamma}^{(k)})^*$ is nonzero. This in turn induces a nonzero map $\mathbb{D}_\lambda^{(k)} \to \mathsf{q}^m (D_{-\gamma}^{(k)})^*$ via precomposition of $h \circ g$ with f.

By Theorem 5.1—which holds either by assumption ($k \geq 2$) or by [15] ($k = 1$)—such a map exists only if $\lambda = \gamma$, and its image is necessarily isomorphic to the head of $\mathbb{D}_\lambda^{(k)}$. In particular, the image must be one-dimensional. However, such a space can appear in the generator of $\ker f$ only if $M = 0$, contradicting the assumption that M is a proper quotient. Hence, the sequence (5.4) does not split, and the desired nontrivial extension class is realized by $M^\sharp$.

The case where M is a proper quotient of $D_\lambda^{(k)}$ is dual to the above, using the duality isomorphism (1.5) and the Yoneda interpretation (§1.3.6).

Finally, the cases where M is a proper quotient of either $\mathbb{W}_\lambda^{(k)} = \mathbb{L}^\bullet \mathcal{D}_{w_0}(\mathbb{D}_\lambda^{(k)})$ or $W_\lambda^{(k)} = D_{w_0 \lambda}^{(k)}$ reduce to the above cases, since we have $\mathbb{L}^\bullet \mathcal{D}_{w_0}(M) = M$, and Theorem 1.16 ensures that $\mathcal{D}_{w_0}$ can be applied to each factor. $\qquad\square$

5.2 Filtration criteria

Definition 5.3. Let $k \in \mathbb{Z}_{>0}$.

A module $M \in \mathfrak{B}_{\mathrm{bdd}}$ is said to admit a $D^{(k)}$-filtration (resp. a $\mathbb{D}^{(k)}$-filtration) if there exists a decreasing separable filtration

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots, \qquad \bigcap_{i \geq 1} M^i = 0$$

by $\widetilde{\mathfrak{b}}$-submodules, such that for each $i \in \mathbb{Z}_{\geq 0}$, the successive quotient M^i / M^{i+1} is isomorphic to $\mathsf{q}^{m_i} D_{\mu_i}^{(k)}$ (resp. $\mathsf{q}^{m_i} \mathbb{D}_{\mu_i}^{(k)}$), for some $m_i \in \mathbb{Z}$ and $\mu_i \in P$, up to a twist by $\mathbb{C}_{l\Lambda_0}$ for some $l \in \mathbb{Z}$.

Similarly, a module $M \in \mathfrak{C}_{\mathrm{bdd}}$ is said to admit a $W^{(k)}$-filtration (resp. a $\mathbb{W}^{(k)}$-filtration) if there exists a decreasing separable filtration

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots, \qquad \bigcap_{i \geq 1} M^i = 0$$

by $\widetilde{\mathfrak{g}}_{\geq 0}$-submodules, such that for each $i \in \mathbb{Z}_{\geq 0}$, the successive quotient M^i / M^{i+1} is isomorphic to $\mathsf{q}^{m_i} W_{\mu_i}^{(k)}$ (resp. $\mathsf{q}^{m_i} \mathbb{W}_{\mu_i}^{(k)}$), for some $m_i \in \mathbb{Z}$ and $\mu_i \in P^+$, again up to a twist by $\mathbb{C}_{l\Lambda_0}$ for some $l \in \mathbb{Z}$.

Suppose that M admits an X-filtration for some $X \in \{\mathbb{D}^{(k)}, D^{(k)}, \mathbb{W}^{(k)}, W^{(k)}\}$. Then we define the collection of elements $(M : X_\lambda)_q \in \mathbb{Z}(\!(q)\!)$ by the expansion

$$\mathrm{gch}\, M = \sum_\lambda (M : X_\lambda)_q \cdot \mathrm{gch}\, X_\lambda, \qquad (5.5)$$

where the sum is taken over $\lambda \in P$ or $\lambda \in P^+$, depending on whether $X = \mathbb{D}^{(k)}, D^{(k)}$ or $X = \mathbb{W}^{(k)}, W^{(k)}$, respectively. By Lemma 2.16, these coefficients are uniquely determined. They record the graded multiplicities of the modules X_λ in M with respect to the X-filtration.

Similarly, for each $M \in \mathfrak{C}_{\mathrm{bdd}}$, we write $[M : V_\lambda]_q \in \mathbb{Z}_{\geq 0}(\!(q)\!)$ for the graded multiplicity of V_λ in M. Namely, we have

$$\mathrm{gch}\, M = \sum_\lambda [M : V_\lambda]_q \cdot \mathrm{ch}\, V_\lambda.$$

The subscript q may be replaced by another variable (e.g., q^{-1}) to indicate a change of grading. The notation $(\cdot)_{q^{-1}}$ is used to indicate inversion of the grading parameter, which often arises in dualities. In particular, we have

$$(M : X_\lambda)_{q^{-1}} = (M : X_\lambda)_q \Big|_{q \mapsto q^{-1}}, \quad [M : V_\lambda]_{q^{-1}} = [M : V_\lambda]_q \Big|_{q \mapsto q^{-1}}. \qquad (5.6)$$

Theorem 5.4 (Level k criterion for filtrations). *Let $k \in \mathbb{Z}_{>0}$. Then the following statements hold:*

1. *A module $M \in \mathfrak{B}_{\mathrm{bdd}}$ admits a $D^{(k)}$-filtration if and only if*

$$\mathrm{ext}^1_{\mathfrak{B}}(\mathbb{D}^{(k)}_\lambda,\ M^\vee) = 0 \qquad \text{for all } \lambda \in P. \qquad (5.7)$$

2. *A module $M \in \mathfrak{B}_{\mathrm{bdd}}$ admits a $\mathbb{D}^{(k)}$-filtration if and only if*

$$\mathrm{ext}^1_{\mathfrak{B}}(M,\ (D^{(k)}_\lambda)^*) = 0 \qquad \text{for all } \lambda \in P. \qquad (5.8)$$

3. *A module $M \in \mathfrak{C}_{\mathrm{bdd}}$ admits a $W^{(k)}$-filtration if and only if*

$$\mathrm{ext}^1_{\mathfrak{B}}(\mathbb{W}^{(k)}_\lambda,\ M^\vee) = 0 \qquad \text{for all } \lambda \in P^+. \qquad (5.9)$$

4. *A module $M \in \mathfrak{C}_{\mathrm{bdd}}$ admits a $\mathbb{W}^{(k)}$-filtration if and only if*

$$\mathrm{ext}^1_{\mathfrak{B}}(M,\ (W^{(k)}_\lambda)^*) = 0 \qquad \text{for all } \lambda \in P^+. \qquad (5.10)$$

Moreover, in each of the above cases, all higher ext^i vanish for $i > 1$.

The case $k = 1$ of Theorem 5.4(2)(4) corresponds to [15, Theorem 5.9(ii)(i)], respectively. The proofs of Theorem 5.4(3)(4) are analogous to those of Theorem 5.4(1)(2), with the use of Theorem 5.1(1) replaced by that of Theorem 5.1(2).

Throughout the proofs of Theorems 5.1 and 5.4, we assume their validity for all strictly smaller values of k, with the base case given by Theorem 5.1(1) at $k = 1$ (Theorem 2.17).

In fact, the implication from Theorem 5.1 at level k to Theorem 5.4 at level k is largely formal, once Proposition 5.2 is established. Accordingly, we devote Sections 4–7 to preparing several auxiliary results that are required to prove Theorem 5.1 at level k from Theorems 5.1 and 5.4 at levels $< k$.

We also note that the induction closes within Theorems 5.1(1) and 5.4(1)(2); Theorems 5.1(2) and 5.4(3)(4) may then be regarded as their corollaries.

Corollary 5.5. *Let $k \in \mathbb{Z}_{>0}$, and assume that Theorem 5.4 holds at level k. Then:*

- *If $M, N \in \mathfrak{C}_{\mathrm{bdd}}$ and $M \oplus N$ admits a $\mathbb{W}^{(k)}$-filtration, then M also admits a $\mathbb{W}^{(k)}$-filtration;*

- *If $M, N \in \mathfrak{B}_{\mathrm{bdd}}$ and $M \oplus N$ admits a $\mathbb{D}^{(k)}$-filtration, then M also admits a $\mathbb{D}^{(k)}$-filtration.*

In particular, the class of $\mathbb{W}^{(k)}$-filtered (resp. $\mathbb{D}^{(k)}$-filtered) modules in $\mathfrak{C}_{\mathrm{bdd}}$ (resp. $\mathfrak{B}_{\mathrm{bdd}}$) is closed under taking direct summands.

Proof. This follows immediately from Theorem 5.4, as the functors $\mathrm{ext}^1_{\mathfrak{C}}$ and $\mathrm{ext}^1_{\mathfrak{B}}$ commute with finite direct sums. $\qquad\square$

Lemma 5.6. *Let $k \in \mathbb{Z}_{>0}$, and assume that Theorem 5.4 holds at level k. Let $M, N \in \mathfrak{B}_{\mathrm{bdd}}$ be two $\mathbb{D}^{(k)}$-filtered modules, and suppose that there is a surjective morphism $f : M \twoheadrightarrow N$. Then $\ker f$ also admits a $\mathbb{D}^{(k)}$-filtration. The analogous statement holds for the filtrations of type $D^{(k)}$, $\mathbb{W}^{(k)}$, or $W^{(k)}$.*

Proof. Since the proofs are similar in all cases, we treat only the case of $\mathbb{D}^{(k)}$-filtrations. Consider the short exact sequence

$$0 \to \ker f \to M \to N \to 0.$$

Applying the functor $\mathrm{ext}^{\bullet}_{\mathfrak{B}}(\bullet, (D^{(k)}_\lambda)^*)$ yields a long exact sequence, from which we extract an exact sequence for each $i \geq 0$:

$$\mathrm{ext}^i_{\mathfrak{B}}(M, (D^{(k)}_\lambda)^*) \longrightarrow \mathrm{ext}^i_{\mathfrak{B}}(\ker f, (D^{(k)}_\lambda)^*) \longrightarrow \mathrm{ext}^{i+1}_{\mathfrak{B}}(N, (D^{(k)}_\lambda)^*).$$

Since M and N both admit $\mathbb{D}^{(k)}$-filtrations, Theorem 5.4 implies

$$\mathrm{ext}^i_{\mathfrak{B}}(M, (D^{(k)}_\lambda)^*) = 0 = \mathrm{ext}^{i+1}_{\mathfrak{B}}(N, (D^{(k)}_\lambda)^*) \qquad \text{for all } i > 0.$$

Therefore,

$$\mathrm{ext}^{>0}_{\mathfrak{B}}(\ker f, (D^{(k)}_\lambda)^*) = 0,$$

and Theorem 5.4 implies that $\ker f$ admits a $\mathbb{D}^{(k)}$-filtration. $\qquad\square$

6 ext^1-vanishing results

We retain the setup from the previous section.

Proposition 6.1. *Let $k \in \mathbb{Z}_{\geq 1}$. For each $\Lambda \in P^+_{\mathrm{af},k}$ and $\mu \in P^+$, we have*

$$\mathrm{ext}^1_{\mathfrak{B}}(^{\theta}L(\Lambda),\ (D^{(k)}_\mu)^*) = 0. \tag{6.1}$$

Proof. We first find $\Lambda' \in P^+_{\mathrm{af},k}$ and $w \in W_{\mathrm{af}}$ such that

$$\mu + k\Lambda_0 = w\Lambda' \quad \text{and hence} \quad D^{(k)}_\mu = \mathcal{D}_w(\mathbb{C}_{\Lambda'}).$$

We rewrite (6.1) as

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,^\theta L(\Lambda), (D^{(k)}_\mu)^*\big) = \mathrm{ext}^1_{\mathfrak{B}}\big(\,^\theta L(\Lambda), \big(\mathcal{D}_w(\mathbb{C}_{\Lambda'})\big)^*\big)$$
$$\cong \mathrm{ext}^1_{\mathfrak{B}}\big(\mathcal{D}_{w^{-1}}(\,^\theta L(\Lambda)), (\mathbb{C}_{\Lambda'})^*\big)$$
$$= \mathrm{ext}^1_{\mathfrak{B}}\big(\,^\theta L(\Lambda), (\mathbb{C}_{\Lambda'})^*\big),$$

where the isomorphism follows from Theorems 1.16 and 2.3.

By the BGG-resolution (1.10), the last term vanishes:

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,^\theta L(\Lambda), \mathbb{C}_{-\Lambda'}\big) = 0,$$

since Λ' does not lie in the set $\{\, s_i \cdot \Lambda + m\delta \mid i \in \mathtt{I}_{\mathrm{af}},\ m \in \mathbb{Z} \,\}$. This completes the proof. $\qquad\square$

Lemma 6.2. *Assume Theorem 5.1(2) for $k = 1$. For $\lambda, \mu \in P$ with $\lambda_+ + \mu_- \neq 0$, we have*

$$\mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{W}_\lambda,\ (\mathbb{W}_\mu)^\vee) = 0.$$

In the case $\lambda_+ + \mu_- = 0$, we have

$$\mathrm{ext}^{\geq 1}_{\mathfrak{B}}(\mathbb{W}_\lambda,\ (\mathbb{W}_\mu)^\vee) = 0.$$

Proof. Suppose $\lambda = u\lambda_-$. Then the length formula [48, (2.4.1)] asserts that for any $\beta \in Q \cap P^+$ with $\langle \alpha_i^\vee, \beta \rangle > 0$ for all $i \in \mathtt{I}$, we have

$$\ell(ut_{-\beta}) = -\ell(u) + \ell(t_{-\beta}).$$

It follows that

$$\mathbb{W}_\lambda \cong \mathsf{q}^{\langle \beta, \lambda_-\rangle} \mathcal{D}_{ut_{-\beta}}(\mathbb{W}_{\lambda_-}),$$

which implies

$$\mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{W}_\lambda, (\mathbb{W}_\mu)^\vee) \cong \mathrm{ext}^\bullet_{\mathfrak{B}}(\mathsf{q}^{\langle \beta, \lambda_-\rangle} \mathcal{D}_{ut_{-\beta}}(\mathbb{W}_{\lambda_-}), (\mathbb{W}_\mu)^\vee)$$
$$\cong \mathsf{q}^{-\langle \beta, \lambda_-\rangle} \mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{W}_{\lambda_-}, (\mathcal{D}_{t_\beta u^{-1}}(\mathbb{W}_\mu))^\vee).$$

Since $\mathcal{D}_{w_0}(\mathbb{W}_{\lambda_-}) \cong \mathbb{W}_{\lambda_-}$, we further obtain

$$\mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{W}_{\lambda_-}, (\mathcal{D}_{t_\beta u^{-1}}(\mathbb{W}_\mu))^\vee) \cong \mathrm{ext}^\bullet_{\mathfrak{B}}(\mathbb{W}_{\lambda_-}, ((\mathcal{D}_{w_0} \circ \mathcal{D}_{t_\beta u^{-1}})(\mathbb{W}_\mu))^\vee).$$

Here, $(\mathcal{D}_{w_0} \circ \mathcal{D}_{t_\beta u^{-1}})(\mathbb{W}_\mu)$ is $\mathfrak{g}$-stable, and is therefore isomorphic to a grading shift of $\mathbb{W}_{\mu_-}$.

Let $R := \mathrm{End}_{\mathfrak{C}}(\mathbb{W}^{(1)}_{\mu_+})$, which is a polynomial ring by Theorem 2.8. For each m, denote by $R_{\leq m}$ the degree-m truncation. Then

$$\mathbb{W}^{(1)}_{\mu_+} \cong \varprojlim_m \big(\mathbb{W}^{(1)}_{\mu_+} \otimes_R R_{\leq m}\big), \qquad (6.2)$$

where $\mathrm{End}_{\mathfrak{C}}(\mathbb{W}^{(1)}_{\mu_+})_{\leq m}$ denotes the degree-m truncation of the endomorphism ring.

Each term in the inverse system (6.2) is a finite self-extension of

$$W^{(1)}_{\mu_+} \otimes \mathbb{C}_{-2\Lambda_0}.$$

By Proposition 2.1, we deduce

$$\text{ext}^\bullet_{\mathfrak{B}}\left(\mathbb{W}_{\lambda_-}, \mathbb{W}^\vee_{\mu_-}\right) \cong \varinjlim_m \text{ext}^\bullet_{\mathfrak{B}}\left(\mathbb{W}_{\lambda_-}, (\mathbb{W}^{(1)}_{\mu_+} \otimes_R R_{\leq m})^* \otimes \mathbb{C}_{-\Lambda_0}\right)$$

$$\cong \varinjlim_m \text{ext}^\bullet_{\mathfrak{C}}\left(\mathbb{W}_{\lambda_-}, (\mathbb{W}^{(1)}_{\mu_+} \otimes_R R_{\leq m})^* \otimes \mathbb{C}_{-\Lambda_0}\right),$$

where the first isomorphism follows from Corollary 1.13.

By Proposition 2.10(1), the first entry of ext in the last expression identifies with $\mathbb{W}^{(1)}_{\lambda_+} \otimes \mathbb{C}_{\Lambda_0}$. Hence the limit vanishes completely when $\lambda_+ + \mu_- \neq 0$, and vanishes in positive degrees when $\lambda_+ + \mu_- = 0$, by Theorem 5.1(2) for $k = 1$.

This completes the proof. □

Proposition 6.3. *Assume Theorem 5.4(1) for $k = 1$. For each $\lambda \in P$, the $\widetilde{\mathfrak{b}}$-module $\mathbb{W}_\lambda \otimes \mathbb{C}_{\Lambda_0}$ admits a $D^{(1)}$-filtration.*

Proof. By Theorem 5.4(1) for $k = 1$, we need to verify

$$\text{ext}^{\geq 1}_{\mathfrak{B}}(\mathbb{W}_\lambda, (\mathbb{D}^{(1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^\vee) = 0 \qquad \lambda, \mu \in P. \tag{6.3}$$

In case $\mu = \mu_+$, this is Proposition 2.10(2) and Lemma 6.2.

We prove the assertion for $\mu = u\mu_+$, where $u \in W$ is taken smallest possible with respect to the Bruhat order, by assuming the validity of (6.3) for the case of smaller u. By Proposition 2.10(3), we have a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow \mathbb{W}_{u\mu} \xrightarrow{f} \mathbb{D}^{(1)}_{u\mu} \otimes \mathbb{C}_{\Lambda_0} \longrightarrow 0, \tag{6.4}$$

where $\ker f$ is filtered by $\{\mathbb{D}^{(1)}_{v\mu} \otimes \mathbb{C}_{\Lambda_0}\}_{v<w}$.

We have

$$\text{ext}^{\geq 1}_{\mathfrak{B}}(\mathbb{W}_\lambda, (\ker f)^\vee) = 0 = \text{ext}^{\geq 1}_{\mathfrak{B}}(\mathbb{W}_\lambda, (\mathbb{W}_{u\mu})^\vee)$$

by induction hypothesis and Lemma 6.2. We have the long exact sequence obtained by applying $\text{hom}_{\mathfrak{B}}(\mathbb{W}_\lambda, \bullet)$ to (6.4):

$$\text{hom}_{\mathfrak{B}}(\mathbb{W}_\lambda, (\mathbb{W}_{u\mu})^\vee) \xrightarrow{g} \text{hom}_{\mathfrak{B}}(\mathbb{W}_\lambda, (\ker f)^\vee) \to \text{ext}^1_{\mathfrak{B}}(\mathbb{W}_\lambda, (\mathbb{D}^{(1)}_{u\mu} \otimes \mathbb{C}_{\Lambda_0})^\vee) \to 0$$

In case $s_i u < u$, then applying $\mathcal{D}_i$ to (6.4) yields

$$0 = \mathbb{L}^{-1}\mathcal{D}_i(\mathbb{D}^{(1)}_{u\mu} \otimes \mathbb{C}_{\Lambda_0}) \to \mathcal{D}_i(\ker f) \longrightarrow \mathbb{W}_{u\mu} \to \mathcal{D}_i(\mathbb{D}^{(1)}_{u\mu} \otimes \mathbb{C}_{\Lambda_0}) = 0$$

by Proposition 2.10(5) (and Theorem 1.14(4) for $M = \mathbb{C}_{\Lambda_0}$), which forces

$$\ker f \subset \mathcal{D}_i(\ker f) = \mathbb{W}_{u\mu}.$$

Thus, the dual form of Theorem 4.1 obtained through (1.5) implies that g is surjective.

This proceeds the induction and we conclude (6.3). This completes the proof. □

Theorem 6.4. *Let $k \in \mathbb{Z}_{\geq 2}$. For each $\Lambda \in P^+_{\text{af},k}$ and $\mu \in P$, we have*

$$\text{ext}^1_{\mathfrak{B}}(^\theta L(\Lambda), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*) = 0. \tag{6.5}$$

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We make several reduction result, that can be seen as a preliminary part of the proof of Theorem 6.4. The proof of Theorem 6.4 is given after Lemma 6.7.

Proposition 6.5. *Theorem 6.4 holds for level k if*

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathbb{W}_\lambda, (\mathbb{C}_{\Lambda''})^*\big) = 0 \tag{6.6}$$

holds for every $\Lambda', \Lambda'' \in P^*_{\mathrm{af},(k-1)}$, $\lambda \in P$.

Proof. By [5], there exist $\Lambda' \in P^+_{\mathrm{af},(k-1)}$ and $\Lambda_i \in P^+_{\mathrm{af},1}$ such that

$$L(\Lambda) \subset L(\Lambda') \otimes L(\Lambda_i).$$

Therefore, the assertion (6.5) is equivalent to the vanishing of

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes {}^{\theta}L(\Lambda_i), (W^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\big) = 0, \tag{6.7}$$

for some $\Lambda' \in P^+_{\mathrm{af},(k-1)}$, $\Lambda_i \in P^+_{\mathrm{af},1}$, and $\mu \in P^+$.

Here, ${}^{\theta}L(\Lambda_i)$ admits a decreasing separable filtration whose associated graded is a direct sum of grading shifts of the modules $\{\,\mathbb{W}^{(1)}_\lambda\,\}_{\lambda \in P^+}$, by Proposition 6.1 and Theorem 5.4 for $k = 1$ (see also [15, Theorem 4.7] for their constructions).

Thus, it suffices to prove

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathbb{W}^{(1)}_\lambda, (W^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\big) = \mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathbb{W}_{\lambda_-}, (W^{(k-1)}_\mu)^*\big) = 0$$

for each $\Lambda' \in P^+_{\mathrm{af},(k-1)}$, $\lambda, \mu \in P^+$ in order to establish Theorem 6.4 for level k.

There exists $\Lambda'' \in P^+_{\mathrm{af},(k-1)}$ and $w \in W_{\mathrm{af}}$ such that $\mu + (k-1)\Lambda_0 = w\Lambda''$. We have $W^{(k-1)}_\mu \cong \mathcal{D}_w(\mathbb{C}_{\Lambda''})$. By Theorem 1.14(4) and Proposition 2.10(4), we have

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathbb{W}_{\lambda_-}, (W^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\big) = \mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathbb{W}_{\lambda_-}, \mathcal{D}_w(\mathbb{C}_{\Lambda''})^*\big)$$
$$\cong \mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathcal{D}_w(\mathbb{W}_{\lambda_-}), (\mathbb{C}_{\Lambda''})^*\big)$$
$$= \mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes \mathbb{W}_{u\lambda}, (\mathbb{C}_{\Lambda''})^*\big)$$

up to grading shift for some $u \in W$.

This proves the result. $\qquad\square$

Corollary 6.6. *Theorem 6.4 holds for level k if*

$$\mathrm{ext}^1_{\mathfrak{B}}\big(\,{}^{\theta}L(\Lambda') \otimes D^{(1)}_\lambda, \ (\mathbb{C}_{\Lambda''-\Lambda_0})^*\big) = 0 \tag{6.8}$$

holds for every $\Lambda', \Lambda'' \in P^*_{\mathrm{af},(k-1)}$, $\lambda \in P$. In addition, Λ' and Λ'' in (6.8) is the same as that of (6.6), and hence up to the choice of λ, we can restricts to $\Lambda', \Lambda'' \in P^*_{\mathrm{af},(k-1)}$ in which we need to verify (6.6).

Proof. By Proposition 6.3, we have

$$\mathbb{W}_\lambda \otimes \mathbb{C}_{\Lambda_0} \cong \varprojlim M(r),$$

where $\{M(r)\}_r$ is an inverse system with surjective transition maps such that each $M(r)$ is finite-dimensional and admits $D^{(1)}$-filtration. Since $\mathbb{W}_\lambda \in \mathfrak{B}_{\mathrm{bdd}}$,

the degree $\leq m$-part of $\mathbb{W}_\lambda \otimes \mathbb{C}_{\Lambda_0}$ is same as that of $M(r)$ for $r \gg 0$ for each $m \in \mathbb{Z}$.

Thus we have

$$\mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda') \otimes \mathbb{W}_\lambda, (\mathbb{C}_{\Lambda''})^*\right) \cong \varinjlim_m \mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda') \otimes M(r), (\mathbb{C}_{\Lambda''})^*\right) \qquad (6.9)$$

by Lemma 1.12. Here the right hand side of (6.9) vanishes by assumption.

This completes the proof. $\qquad\square$

Lemma 6.7. *Theorem 6.4 holds for level $k = 2$.*

Proof. It suffices to verify the condition (6.8) in Corollary 6.6 for $k = 2$. To this end, we apply an affine Dynkin diagram automorphism so that Λ'' is rearranged into Λ_0 in (6.6). This reduces the choice of $\Lambda'' \in P_{\mathrm{af},1}^+$ to Λ_0.

We have

$$\mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda') \otimes D_\lambda^{(1)}, (\mathbb{C}_{\Lambda_0 - \Lambda_0})^*\right) = \mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda'), (D_\lambda^{(1)})^*\right) \qquad (6.10)$$

for each $\Lambda' \in P_{\mathrm{af},1}^+$. This vanishes by Proposition 6.1.

This proves the assertion. $\qquad\square$

Proof of Theorem 6.4. We prove (6.5) by induction on k. The case $k = 2$ is proved in Lemma 6.7.

We consider the case of $k > 2$ by assuming the validity of the assertion for lower values of k. We now find $x \in W_{\mathrm{af}}$ and $\Lambda_j \in P_{\mathrm{af},1}^+$ such that

$$\lambda + \Lambda_0 = x\Lambda_j \quad \text{and hence} \quad D_\lambda^{(1)} \cong \mathcal{D}_x(\mathbb{C}_{\Lambda_j}).$$

We rewrite (6.8) as

$$\mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda') \otimes D_\lambda^{(1)}, (\mathbb{C}_{\Lambda'' - \Lambda_0})^*\right) = \mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda') \otimes \mathcal{D}_x(\mathbb{C}_{\Lambda_j}), (\mathbb{C}_{\Lambda'' - \Lambda_0})^*\right) \tag{6.11}$$

$$\cong \mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda') \otimes \mathbb{C}_{\Lambda_j}, (\mathbb{L}^\bullet \mathcal{D}_{x^{-1}}(\mathbb{C}_{\Lambda'' - \Lambda_0}))^*\right)$$

$$\cong \mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda'), (\mathbb{L}^\bullet \mathcal{D}_{x^{-1}}(\mathbb{C}_{\Lambda'' - \Lambda_0}) \otimes \mathbb{C}_{\Lambda_j})^*\right).$$

To the last expression, we may apply an affine Dynkin diagram automorphism that sends Λ_j to Λ_0.

Therefore, if the complex

$$\mathbb{L}^\bullet \mathcal{D}_{x^{-1}}(\mathbb{C}_{\Lambda'' - \Lambda_0})$$

defines an actual Demazure module and we can drop $\mathbb{L}^{>0}$, then we obtain

$$\mathrm{ext}_{\mathfrak{B}}^1 \left({}^\theta L(\Lambda'), (\mathcal{D}_{x^{-1}}(\mathbb{C}_{\Lambda'' - \Lambda_0}) \otimes \mathbb{C}_{\Lambda_j})^*\right) = 0$$

from the level $(k-1)$ case of (6.5).

This scenario applies when $\Lambda'' - \Lambda_0 \in P_{\mathrm{af},(k-1)}^+$.

Thus, we are reduced to the case where $\Lambda'' - \Lambda_0 \notin P_{\mathrm{af},(k-1)}^+$ (but still $\Lambda'' \in P_{\mathrm{af},k}^+$). By Proposition 3.3, we have a short exact sequence of $\widetilde{\mathfrak{b}}$-modules:

$$0 \longrightarrow \mathsf{q} D_{\Lambda'' + \alpha_0}^{(k-1)} \longrightarrow D_{\Lambda''}^{(k-1)} \longrightarrow \mathbb{C}_{\Lambda'' - \Lambda_0} \longrightarrow 0. \qquad (6.12)$$

As in (6.11), we have

$$\operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (D^{(k-1)}_{\underline{\Lambda''}})^*\big) \equiv \operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes \mathcal{D}_x(\mathbb{C}_{\Lambda_j}), (D^{(k-1)}_{\underline{\Lambda''}})^*\big)$$

$$\cong \operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes \mathbb{C}_{\Lambda_j}, (\mathcal{D}_{x^{-1}}(D^{(k-1)}_{\underline{\Lambda''}}))^*\big)$$

$$\cong \operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda'), (\mathcal{D}_{x^{-1}}(D^{(k-1)}_{\underline{\Lambda''}}) \otimes \mathbb{C}_{\Lambda_j})^*\big).$$

We apply an affine Dynkin diagram automorphism that sends Λ_j to Λ_0, using Theorem 1.16 and Corollary 2.12. Then, by applying (6.5) for level $(k-1)$, we deduce

$$\operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (D^{(k-1)}_{\underline{\Lambda''}})^*\big) = 0.$$

We now consider the effect of applying the functor

$$\hom_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes \mathcal{D}_x(\mathbb{C}_{\Lambda_j}), (\bullet)^*\big)$$

to the short exact sequence (6.12). The vanishing of (6.11), namely

$$\operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (\mathbb{C}_{\Lambda''-\Lambda_0})^*\big) = 0,$$

is then equivalent to the surjectivity of the map ψ:

$$\hom_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (D^{(k-1)}_{\underline{\Lambda''}})^*\big) \xrightarrow{\psi} \hom_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (\mathsf{q}D^{(k-1)}_{\underline{\Lambda''+\alpha_0}})^*\big)$$

$$\longrightarrow \operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (\mathbb{C}_{\Lambda''-\Lambda_0})^*\big) \longrightarrow \operatorname{ext}^1_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (D^{(k-1)}_{\underline{\Lambda''}})^*\big) = 0.$$

We have $D^{(1)}_\lambda \subset \mathcal{D}_0(D^{(1)}_\lambda)$ and $\mathbb{L}^{<0}\mathcal{D}_0(D^{(1)}_\lambda) = 0$ as $D^{(1)}_\lambda$ is a level-one Demazure module. Theorem 1.14 implies

$$^\theta L(\Lambda') \otimes D^{(1)}_\lambda \subset \mathbb{L}^\bullet \mathcal{D}_0\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda\big) \cong {}^\theta L(\Lambda') \otimes \mathbb{L}^\bullet \mathcal{D}_0(D^{(1)}_\lambda).$$

Moreover, we have

$$\mathbb{L}^\bullet \mathcal{D}_0(\mathsf{q}D^{(k-1)}_{\underline{\Lambda''+\alpha_0}}) = D^{(k-1)}_{\underline{\Lambda''}}.$$

Hence, by Theorem 1.16, the map ψ coincides with

$$\hom_{\mathfrak{B}}\big(\mathcal{D}_0({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda), (D^{(k-1)}_{\underline{\Lambda''+\alpha_0}})^*\big) \longrightarrow \hom_{\mathfrak{B}}\big({}^\theta L(\Lambda') \otimes D^{(1)}_\lambda, (D^{(k-1)}_{\underline{\Lambda''+\alpha_0}})^*\big).$$

Hence, Corollary 4.3 asserts that ψ must be surjective.

Therefore, (6.5) for level $(k-1)$ implies (6.5) for level k. This proceeds the induction and we conclude the assertion. $\qquad\square$

7   Some filtration results

Keep the setting of the previous section.

Proposition 7.1. *Let $k, l \in \mathbb{Z}_{>0}$ be such that $k \geq l$. Assume that Theorem 5.4 holds for all levels $< k$. Then for each $\lambda \in P^+$, the module $\mathbb{D}^{(k)}_\lambda \otimes \mathbb{C}_{(k-l)\Lambda_0}$ admits a $\mathbb{D}^{(l)}$-filtration.*

Proof. The case $k = l$ is trivial by definition. Thus, we assume $k > l$ in the following. In view of Theorem 5.4, which holds for all levels $< k$ by assumption, it suffices to prove that

$$\text{ext}^1_{\mathfrak{B}} \left(\mathbb{D}^{(l)}_{\lambda}, (D^{(l-1)}_{\mu} \otimes \mathbb{C}_{\Lambda_0})^* \right) = 0 \quad \text{for all } \lambda, \mu \in P.$$

Hence, we may assume $k = l$ in what follows. Thus, it is enough to prove

$$\text{ext}^1_{\mathfrak{B}} \left(\mathbb{D}^{(k)}_{\lambda}, (D^{(k-1)}_{\mu} \otimes \mathbb{C}_{\Lambda_0})^* \right) = 0 \quad \text{for all } \lambda, \mu \in P. \tag{7.1}$$

For $\Lambda \in P^+_{\text{af},k}$, let $W^{\Lambda}_{\text{af}} \subset W_{\text{af}}$ denote the set of maximal length coset representatives for $W_{\text{af}}/\{w \in W_{\text{af}} \mid w\Lambda = \Lambda\}$. For each $\Lambda \in P^+_{\text{af},k}$, $m \in \mathbb{Z}_{\geq 0}$, and $i \in \mathtt{I}_{\text{af}}$, define

$$L^m(\Lambda) := \sum_{\ell(v) \geq m} {}^{\theta}L(\Lambda)^v, \qquad L^m_i(\Lambda) := L^{m+1}(\Lambda) + \sum_{\substack{\ell(v)=m \\ s_i v < v}} {}^{\theta}L(\Lambda)^v,$$

where v ranges over W^{Λ}_{af} subject to the indicated conditions.

By Theorem 2.3 and Corollary 2.4, we have

$$\mathbb{L}^{\bullet}\mathcal{D}_i(L^m(\Lambda)) = \sum_{\substack{w \in W^{\Lambda}_{\text{af}} \\ \ell(w)=m-1 \\ s_i w > w}} {}^{\theta}L(\Lambda)^w + L^m(\Lambda).$$

In view of Theorem 1.5 (see also [43, Theorem C]), it follows that

$$\frac{\mathcal{D}_i(L^m(\Lambda))}{L^m(\Lambda)} \cong \bigoplus_{\substack{w \in W^{\Lambda}_{\text{af}} \\ \ell(w)=m-1 \\ s_i w > w}} \mathsf{q}^{\langle d, \Lambda - w\Lambda \rangle} \mathbb{D}^{(k)}_{-w\Lambda}. \tag{7.2}$$

In particular, we deduce that

$$\mathbb{L}^{\bullet}\mathcal{D}_i(L^m(\Lambda)) = \mathbb{L}^{\bullet}\mathcal{D}_i(L^m_i(\Lambda)), \tag{7.3}$$

since the condition $s_i v \not< v$ implies $s_i v > v$ for each $v \in W^{\Lambda}_{\text{af}}$.

We prove the following assertion by induction on m:

$(\bigstar)^m_1$ $L^m(\Lambda) \otimes \mathbb{C}_{\Lambda_0}$ admits a $\mathbb{D}^{(k-1)}$-filtration;

$(\bigstar)^m_2$ For every $w \in W^{\Lambda}_{\text{af}}$ such that $\ell(w) < m$, $\mathbb{D}^{(k)}_{-w\Lambda} \otimes \mathbb{C}_{\Lambda_0}$ admits a $\mathbb{D}^{(k-1)}$-filtration.

Here we note that $(\bigstar)^m_1$ is equivalent to

$$\text{ext}^1_{\mathfrak{B}} \left(L^{m-1}_i(\Lambda) \otimes \mathbb{C}_{\Lambda_0}, (D^{(k-1)}_{\mu})^* \right) \cong \text{ext}^1_{\mathfrak{B}} \left(L^{m-1}_i(\Lambda), (D^{(k-1)}_{\mu} \otimes \mathbb{C}_{\Lambda_0})^* \right) = 0,$$

and $(\bigstar)^m_2$ is equivalent to

$$\text{ext}^1_{\mathfrak{B}} \left(\mathbb{D}^{(k)}_{-w\Lambda} \otimes \mathbb{C}_{\Lambda_0}, (D^{(k-1)}_{\mu})^* \right) \cong \text{ext}^1_{\mathfrak{B}} \left(\mathbb{D}^{(k)}_{-w\Lambda}, (D^{(k-1)}_{\mu} \otimes \mathbb{C}_{\Lambda_0})^* \right) = 0$$

by Theorem 5.4(2).

The base case $(\bigstar)^0$ holds by the definition of $\{\mathbb{D}^{(k)}_{-\overline{w\Lambda}}\}_{w \in W^\Lambda_{\mathrm{af}}}$ since $(\bigstar)^0_2$ is vacuous. (In addition, $(\bigstar)^1_2$ is vacuous or holds by definition.)

By Theorem 2.3 and Corollary 2.4, we have a short exact sequence

$$0 \longrightarrow L^m(\Lambda) \longrightarrow L^{m-1}(\Lambda) \longrightarrow \bigoplus_{\substack{w \in W^\Lambda_{\mathrm{af}} \\ \ell(w) = m-1}} \mathsf{q}^{\langle d,\, \Lambda - w\Lambda \rangle} \mathbb{D}^{(k)}_{-\overline{w\Lambda}} \longrightarrow 0.$$

Applying Lemma 5.6, we see that $(\bigstar)^{m-1}_1$ and $(\bigstar)^m_2$ together imply $(\bigstar)^m_1$.

Thus, it remains to prove $(\bigstar)^m_2$ assuming $(\bigstar)^{m-1}$.

Assume now that $(\bigstar)^{m-1}$ holds. By (7.2), and $(\bigstar)^{m-1}$, we have

$$\mathrm{ext}^1_{\mathfrak{B}}(L^{m-1}(\Lambda), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*) = 0 = \bigoplus_{\substack{w \in W^\Lambda_{\mathrm{af}} \\ \ell(w) = m-2}} \mathrm{ext}^1_{\mathfrak{B}}(\mathbb{D}^{(k)}_{-\overline{w\Lambda}}, D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*)$$

for every $\mu \in P$. By the long exact sequence of ext applied to

$$0 \to L^{m-1}(\Lambda) \to \mathcal{D}_i(L^{m-1}(\Lambda)) \to \bigoplus_{\substack{w \in W^\Lambda_{\mathrm{af}} \\ \ell(w) = m-2 \\ s_i w > w}} \mathsf{q}^{\langle d, \Lambda - w\Lambda \rangle} \mathbb{D}^{(k)}_{-\overline{w\Lambda}} \to 0$$

and (7.3), we obtain

$$\mathrm{ext}^1_{\mathfrak{B}}(\mathcal{D}_i(L^{m-1}_i(\Lambda)), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*) = \mathrm{ext}^1_{\mathfrak{B}}(\mathcal{D}_i(L^{m-1}(\Lambda)), D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*) = 0 \tag{7.4}$$

for every $\mu \in P$.

By Corollary 2.4 and Corollary 4.3, we have a surjection

$$\mathrm{hom}_{\mathfrak{B}}\big(\mathcal{D}_i(L^{m-1}_i(\Lambda)), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\big) \twoheadrightarrow \mathrm{hom}_{\mathfrak{B}}\big(L^{m-1}_i(\Lambda), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\big)$$

for each $i \in \mathtt{I}_{\mathrm{af}}$ and $\mu \in P$.

This yields a portion of the long exact sequence:

$$\mathrm{hom}_{\mathfrak{B}}\left(\mathcal{D}_i(L^{m-1}_i(\Lambda)), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\right) \longrightarrow \mathrm{hom}_{\mathfrak{B}}\left(L^{m-1}_i(\Lambda), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\right)$$

$$\longrightarrow \mathrm{ext}^1_{\mathfrak{B}}\left(\frac{\mathcal{D}_i(L^{m-1}_i(\Lambda))}{L^{m-1}_i(\Lambda)}, (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\right)$$

$$\longrightarrow \mathrm{ext}^1_{\mathfrak{B}}\left(\mathcal{D}_i(L^{m-1}_i(\Lambda)), (D^{(k-1)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\right) = 0,$$

associated to the short exact sequence

$$0 \longrightarrow L^{m-1}_i(\Lambda) \longrightarrow \mathcal{D}_i(L^{m-1}_i(\Lambda)) \longrightarrow \frac{\mathcal{D}_i(L^{m-1}_i(\Lambda))}{L^{m-1}_i(\Lambda)} \longrightarrow 0. \tag{7.5}$$

From this, we deduce that

$$\frac{\mathcal{D}_i(L^{m-1}(\Lambda))}{L^{m-1}_i(\Lambda)} \otimes \mathbb{C}_{\Lambda_0}$$

admits a $\mathbb{D}^{(k-1)}$-filtration by Theorem 5.4 at level $k-1$.

Moreover, we have a short exact sequence

$$0 \longrightarrow \frac{L^{m-1}(\Lambda)}{L_i^{m-1}(\Lambda)} \longrightarrow \frac{\mathcal{D}_i(L_i^{m-1}(\Lambda))}{L_i^{m-1}(\Lambda)} \longrightarrow \bigoplus_{\substack{w \in W_{\mathrm{af}}^\Lambda \\ \ell(w)=m-2 \\ s_i w > w}} \mathsf{q}^{\langle d, \Lambda - w\Lambda \rangle} \mathbb{D}_{-w\Lambda}^{(k)} \longrightarrow 0.$$

By $(\bigstar)^{m-1}$ and Lemma 5.6, we conclude that

$$\frac{L^{m-1}(\Lambda)}{L_i^{m-1}(\Lambda)} \otimes \mathbb{C}_{\Lambda_0}$$

admits a $\mathbb{D}^{(k-1)}$-filtration. In fact, we have the decomposition

$$\frac{L^{m-1}(\Lambda)}{L_i^{m-1}(\Lambda)} \cong \bigoplus_{\substack{w \in W_{\mathrm{af}}^\Lambda \\ \ell(w)=m-1 \\ s_i w > w}} \mathsf{q}^{\langle d, \Lambda - w\Lambda \rangle} \mathbb{D}_{-w\Lambda}^{(k)}. \tag{7.6}$$

Since every $w \in W_{\mathrm{af}}$ satisfies $s_i w > w$ for some $i \in \mathtt{I}_{\mathrm{af}}$, it follows that every $\mathbb{D}_{-w\Lambda}^{(k)}$ with $\ell(w) = m-1$ appears on the right-hand side of (7.6) for some $i \in \mathtt{I}_{\mathrm{af}}$. Thus, Corollary 5.5 implies $(\bigstar)_2^m$ from $(\bigstar)^{m-1}$.

This completes the induction, and we obtain $(\bigstar)$.

Therefore, each $\mathbb{D}_\lambda^{(k)} \otimes \mathbb{C}_{\Lambda_0}$ admits a $\mathbb{D}^{(k-1)}$-filtration. It follows that each $\mathbb{D}_\lambda^{(k)} \otimes \mathbb{C}_{(k-l)\Lambda_0}$ admits a $\mathbb{D}^{(l)}$-filtration for every $1 \le l \le k$, as desired. $\qquad \square$

Theorem 7.2. *Let $k \in \mathbb{Z}_{>0}$, and assume that Theorem 5.4 holds for all levels $< k$. Then for each $\Lambda \in P_{\mathrm{af},k}^+$ and $w \in W_{\mathrm{af}}$, we have*

$$\mathrm{ext}_{\mathfrak{B}}^\bullet \left({}^\theta L(\Lambda)^w, \ (\mathbb{C}_{\mu+k\Lambda_0})^* \right) = 0 \qquad whenever \ -\mu \not\preceq -\overline{w\Lambda}. \tag{7.7}$$

In particular, for every $k' \in \mathbb{Z}_{>0}$ and $\mu \in P$, we have

$$\mathrm{ext}_{\mathfrak{B}}^\bullet \left({}^\theta L(\Lambda)^w, \ (D_\mu^{(k')} \otimes \mathbb{C}_{(k-k')\Lambda_0})^* \right) = 0 \qquad whenever \ -\mu \not\preceq -\overline{w\Lambda}.$$

Proof. The module ${}^\theta L(\Lambda)^w$ admits a $\mathbb{D}^{(k)}$-filtration by Corollary 1.7. By repeatedly applying Proposition 7.1, this filtration can be refined into a $\mathbb{D}^{(1)}$-filtration. Since ${}^\theta L(\Lambda)^w$ has a cyclic $\widetilde{\mathfrak{b}}$-generator of $\widetilde{\mathfrak{h}}$-weight $-w\Lambda$, its associated graded is a direct sum of character twists of modules of the form $\mathbb{D}_\nu^{(1)}$, where $-\overline{w\Lambda} \preceq -\nu$, by Corollary 2.19.

In particular, we have a projective system $\{M(r)\}_r$ of surjective transition maps such that each $M(r)$ admits a finite filtration by the grading shifts of $\mathbb{D}_\nu^{(1)}$ with $-\overline{w\Lambda} \preceq -\nu$, and

$${}^\theta L(\Lambda)^w \cong \varprojlim_r M(r).$$

By Lemma 2.18 and Lemma 1.12, we deduce that

$$\mathrm{ext}_{\mathfrak{B}}^\bullet \left({}^\theta L(\Lambda)^w, (\mathbb{C}_{\mu+k\Lambda_0})^* \right) \cong \varinjlim_r \mathrm{ext}_{\mathfrak{B}}^\bullet \left(M(r), (\mathbb{C}_{\mu+k\Lambda_0})^* \right) = 0,$$

whenever $-\mu \not\preceq -\overline{w\Lambda}$. This proves the first assertion.

For the second assertion, note that by Lemma 2.22 and the definition of Demazure modules, $D_\mu^{(k')}$ can be obtained as a successive extension of one-dimensional $\widetilde{\mathfrak{b}}$-modules of the form $\{\mathsf{q}^m \mathbb{C}_{\gamma+k'\Lambda_0}\}_{\gamma \preceq \mu,\, m \in \mathbb{Z}}$ starting from $\mathbb{C}_{\mu+k'\Lambda_0}$. Since $\dim D_\mu^{(k')} < \infty$, this process terminates after finitely many steps. Applying the functor $\mathrm{ext}_{\mathfrak{B}}^\bullet \left({}^\theta L(\Lambda)^w, \bullet^*\right)$ and (7.7) to these short exact sequences, we deduce inductively the vanishing for the larger modules as well. This yields the second assertion.

This completes the proof. $\qquad\square$

8   Proofs of the main theorems

We retain the setting of the previous section. This section is devoted entirely to the proofs of Theorems 5.1 and 5.4. Since our approach proceeds by induction on the level k (with the base case reducing to [15]), we assume that both Theorem 5.1 and Theorem 5.4 hold for all levels strictly less than k.

The combination of Theorems 8.1 and 8.2 then establishes Theorems 5.1 and 5.4 for arbitrary k by induction from the case $k = 1$.

Theorem 8.1. *Theorems 5.1 for level $< k$ and Theorem 5.4 for level $< k$ implies Theorem 5.1 for level k.*

Proof. We prove Theorem 5.1(1) for level k. Let $\Lambda, \Lambda' \in P_{\mathrm{af},k}^+$ and $w, v \in W_{\mathrm{af}}$ such that $\mu + k\Lambda_0 = v\Lambda'$. By rearranging w and v if necessary, we may assume that w is the minimal element among those satisfying $w\Lambda = w'\Lambda$ for some w', and that v is the maximal element among those satisfying $\mu + k\Lambda_0 = v'\Lambda'$ for some v'.

By Theorem 1.16, we have

$$\mathrm{ext}_{\mathfrak{B}}^\bullet({}^\theta L(\Lambda)^w, (D_\mu^{(k)})^*) \cong \mathrm{ext}_{\mathfrak{B}}^\bullet({}^\theta L(\Lambda)^w, \mathcal{D}_v(\mathbb{C}_{\Lambda'})^*)$$
$$\cong \mathrm{ext}_{\mathfrak{B}}^\bullet \left(\mathcal{D}_{v^{-1}}\left({}^\theta L(\Lambda)^w\right), (\mathbb{C}_{\Lambda'})^*\right).$$

Thanks to Theorem 2.3, we have

$$\mathcal{D}_{v^{-1}}\left({}^\theta L(\Lambda)^w\right) = {}^\theta L(\Lambda)$$

if and only if a reduced expression of v^{-1} contains a subexpression whose product equals w^{-1}. This occurs precisely when $w^{-1} \leq v^{-1}$ in the Bruhat order, which is equivalent to $w \leq v$. Hence, we conclude that

$$\mathrm{ext}_{\mathfrak{B}}^\bullet({}^\theta L(\Lambda)^w, (D_\mu^{(k)})^*) \cong \begin{cases} \mathrm{ext}_{\mathfrak{B}}^\bullet({}^\theta L(\Lambda), (\mathbb{C}_{\Lambda'})^*) & \text{if } w \leq v, \\ \mathrm{ext}_{\mathfrak{B}}^\bullet({}^\theta L(\Lambda)^u, (\mathbb{C}_{\Lambda'})^*) & \text{otherwise,} \end{cases} \qquad (8.1)$$

for some $u \in W_{\mathrm{af}}$ with $u \neq e$ and $u\Lambda \neq \Lambda$, by Proposition 1.4.

By Theorem 7.2 (which relies on Theorem 5.4 for levels $< k$), together with the fact that $\overline{WP_{\mathrm{af},k}^+}$ is closed under taking smaller elements with respect to $\prec$, we find that

$$\mathrm{ext}_{\mathfrak{B}}^\bullet({}^\theta L(\Lambda)^u, (\mathbb{C}_{\Lambda'})^*) \equiv 0 \qquad \text{if } u\Lambda \notin W\Lambda. \qquad (8.2)$$

We now examine the case $u\Lambda \in W\lambda$, which complements (8.2).

We apply Proposition 7.1 to ${}^\theta L(\Lambda)^u$ (which requires Theorem 5.4 for levels $< k$), thereby obtaining a decreasing separable filtration of ${}^\theta L(\Lambda)^u$ whose associated graded pieces are grading shifts of $\{\mathbb{D}_\mu^{(k-1)} \otimes \mathbb{C}_{\Lambda_0}\}_{\mu \in P}$. Hence we may write

$$
{}^\theta L(\Lambda)^u \;\cong\; \varprojlim_r M(r),
$$

where $\{M(r)\}_r$ is a projective system with surjective transition maps such that each $M(r) \otimes \mathbb{C}_{\Lambda_0}$ admits a finite $\mathbb{D}^{(k-1)}$-filtration.

In the case $\Lambda' - \Lambda_0 \in P_{\mathrm{af},(k-1)}^+$, we have

$$
\mathrm{ext}_{\mathfrak{B}}^{>0}(\mathbb{D}_\mu^{(k-1)} \otimes \mathbb{C}_{-\Lambda_0}, (\mathbb{C}_{\Lambda'})^*) \cong \mathrm{ext}_{\mathfrak{B}}^{>0}(\mathbb{D}_\mu^{(k-1)}, (\mathbb{C}_{\Lambda'-\Lambda_0})^*) = 0
$$

by Theorem 5.1 for level $k-1$ and the identification

$$
\mathbb{C}_{\Lambda'} \cong D_{\overline{\Lambda'}}^{(k)} \cong D_{\overline{\Lambda'}}^{(k-1)} \otimes \mathbb{C}_{\Lambda_0}.
$$

It follows that

$$
\mathrm{ext}_{\mathfrak{B}}^{>0}({}^\theta L(\Lambda)^u, (\mathbb{C}_{\Lambda'})^*) = \varinjlim_r \mathrm{ext}_{\mathfrak{B}}^{>0}(M(r), (\mathbb{C}_{\Lambda'})^*) \equiv 0 \qquad \text{for } u \in W
$$

by Lemma 1.12.

In the case where $\Lambda' - \Lambda_0 \notin P_{\mathrm{af},(k-1)}^+$, we necessarily have $\langle \vartheta^\vee, \overline{\Lambda'} \rangle = k$. In this situation, we apply Proposition 3.3 (with Λ replaced by Λ') and note that $\Lambda' = (\Lambda' - \Lambda_0) + \Lambda_0$ and $\overline{\Lambda'} = \overline{\Lambda' - \Lambda_0}$. It then follows that

$$
\mathrm{ext}_{\mathfrak{B}}^{i}\big({}^\theta L(\Lambda)^u, (\mathsf{q} D_{s_0(\Lambda'-\Lambda_0)}^{(k-1)} \otimes \mathbb{C}_{\Lambda_0})^*\big) \longrightarrow \mathrm{ext}_{\mathfrak{B}}^{i+1}\big({}^\theta L(\Lambda)^u, (\mathbb{C}_{\Lambda'})^*\big)
$$
$$
\longrightarrow \mathrm{ext}_{\mathfrak{B}}^{i+1}\big({}^\theta L(\Lambda)^u, (D_{\overline{\Lambda'}}^{(k-1)} \otimes \mathbb{C}_{\Lambda_0})^*\big) \quad (8.3)
$$

is exact for each $i \geq 0$.

As in the discussion for the case $\Lambda' - \Lambda_0 \in P_{\mathrm{af},(k-1)}^+$, Proposition 7.1 together with Theorem 5.1 for level $k-1$ imply that the middle term in (8.3) vanishes for all $i > 0$. Moreover, we have

$$
{}^\theta L(\Lambda)^u \otimes \mathbb{C}_{\Lambda_0} \subset \mathcal{D}_0\big({}^\theta L(\Lambda)^u \otimes \mathbb{C}_{\Lambda_0}\big) \equiv \mathbb{L}^\bullet \mathcal{D}_0\big({}^\theta L(\Lambda)^u \otimes \mathbb{C}_{\Lambda_0}\big),
$$

by Proposition 2.10(5) and

$$
\mathbb{L}^{<0} \mathcal{D}_0\big({}^\theta L(\Lambda)^u \otimes \mathbb{C}_{\Lambda_0}\big) \;\cong\; \varprojlim_r \mathbb{L}^\bullet \mathcal{D}_0\big(M(r)\big).
$$

Thus there is a surjection

$$
\hom_{\mathfrak{B}}\big({}^\theta L(\Lambda)^u \otimes \mathbb{C}_{\Lambda_0},\; (D_{\overline{\Lambda'}}^{(k-1)})^*\big) \longrightarrow \hom_{\mathfrak{B}}\big({}^\theta L(\Lambda)^u \otimes \mathbb{C}_{\Lambda_0},\; (\mathsf{q} D_{s_0(\Lambda'-\Lambda_0)}^{(k-1)})^*\big)
$$

by Theorem 4.1 and Theorem 1.16. Hence the middle term in (8.3) also vanishes for $i = 0$.

Combining the above case-by-case analysis of $u\Lambda \notin W\lambda$ and $u\Lambda \in W\lambda$ (with the subcases $\Lambda' - \Lambda_0 \in P_{\mathrm{af},(k-1)}^+$ and $\Lambda' - \Lambda_0 \notin P_{\mathrm{af},(k-1)}^+$), we obtain

$$
\mathrm{ext}_{\mathfrak{B}}^{>0}\big({}^\theta L(\Lambda)^u, (\mathbb{C}_{\Lambda'})^*\big) = 0 \qquad \text{for all } u \in W_{\mathrm{af}}.
$$

Since $^\theta L(\Lambda)^u$ is a cyclic $\widetilde{\mathfrak{b}}$-module whose head is $\mathbb{C}_{-u\Lambda}$, the corresponding hom space vanishes unless $u\Lambda = \Lambda'$. We therefore conclude that

$$\mathrm{ext}^i_{\mathfrak{B}}\big(^\theta L(\Lambda)^u, (\mathbb{C}_{\Lambda'})^*\big) \cong \mathbb{C}^{\delta_{i0}\delta_{u\Lambda,\Lambda'}}. \tag{8.4}$$

Summarizing the above, we obtain

$$\mathrm{ext}^\bullet_{\mathfrak{B}}(^\theta L(\Lambda)^w, (D_\mu^{(k)})^*) \cong \begin{cases} \mathrm{ext}^\bullet_{\mathfrak{B}}(^\theta L(\Lambda), (\mathbb{C}_{\Lambda'})^*) & \text{if } w \le v, \\ 0 & \text{otherwise} \end{cases}. \tag{8.5}$$

We have

$$\Lambda' = x \cdot \Lambda = x(\Lambda + \rho) - \rho \qquad \text{for } x \in W_\mathrm{af} \tag{8.6}$$

if and only if $x = e$ and $\Lambda = \Lambda'$, since $\rho(K) = h^\vee > 0$ (the dual Coxeter number). Note that

$$P^+_{\mathrm{af},k} + h^\vee \Lambda_0 \subset P^+_{(k+h^\vee)},$$

and that $P^+_{(k+h^\vee)}$ is the fundamental domain for the action of W_af on $P_{(k+h^\vee)}$. Therefore, by the BGG resolution (1.10), we obtain

$$\mathrm{ext}^i_{\mathfrak{B}}(^\theta L(\Lambda)^w, (D_\mu^{(k)})^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, \ w \le v, \ \Lambda = \Lambda', \\ 0 & \text{otherwise} \end{cases}. \tag{8.7}$$

Since the maps in (8.7) are induced from the right-hand side of (8.5) through successive applications of the Demazure functors, the inclusion

$$^\theta L(\Lambda)^w \subset {}^\theta L(\Lambda)^{w'} \qquad (w' \in W_\mathrm{af})$$

induces an isomorphism

$$\mathrm{hom}_{\mathfrak{B}}\big(^\theta L(\Lambda)^w, \ (D_\mu^{(k)})^*\big) \cong \mathrm{hom}_{\mathfrak{B}}\big(^\theta L(\Lambda)^{w'}, \ (D_\mu^{(k)})^*\big) \tag{8.8}$$

whenever $w' \le w \le v$. Since $\{^\theta L(\Lambda)^u\}_{u \in W_\mathrm{af}}$ gives rise to a $\mathbb{D}^{(k)}$-filtration of $^\theta L(\Lambda)$ described in Corollary 1.7, taking the associated graded of this filtration yields

$$\mathrm{ext}^i_{\mathfrak{B}}(\mathbb{D}_\lambda^{(k)}, (D_\mu^{(k)})^*) \cong \begin{cases} \mathbb{C} & \text{if } i = 0, \ \lambda + \mu = 0, \\ 0 & \text{otherwise} \end{cases}, \tag{8.9}$$

where hom-part follows from (8.8), and $\mathrm{ext}^{>0}$-part follows from (8.7). This completes the proof of Theorem 5.1(1) for level k.

We now prove Theorem 5.1(2) for level k. By Proposition 2.1, we have

$$\mathrm{ext}^\bullet_{\mathfrak{C}}\big(\mathbb{W}_\lambda^{(k)}, (W_\mu^{(k)})^*\big) = \mathrm{ext}^\bullet_{\mathfrak{B}}\big(\mathbb{W}_\lambda^{(k)}, (W_\mu^{(k)})^*\big) \qquad \text{for } \lambda, \mu \in P^+.$$

Note that $W_\mu^{(k)} = D_{w_0\mu}^{(k)}$, and that $\mathbb{W}_\lambda^{(k)}$ admits a filtration by $\mathbb{D}_{w\lambda}^{(k)}$ with multiplicity one, by Proposition 2.10(1)(3). Hence, by the $\mathrm{ext}^{>0}$-vanishing part of the first assertion, we obtain

$$\mathrm{ext}^\bullet_{\mathfrak{B}}\big(\mathbb{W}_\lambda^{(k)}, (W_\mu^{(k)})^*\big) \cong \bigoplus_{\nu \in W\lambda} \mathrm{hom}_{\mathfrak{B}}\big(\mathbb{D}_\nu^{(k)}, (D_{w_0\mu}^{(k)})^*\big).$$

This further implies

$$\mathrm{ext}_{\mathfrak{B}}^i\big(\mathbb{W}_\lambda^{(k)}, (W_\mu^{(k)})^*\big) \cong \begin{cases} \mathbb{C} & \text{if } \lambda = \mu, \ i = 0, \\ 0 & \text{otherwise} \end{cases}$$

by the hom-part of the first assertion. This completes the proof of Theorem 5.1(2) for level k. $\qquad\square$

Theorem 8.2. *Theorems 5.1 for level $\leq k$ and Theorem 5.4 for level $< k$ implies Theorem 5.4 for level k.*

Proof. If the module M admits one of the four types of filtrations, then we have a presentation

$$M \cong \varprojlim_r M(r) \in \mathfrak{B}_{\mathrm{bdd}}$$

by a projective system $\{M(r)\}_r$ with surjective transition maps such that each term $M(r)$ admits the same type of filtration.

Thus Theorem 5.1 ensures the desired $\mathrm{Ext}^{>0}$-vanishing by taking limits, since the presentation $M = \varprojlim_r M(r)$ induces a inductive limit which converges to the desired space, and individual term in the inductive limit is zero at $\mathrm{Ext}^{>0}$-terms.

In general, if $\mathrm{Ext}^{>0}(M, N) = 0$ for some modules M and N, then in particular $\mathrm{Ext}^1(M, N) = 0$. Therefore, it suffices to show that the vanishing of Ext^1 with respect to one of the four series of modules implies the existence of the corresponding filtration.

Since the cases of $\mathbb{D}^{(k)}$ and $\mathbb{W}^{(k)}$ are analogous to those of $D^{(k)}$ and $W^{(k)}$, we focus on the cases of $\mathbb{D}^{(k)}$ and $D^{(k)}$, and leave the details of the remaining cases to the reader (cf. Proof of Proposition 5.2).

We prove Theorem 5.4(1) for level k. Suppose $M \in \mathfrak{B}_{\mathrm{bdd}}$ satisfies

$$\mathrm{ext}_{\mathfrak{B}}^1(\mathbb{D}_\lambda^{(k)}, M^\vee) = 0 \qquad \text{for all } \lambda \in P.$$

Then M is generated by (any $\widetilde{\mathfrak{h}}$-splitting of) its head. Hence, we can find a projective cover $f_1 \colon Q \twoheadrightarrow M$ in $\mathfrak{B}$ whose d-grading is bounded from below. A projective module of $\mathfrak{B}_{\mathrm{bdd}}$ is the direct sum of the projective covers of one-dimensional modules, presented in §1.3.1. For each indecomposable direct summand of Q of the form $\mathsf{q}^m Q_{\mu+k\Lambda_0}$ (with $m \in \mathbb{Z}$, $\mu \in P$), we also have a map

$$f_2 \colon \mathsf{q}^m Q_{\mu+k\Lambda_0} \longrightarrow \mathsf{q}^m D_\mu^{(k)}.$$

We rearrange the direct summand $\mathsf{q}^m Q_{\mu+k\Lambda_0}$ if necessary to assume that m is smallest possible. We have $M_j = 0$ for all $j < m$, and hence $M_m \neq 0$ is the lowest d-degree component of M. We then define the maximal quotient

$$M' := \mathsf{q}^m Q_{\mu+k\Lambda_0} \big/ \left(\ker f_1 \cap \mathsf{q}^m Q_{\mu+k\Lambda_0} + \ker f_2\right),$$

which is also the maximal quotient of $\mathsf{q}^m D_\mu^{(k)}$ that admits a (compatible) surjection from M. Set $M'' := \ker(M \to M')$.

Assume, for contradiction, that M' is a proper quotient of $\mathsf{q}^m D_\mu^{(k)}$. Then, by Proposition 5.2, we have

$$\mathrm{ext}_{\mathfrak{B}}^1(\mathbb{D}_\gamma^{(k)}, (M')^*) \neq 0$$

for some $\gamma \in P$. Consider (a portion of) the long exact sequence:

$$\hom_{\mathfrak{B}}(\mathbb{D}_{\gamma}^{(k)}, (M'')^{\vee}) \longrightarrow \text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, (M')^*) \longrightarrow \text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, M^{\vee}).$$

The nonzero element in $\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, (M')^*)$ obtained from Proposition 5.2 corresponds to an extension of M' by a one-dimensional $\widetilde{\mathfrak{b}}$-module of the form $\mathbb{C}_{-\gamma+m'\delta}$ for some $m' \in \mathbb{Z}$.

If this extension were also a quotient of M, it would contradict the maximality of M'. Therefore, this extension class cannot lie in the image of

$$\hom_{\mathfrak{B}}(\mathbb{D}_{\gamma}^{(k)}, (M'')^{\vee}) \longrightarrow \text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, (M')^*).$$

Hence, the resulting class in $\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, (M')^*)$ maps nontrivially to $\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, M^{\vee})$, implying that

$$\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\gamma}^{(k)}, M^{\vee}) \neq 0,$$

which contradicts the assumption.

Therefore, M' cannot be a proper quotient of $\mathsf{q}^m D_{\mu}^{(k)}$.

As a consequence, we have $M' = \mathsf{q}^m D_{\mu}^{(k)}$. Consider (a portion of) the long exact sequence

$$\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\lambda}^{(k)}, M^{\vee}) \longrightarrow \text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\lambda}^{(k)}, (M'')^{\vee}) \longrightarrow \text{ext}_{\mathfrak{B}}^2(\mathbb{D}_{\lambda}^{(k)}, (D_{\mu}^{(k)})^*)$$

for every $\lambda \in P$. Since we have $\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\lambda}^{(k)}, M^{\vee}) = 0$ and $\text{ext}_{\mathfrak{B}}^2(\mathbb{D}_{\lambda}^{(k)}, (D_{\mu}^{(k)})^*) = 0$, it follows that

$$\text{ext}_{\mathfrak{B}}^1(\mathbb{D}_{\lambda}^{(k)}, (M'')^{\vee}) = 0 \qquad \text{for all } \lambda \in P.$$

Thus the problem reduces to the case of M''. By construction, the d-degree m component of M'' is strictly smaller than that of M. Consequently, replacing M by M'' strictly decreases the dimension of the lowest d-degree component of M.

By repeating this procedure, we obtain a decreasing filtration

$$M = M^0 \supset M^1 \supset M^2 \supset \cdots$$

such that each quotient M^i/M^{i-1} is isomorphic to a grading shift of some $D_{\gamma}^{(k)}$ with $\gamma \in P$. Since $M \in \mathfrak{B}_{\text{bdd}}$, the lowest d-degree component of M, as well as those of its subquotients, is finite-dimensional.

It follows that for each $m \in \mathbb{Z}$ there exists $i_m \gg 0$ such that $(M^i)_j = 0$ for all $j < m$ whenever $i > i_m$. This implies that the filtration $\{M^i\}_i$ is separable. Therefore,

$$M \cong \varprojlim_i M/M^i \in \mathfrak{B}_{\text{bdd}}.$$

Hence Theorem 5.4(1) holds for level k.

We consider Theorem 5.4(2) for level k. By (1.5) and the Yoneda interpretation (§1.3.6), our condition on M is equivalent to

$$\text{ext}_{\mathfrak{B}}^1\big(M, (D_{\lambda}^{(k)})^*\big) \cong \text{ext}_{\mathfrak{B}}^1\big(D_{\lambda}^{(k)}, M^{\vee}\big) = 0 \qquad (\lambda \in P),$$

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while the non-vanishing condition of ext^1 in Proposition 5.2 becomes

$$0 \neq \bigoplus_{\mu \in P} \mathrm{ext}_{\mathfrak{B}}^1 \big(M', (D_\mu^{(k)})^*\big) \cong \bigoplus_{\mu \in P} \mathrm{ext}_{\mathfrak{B}}^1 \big(D_\mu^{(k)}, (M')^\vee\big)$$

for a proper quotient M' of $\mathbb{D}_\lambda^{(k)}$.

Thus, arguments parallel to those in the proof of Theorem 5.4(1) for level k imply Theorem 5.4(2) for level k.

This completes the proof. $\qquad\square$

9   Branching rules

We continue to work in the setting of the previous section.

Theorem 9.1. *Let $k \in \mathbb{Z}_{>0}$ and $\lambda \in P$. Then:*

1. *The graded $\widetilde{\mathfrak{b}}$-module $\mathbb{D}_\lambda^{(k+1)} \otimes \mathbb{C}_{\Lambda_0}$ admits a $\mathbb{D}^{(k)}$-filtration;*

2. *The graded $\widetilde{\mathfrak{b}}$-module $D_\lambda^{(k)} \otimes \mathbb{C}_{\Lambda_0}$ admits a $D^{(k+1)}$-filtration.*

Moreover, for every $\mu \in P$, we have the equality:

$$(\mathbb{D}_\lambda^{(k+1)} : \mathbb{D}_\mu^{(k)})_q = (D_\mu^{(k)} : D_\lambda^{(k+1)})_q.$$

Remark 9.2. Theorem 9.1(2) was established by Joseph [35] (see also Naoi [51, Remark 4.15]) in the case where $\mathfrak{g}$ is of type ADE.

Proof of Theorem 9.1. In view of Theorem 5.4, the first two assertions are equivalent to

$$\mathrm{ext}_{\mathfrak{B}}^1(\mathbb{D}_\lambda^{(k+1)}, (D_\mu^{(k)} \otimes \mathbb{C}_{\Lambda_0})^*) = 0 \qquad \text{for all } \lambda, \mu \in P, \tag{9.1}$$

which is precisely equation (7.1).

The final assertion follows from the fact that

$$\left\langle \mathbb{D}_\lambda^{(k+1)}, \ D_\mu^{(k+1)} \right\rangle_{\mathrm{EP}} = \delta_{\lambda,\mu} = \left\langle \mathbb{D}_\lambda^{(k)}, \ D_\mu^{(k)} \right\rangle_{\mathrm{EP}} \qquad \text{for all } \lambda, \mu \in P,$$

by Theorem 5.1. Hence, the transition matrices between their graded characters are transposes of each other by Lemma 2.16. $\qquad\square$

Corollary 9.3. *Let $k \in \mathbb{Z}_{>0}$ and $\lambda \in P$. Then the module $D_\lambda^{(k)}$, regarded as a $\mathfrak{b}$-module, admits a filtration by Demazure modules of $\mathfrak{g}$.*

Proof. By repeatedly applying Theorem 9.1, we obtain a $D^{(l)}$-filtration of $D_\lambda^{(k)}$ such that $\langle \vartheta^\vee, \mu_+ \rangle < l$ for every $\mu \in \overline{\Psi(D_\lambda^{(k)})}$.

Each filtration piece of the form $D_\nu^{(l)}$ (for some $\nu \in P$, up to grading shifts) is contained in V_{ν_+}, since $\nu_+ + l\Lambda_0 \in P_l^+$. In particular, each such $D_\nu^{(l)}$ is a Demazure module of $\mathfrak{g}$.

This proves the claim. $\qquad\square$

Lemma 9.4. *Let $k \in \mathbb{Z}_{>0}$. For each $\lambda \in P^+$, the module $P_{\lambda - k\Lambda_0}$ admits a $\mathbb{W}^{(k)}$-filtration.*

Proof. We have $\mathrm{ext}^1_{\mathfrak{C}}(P_\lambda, (W^{(k)}_\nu \otimes \mathbb{C}_{-k\Lambda_0})^*) = 0$ for every $\nu \in P^+$. Hence, by Theorem 5.4, the module $P_{\lambda - k\Lambda_0}$ admits a $\mathbb{W}^{(k)}$-filtration. $\qquad\square$

Theorem 9.5. *Let $k \in \mathbb{Z}_{>0}$ and $\lambda \in P^+$. Then:*

1. *The graded $\widetilde{\mathfrak{g}}_{\geq 0}$-module $\mathbb{W}^{(k+1)}_\lambda \otimes \mathbb{C}_{\Lambda_0}$ admits a $\mathbb{W}^{(k)}$-filtration;*

2. *The graded $\widetilde{\mathfrak{g}}_{\geq 0}$-module $W^{(k)}_\lambda \otimes \mathbb{C}_{\Lambda_0}$ admits a $W^{(k+1)}$-filtration.*

Moreover, for each $\mu \in P^+$, we have

$$(P_\lambda : \mathbb{W}^{(k)}_\mu)_q = [W^{(k)}_\mu : V_\lambda]_q \quad and \quad (\mathbb{W}^{(k+1)}_\lambda : \mathbb{W}^{(k)}_\mu)_q = (W^{(k)}_\mu : W^{(k+1)}_\lambda)_q. \tag{9.2}$$

Proof. In view of Theorem 5.4, the first two assertions are equivalent to

$$\mathrm{ext}^{>0}_{\mathfrak{C}}\left(\mathbb{W}^{(k+1)}_\lambda, (W^{(k)}_\mu \otimes \mathbb{C}_{\Lambda_0})^*\right) = 0 \qquad \text{for all } \lambda, \mu \in P^+. \tag{9.3}$$

Since we have

$$W^{(k)}_\mu = D^{(k)}_{w_0\mu} = \mathbb{L}^\bullet \mathcal{D}_{w_0}\left(D^{(k)}_\mu\right) \quad \text{and} \quad \mathbb{W}^{(k)}_\lambda = \mathbb{L}^\bullet \mathcal{D}_{w_0}\left(\mathbb{D}^{(k)}_\lambda\right),$$

Theorem 1.16 reduces (9.3) to a special case of (9.1).

The second assertion follows from the equalities

$$\left\langle \mathbb{W}^{(k+1)}_\lambda, W^{(k+1)}_\mu \right\rangle_{\mathrm{EP}} = \delta_{\lambda,\mu} = \left\langle \mathbb{W}^{(k)}_\lambda, W^{(k)}_\mu \right\rangle_{\mathrm{EP}},$$
$$\langle P_\lambda, V_\mu \rangle_{\mathrm{EP}} = \delta_{\lambda,\mu} \qquad \text{for all } \lambda, \mu \in P^+,$$

which follow from Theorem 5.1 and Proposition 2.1. Hence, the transition matrices between their graded characters are transposes of each other by Lemma 2.16. $\qquad\square$

Corollary 9.6. *Let $k \in \mathbb{Z}_{>0}$, and assume Theorem 5.1 and Theorem 5.4 hold at level k. Then:*

1. *If $M \in \mathfrak{B}_{\mathrm{bdd}}$ admits a $\mathbb{D}^{(k)}$-filtration, then for each $\lambda \in P$, we have*

$$(M : \mathbb{D}^{(k)}_\lambda)_{q^{-1}} = \mathrm{gdim} \, \hom_{\mathfrak{B}}\left(M, (D^{(k)}_{-\lambda})^*\right).$$

   Similarly, if $M \in \mathfrak{B}_{\mathrm{bdd}}$ admits a $D^{(k)}$-filtration, then

$$(M : D^{(k)}_\lambda)_{q^{-1}} = \mathrm{gdim} \, \hom_{\mathfrak{B}}\left(\mathbb{D}^{(k)}_{-\lambda}, M^\vee\right).$$

2. *If $M \in \mathfrak{C}_{\mathrm{bdd}}$ admits a $\mathbb{W}^{(k)}$-filtration, then for each $\lambda \in P^+$, we have*

$$(M : \mathbb{W}^{(k)}_\lambda)_{q^{-1}} = \mathrm{gdim} \, \hom_{\mathfrak{B}}\left(M, (W^{(k)}_{-\lambda_-})^*\right).$$

   Similarly, if $M \in \mathfrak{C}_{\mathrm{bdd}}$ admits a $W^{(k)}$-filtration, then

$$(M : W^{(k)}_\lambda)_{q^{-1}} = \mathrm{gdim} \, \hom_{\mathfrak{B}}\left(\mathbb{W}^{(k)}_{-\lambda_-}, M^\vee\right).$$

Proof. Since the proofs in all cases are similar (and simpler for the cases involving $W^{(k)}$ and $D^{(k)}$), we focus on the case of a $\mathbb{D}^{(k)}$-filtration.

Assume $M \in \mathfrak{B}_{\mathrm{bdd}}$. Then there exists a quotient map

$$f : M \twoheadrightarrow \mathsf{q}^m \mathbb{D}_\mu^{(k)}$$

for some $m \in \mathbb{Z}$ and $\mu \in P$. By Lemma 5.6, we obtain the following isomorphism for each $\lambda \in P$:

$$\hom_{\mathfrak{B}} \left(\ker f, \ (D_\lambda^{(k)})^* \right) \oplus \mathsf{q}^{-m} \mathbb{C}^{\delta_{\lambda+\mu,0}} \ \cong \ \hom_{\mathfrak{B}} \left(M, \ (D_\lambda^{(k)})^* \right). \qquad (9.4)$$

In addition, we have

$$\mathrm{ext}_{\mathfrak{B}}^{>0} \left(\ker f, (D_\lambda^{(k)})^* \right) = 0 \qquad \text{for all } \lambda \in P.$$

Using this, we inductively construct a sequence of surjective maps:

$$\cdots \longrightarrow M_3 \longrightarrow M_2 \longrightarrow M_1, \qquad (9.5)$$

where each M_i is $\mathbb{D}^{(k)}$-filtered. Here, we can choose M_i such that the minimal d-degree part of $\ker(M \to M_{i-1})$ maps nontrivially to $\ker(M_i \to M_{i-1})$.

Since $M \in \mathfrak{B}_{\mathrm{bdd}}$, for each $m \in \mathbb{Z}$, there exists $i_m \gg 0$ such that the kernel of the map $M \to M_i$ is concentrated in degrees $> m$ for all $i > i_m$. This implies that

$$M = \varprojlim_i M_i,$$

and the claim follows by repeatedly applying (9.4), as required. $\qquad \square$

10 Applications

Keep the setting of the previous section.

Lemma 10.1. *Let $k \in \mathbb{Z}_{>0}$ and $\lambda \in P^+$. If $\langle \vartheta^\vee, \lambda \rangle < k$, then we have an isomorphism*

$$\mathbb{W}_\lambda^{(k)} \otimes \mathbb{C}_{\Lambda_0} \cong {}^\theta L(-\lambda_- + (k-1)\Lambda_0)$$

as $\widetilde{\mathfrak{b}}$-modules.

Proof. Consider the surjection

$${}^\theta L(-\lambda_- + k\Lambda_0) \longrightarrow \mathbb{W}_\lambda^{(k)}. \qquad (10.1)$$

Its kernel is spanned by ${}^\theta L(-\lambda_- + k\Lambda_0)^u$, where u runs over elements in $\bigcup_{\beta \neq 0} W t_\beta W$. In particular, this kernel contains the submodule generated by the θ-twist of the vector $\mathbf{v}_{s_0(-\lambda_- + k\Lambda_0)}$ as a $\widetilde{\mathfrak{b}}$-module.

The defining relations of ${}^\theta L(-\lambda_- + (k-1)\Lambda_0)$ is given by (1.10) since

$${}^\theta M(-\lambda_- + (k-1)\Lambda_0) \to {}^\theta L(-\lambda_- + (k-1)\Lambda_0) \to 0$$

can be regarded as the projective cover in $\mathfrak{B}$.

The defining relations of ${}^\theta L(-\lambda_- + (k-1)\Lambda_0)$, as given in (1.10), have $\widetilde{\mathfrak{h}}$-weights

$$s_i(-\lambda_- + (k-1)\Lambda_0) + \alpha_i \qquad (i \in \mathtt{I}_{\mathrm{af}}).$$

For $i \neq 0$, they are d-degree zero and are outside of

$$\Psi(V^*_{\lambda - (k-1)\Lambda_0}) = \Psi({}^\theta L(-\lambda_- + (k-1)\Lambda_0)_0),$$

and hence they gives rise to a relation in the degree zero part of $\mathbb{W}^{(k)}_\lambda$. For $i = 0$, this corresponds to an extremal weight of ${}^\theta L(-\lambda_- + k\Lambda_0) \otimes \mathbb{C}_{\Lambda_0}$, that gives rise to a cyclic generator of the $\widetilde{\mathfrak{b}}$-module ${}^\theta L(-\lambda_- + k\Lambda_0)^{s_0} \otimes \mathbb{C}_{\Lambda_0}$. We have $s_0(-\lambda_- + k\Lambda_0) \neq -\lambda_- + k\Lambda_0$, since

$$-\langle \vartheta^\vee, \lambda_- \rangle = \langle \vartheta^\vee, \lambda \rangle < k.$$

Hence, we have ${}^\theta L(-\lambda_- + k\Lambda_0)^{s_0} \subsetneq {}^\theta L(-\lambda_- + k\Lambda_0)$.

Therefore, this vector again gives rise to zero in $\mathbb{W}^{(k)}_\lambda$ when it is realized as a quotient of ${}^\theta L(-\lambda_- + k\Lambda_0)$. This yields a surjection

$$ {}^\theta L(-\lambda_- + (k-1)\Lambda_0) \longrightarrow \mathbb{W}^{(k)}_\lambda \otimes \mathbb{C}_{\Lambda_0}. \qquad (10.2)$$

Note that $s_0 \not\leq w \in W_{\mathrm{af}}$ if and only if $w \in W$. By Corollary 2.5, any $\mathfrak{g}$-stable proper thick Demazure submodule $L(\Lambda)^w$ of $L(\Lambda)$ must satisfy $w \notin W$ and hence $w \geq s_0$. Thus, the kernel of (10.1) is contained in ${}^\theta L(\Lambda)^{s_0}$ with $\Lambda = -\lambda_- + k\Lambda_0$.

Consequently, the surjection (10.2) must be an isomorphism, as required. $\qquad \square$

Remark 10.2. The inner product $\langle \bullet, \bullet \rangle_{\mathrm{EP}}$, calculated through $\mathrm{ext}^\bullet_{\mathfrak{B}}$, coincides with the ($t = 0$ specialization of the) Cherednik inner product (up to constant factor, see [18, Appendix]). Under this inner product, the $t = 0$ specializations of Macdonald polynomials become orthogonal up to normalization factors.

By the Frenkel–Kac construction ([26, 61]) and Lemma 10.1, the graded character $\mathrm{gch}\, \mathbb{W}^{(2)}_\lambda$ is essentially a theta function when $\langle \vartheta^\vee, \lambda \rangle \leq 1$. Hence, the case $k = 2$ of Theorem 5.1(2) asserts that theta functions arise as duals of Demazure characters with respect to the Hall inner product (which is the Euler-Poincaré characteristic calculated through $\mathrm{ext}^\bullet_{\mathfrak{c}}$).

In the same vein, the family $\{\mathrm{gch}\, \mathbb{W}^{(k)}_\lambda\}_{\lambda \in P^+}$ for $k \geq 2$ can be viewed as a natural enhancement of certain spaces of modular forms, as discussed in [36, §13].

Remark 10.3. The proofs of Lemma 10.1 and Corollary 9.3 imply the identities

$$V_\lambda = W^{(k)}_\lambda \quad \text{for} \quad k \gg 0, \qquad P_\lambda = \varprojlim_k \mathbb{W}^{(k)}_\lambda \quad \text{for each} \quad \lambda \in P^+.$$

Accordingly, we can interpret asymptotic behaviors of (5.5) as

$$\lim_{k \to \infty} (\bullet : W^{(k)}_\mu)_q = [\bullet : V_\mu]_q, \qquad \lim_{k \to \infty} (\mathbb{W}^{(k)}_\lambda : \bullet)_q = (P_\lambda : \bullet)_q,$$

whenever these expressions are well-defined.

We define the level k restricted Kostka polynomials as

$$X^{(k)}_{\mu,\Lambda}(q) := ({}^\theta L(\Lambda) : \mathbb{W}^{(1)}_{-\mu_-})_q \in \mathbb{Z}_{\geq 0}[q] \qquad \mu \in P^+, \Lambda \in P^+_{\mathrm{af},k}. \qquad (10.3)$$

In case $\langle d, \Lambda \rangle = 0$, this polynomial is the level k restricted Kostka polynomials defined by [55, 28, 29, 30, 59] in terms of affine crystals [37, 38] by [44, Corollaries 5.12 and 3.6]. Here we further reinterpret these polynomials using our results:

Corollary 10.4. *Assume that $\mathfrak{g}$ is of type* ADE. *Let $k \in \mathbb{Z}_{>0}$ and let $\Lambda \in P_{\mathrm{af},k}^+$ satisfy $\langle d, \Lambda \rangle = 0$. For each $\mu \in P^+$, we have*

$$X_{\mu,\Lambda}^{(k)}(q) = (W_\mu^{(1)} : W_{\overline{\Lambda}}^{(k+1)})_q \in \mathbb{Z}_{\geq 0}[q]. \tag{10.4}$$

Moreover, the right-hand side of (10.4) makes sense for arbitrary types of $\mathfrak{g}$, and always exhibits positivity.

Corollary 10.5. *Let $k \in \mathbb{Z}_{>0}$. Let $W \in \mathfrak{C}_{\mathrm{bdd}}$ be a finite-dimensional $W^{(k)}$-filtered module on which K acts by k, and assume that*

$$\bigcup_{w \in W_{\mathrm{af}}} \mathcal{D}_w(W) \cong \bigoplus_{\Lambda \in P_{\mathrm{af}}^+} L(\Lambda)^{\oplus m_\Lambda(W)} \qquad \text{with } m_\Lambda(W) \in \mathbb{Z}.$$

Then, for each $\lambda \in P^+$, we have

$$\sum_{\overline{\Lambda} = \lambda} m_\Lambda(W)\, q^{\langle d, \Lambda \rangle} = (W : W_\lambda^{(k+1)})_q. \tag{10.5}$$

Remark 10.6. **(1)** Since the right-hand side of (10.4) is well-defined for every $\Lambda \in P_{\mathrm{af}}$ with $\overline{\Lambda} \in P^+$, Corollary 10.4 embeds the set of level k restricted Kostka polynomials into the following family of polynomials indexed by $\lambda, \mu \in P^+$:

$$(\mathbb{W}_\lambda^{(k+1)} : \mathbb{W}_\mu^{(1)})_q \overset{(9.2)}{=} (W_\mu^{(1)} : W_\lambda^{(k+1)})_q \in \mathbb{Z}_{\geq 0}[q].$$

(2) Suppose $\mathfrak{g}$ is not of type E_7, E_8, or F_4. Then, the graded character $\mathrm{gch}\, W_{l\lambda}^{(l)}$ ($l \leq k$, $\lambda \in P^+$) coincides with the character of the tensor product B of Kirillov–Reshetikhin crystals of level l (see [60, Theorem 5.1] and [25, 53, 4, 54]). It follows that the left-hand side of (10.5) equals the character of the set of level k restricted highest weight elements in B (cf. [30, (3.9)] or [44, Definition 5.5]).

More generally, for certain tensor products of Kirillov–Reshetikhin crystals (together with a construction of the corresponding module W), the left-hand side of (10.5) again represents the character of the set of level k restricted highest weight elements [52]. At least in these cases, identity (10.5) provides a module-theoretic interpretation of level k restricted generalized Kostka polynomials, thereby extending Corollary 10.4 (see also [55, 28, 29, 30]).

Proof of Corollary 10.4. In view of Lemma 10.1, the right hand side of (10.3) satisfies
$$({}^\theta L(\Lambda) : \mathbb{W}_{-\mu_-}^{(1)})_q = (\mathbb{W}_{-(\overline{\Lambda})_-}^{(k+1)} : \mathbb{W}_{-\mu_-}^{(1)})_q.$$

We now apply Theorem 9.5 to deduce

$$(\mathbb{W}_{-(\overline{\Lambda})_-}^{(k+1)} : \mathbb{W}_{-\mu_-}^{(1)})_q = (\mathbb{W}_{\overline{\Lambda}}^{(k+1)} : \mathbb{W}_\mu^{(1)})_q = (W_\mu^{(1)} : W_{\overline{\Lambda}}^{(k+1)})_q \in \mathbb{Z}_{\geq 0}[q].$$

The last equality holds for arbitrary types of $\mathfrak{g}$, completing the proof. $\qquad\square$

Proof of Corollary 10.5. By Theorem 2.2, we have $\mathbb{L}^\bullet \mathcal{D}_w(W) \cong \mathcal{D}_w(W)$ for each $w \in W_{\mathrm{af}}$.

Applying Theorems 1.16 and 5.1, we obtain:

$$\mathrm{ext}^i_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), W^*\right) \cong \mathrm{ext}^i_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), \mathcal{D}_w(W)^*\right)$$

$$\cong \varprojlim_w \mathrm{ext}^i_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), \mathcal{D}_w(W)^*\right)$$

$$= \bigoplus_{\Lambda \in P^+_{\mathrm{af}}} \mathrm{ext}^i_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), {}^\theta L(\Lambda)\right)^{\oplus m(\Lambda)}$$

$$= \bigoplus_{\Lambda \in P^+_{\mathrm{af}}} \varprojlim_w \mathrm{ext}^i_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), \mathcal{D}_w(\mathbb{C}_\Lambda)^*\right)^{\oplus m(\Lambda)}$$

$$\cong \bigoplus_{\Lambda \in P^+_{\mathrm{af}}} \mathrm{ext}^i_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), \mathbb{C}^*_\Lambda\right)^{\oplus m(\Lambda)}$$

$$\cong \begin{cases} \displaystyle\bigoplus_{\overline{\Lambda} = \lambda} \mathbb{C}^{\oplus m_\Lambda(W)}_{-\langle d, \Lambda\rangle} & (i = 0), \\ 0 & (i > 0). \end{cases}$$

Here, the last isomorphism follows from the BGG resolution (1.10).

On the other hand, by Lemma 10.1 and Corollary 9.6, we obtain:

$$\mathrm{gdim}\,\mathrm{hom}_{\mathfrak{C}}\left({}^\theta L(\lambda + k\Lambda_0), W^*\right) = \mathrm{gdim}\,\mathrm{hom}_{\mathfrak{C}}\left(\mathbb{W}^{(k+1)}_{-\lambda_-}, W^*\right) = (W : W^{(k+1)}_\lambda)_{q^{-1}}.$$

Comparing this expression with the graded dimension computed above, we conclude the assertion after substituting q^{-1} to q (cf. (5.6)). $\square$

Corollary 10.7. *Let $k \in \mathbb{Z}_{>0}$ and $\lambda, \mu \in P^+$. If $\langle \vartheta^\vee, \lambda\rangle < k$, then*

$$(W^{(k)}_\mu : W^{(k+1)}_\lambda)_q = \begin{cases} q^m & \text{if } \exists\, w \in W_{\mathrm{af}} \text{ such that } \mu + k\Lambda_0 - m\delta = w(\lambda + k\Lambda_0), \\ 0 & \text{otherwise.} \end{cases}$$

In addition, the module $W^{(k+1)}_\lambda$ appears only in the socle of $W^{(k)}_\mu$ (with respect to the $W^{(k+1)}$-filtration).

Proof. By Lemma 10.1 and the definition of $\mathbb{W}^{(k)}$-modules, we have

$$(\mathbb{W}^{(k+1)}_\lambda : \mathbb{W}^{(k)}_\mu)_q = ({}^\theta L(-\lambda_- + k\Lambda_0) : \mathbb{W}^{(k)}_\mu)_q$$

$$= \begin{cases} q^m & \text{if } \exists\, w \in W_{\mathrm{af}} \text{ such that } \mu + k\Lambda_0 - m\delta = w(\lambda + k\Lambda_0), \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 9.1 yields the claimed multiplicity.

Now observe that $\lambda + k\Lambda_0 \in P^+_k$ implies $W^{(k)}_\lambda = V_\lambda$. Since we have

$$V_\lambda \subset W^{(k+1)}_\lambda \subset W^{(k)}_\lambda,$$

it follows that

$$V_\lambda = W^{(k+1)}_\lambda = W^{(k)}_\lambda \subset \mathsf{q}^{-m} W^{(k)}_\mu \subset L(\lambda + k\Lambda_0)$$

as $\widetilde{\mathfrak{b}}$-modules. This shows that $W^{(k+1)}_\lambda$ appears in the socle of $W^{(k)}_\mu$, and that it is the only contribution of $W^{(k+1)}_\lambda$ in the $W^{(k+1)}$-filtration of $W^{(k)}_\mu$, as desired. $\square$

Remark 10.8. Let $k \in \mathbb{Z}_{>0}$ and $\lambda \in P^+$ with $\langle \vartheta^\vee, \lambda \rangle < k$. Then we have a short exact sequence of $\widetilde{\mathfrak{b}}$-modules:

$$0 \to \mathbb{C}_\lambda \longrightarrow \mathcal{D}_0(\mathbb{C}_{\lambda+k\Lambda_0}) \otimes \mathbb{C}_{-k\Lambda_0} \longrightarrow \mathsf{q}^{-1}\mathcal{D}_0(\mathbb{C}_{\lambda+\vartheta+(k+1)\Lambda_0}) \otimes \mathbb{C}_{-(k+1)\Lambda_0} \to 0.$$

Applying $\mathcal{D}_{w_0}$ to this sequence, we obtain another short exact sequence:

$$0 \to \mathsf{q}^\bullet W_\lambda^{(k+1)} \longrightarrow W_\mu^{(k)} \longrightarrow W_\mu^{(k+1)} \otimes \mathbb{C}_{-\Lambda_0} \to 0$$

for some $\bullet \in \mathbb{Z}$, where
$$\mu = \lambda - \langle \vartheta^\vee, \lambda \rangle \vartheta + k\vartheta.$$

The multiplicity computation in Corollary 10.7 extends this to arbitrary $\mu \in P^+$.

The following result is a higher-level analogue of the main result of [44]:

Corollary 10.9. *Let $k, l \in \mathbb{Z}_{>0}$ be such that $k \geq l - 1$. For each $\Lambda \in P_k^+$, the module ${}^\theta L(\Lambda)$ admits a $\mathbb{W}^{(l)}$-filtration.*

Proof. This follows by combining Lemma 10.1 with part (1) of Theorem 9.5. $\quad\square$

Corollary 10.10. *Let $k, l \in \mathbb{Z}_{>0}$. For each $\lambda \in P$, the module $\mathbb{W}_\lambda^{(k)}$ admits a $W^{(l)}$-filtration.*

Proof. By Proposition 7.1, we have

$$\mathrm{ext}_\mathfrak{C}^1 \left(\mathbb{W}_\lambda^{(k)}, (W_\mu^{(1)} \otimes \mathbb{C}_{(k-1)\Lambda_0})^* \right) = 0 \qquad \text{for all } \lambda, \mu \in P^+.$$

In particular, this implies that each $W_\mu^{(1)}$ admits a $W^{(k)}$-filtration.

Moreover, Theorem 5.4 shows that $\mathbb{W}_\lambda^{(k)}$ admits a $\mathbb{W}^{(1)}$-filtration. Since $\mathbb{W}_\lambda^{(1)}$ in turn admits a $W^{(1)}$-filtration, we conclude that $\mathbb{W}_\lambda^{(k)}$ admits a $W^{(l)}$-filtration. $\quad\square$

11 Discussion on the highest weight structure

We retain the setting of the previous section. Let $(\bullet, \bullet)_0$ be the W-invariant bilinear form on $\mathfrak{h}$ normalized by $(\vartheta^\vee, \vartheta^\vee)_0 = 2$. We extend it to a W_{af}-invariant bilinear form $(\bullet, \bullet)$ on $\widetilde{\mathfrak{h}}$ by setting

$$(\lambda+k\Lambda_0+m\delta,\ \mu+k'\Lambda_0+m'\delta) := (\lambda,\mu)_0+km'+k'm \qquad (\lambda, \mu \in P;\ k, k', m, m' \in \mathbb{Z}).$$

We define a W_{af}-invariant quadratic form $\mathbf{q}$ on P_{af} by

$$\mathbf{q}(\lambda + k\Lambda_0 + m\delta) := \big(\lambda + k\Lambda_0 + m\delta,\ \lambda + k\Lambda_0 + m\delta\big) \qquad (\lambda \in P;\ m \in \mathbb{Z}).$$

Since we have
$$\mathbf{q}(\lambda + k\Lambda_0 + m\delta) = (\lambda, \lambda)_0 - 2km,$$
we obtain the inequality

$$\mathbf{q}(\Lambda - \delta) < \mathbf{q}(\Lambda) \qquad \text{for all } \Lambda \in \bigsqcup_{k>0} P_{\mathrm{af},k}.$$

Proposition 11.1 ([36, Proposition 12.5(d)]). *Let $k \in \mathbb{Z}_{>0}$. For each $\Lambda \in P_{\mathrm{af},k}^+$, every $\widetilde{\mathfrak{h}}$-weight Λ' occurring in $L(\Lambda)$ satisfies $\mathbf{q}(\Lambda') \leq \mathbf{q}(\Lambda)$, with equality if and only if $\Lambda' \in W_{\mathrm{af}}\Lambda$.* $\square$

Definition 11.2. We define a partial order $\blacktriangleleft$ on P_{af} by declaring:

$$\Lambda' \blacktriangleleft \Lambda \iff \mathbf{q}(\Lambda') < \mathbf{q}(\Lambda) \quad \text{or}$$
$$\mathbf{q}(\Lambda') = \mathbf{q}(\Lambda) \text{ and } \Lambda' \in \Lambda + \sum_{i \in \mathtt{I}_{\mathrm{af}}} \mathbb{Z}_{\geq 0} \alpha_i.$$

Remark 11.3. Two elements $\Lambda, \Lambda' \in P_{\mathrm{af}}$ can be comparable with respect to $\blacktriangleleft$ only if $\Lambda(K) = \Lambda'(K)$.

Proposition 11.4. *For each $k \in \mathbb{Z}_{>0}$, $\lambda \in P$, and $m \in \mathbb{Z}$, we have:*

1. *The module $\mathsf{q}^m D_\lambda^{(k)}$ is the largest $\widetilde{\mathfrak{b}}$-module generated by a $\widetilde{\mathfrak{h}}$-weight of the form $\lambda + k\Lambda_0 + m\delta$, whose $\widetilde{\mathfrak{h}}$-weights Λ satisfy*

$$\Lambda \blacktriangleleft \lambda + k\Lambda_0 + m\delta. \tag{11.1}$$

2. *The module $\mathsf{q}^m (\mathbb{D}_\lambda^{(k)})^\vee$ is the largest $\widetilde{\mathfrak{b}}$-module cogenerated by a $\widetilde{\mathfrak{h}}$-weight of the form $-\lambda + k\Lambda_0 + m\delta$, whose $\widetilde{\mathfrak{h}}$-weights Λ satisfy*

$$\Lambda \blacktriangleleft -\lambda + k\Lambda_0 + m\delta. \tag{11.2}$$

Proof. We prove the first assertion. By definition, we have

$$\mathsf{q}^m D_\lambda^{(k)} \subset \mathsf{q}^m L(w(\lambda + k\Lambda_0)),$$

where $w \in W_{\mathrm{af}}$ is chosen so that $w(\lambda + k\Lambda_0) \in P_{\mathrm{af}}^+$. By Proposition 11.1, all $\widetilde{\mathfrak{h}}$-weights appearing in $\mathsf{q}^m D_\lambda^{(k)}$ satisfy (11.1). Moreover, by definition, $\mathsf{q}^m D_\lambda^{(k)}$ is generated by a vector of $\widetilde{\mathfrak{h}}$-weight $\lambda + k\Lambda_0 + m\delta$.

It remains to show the maximality of $\mathsf{q}^m D_\lambda^{(k)}$. From the description in (2.3), the defining relations of $D_\lambda^{(k)}$ as a $U(\widetilde{\mathfrak{b}})$-module are given by reflections of the form

$$\lambda + k\Lambda_0 + m_\alpha \alpha, \qquad m_\alpha = \begin{cases} \langle \alpha^\vee, \lambda + k\Lambda_0 \rangle + 1 & \text{if } \langle \alpha^\vee, \lambda + k\Lambda_0 \rangle \leq 0, \\ 1 & \text{otherwise} \end{cases} \tag{11.3}$$

for each real root $\alpha \in \Delta_{\mathrm{af}}^+$.

By convexity of the level sets of $\mathbf{q}$ and Proposition 11.1, we deduce that the weights appearing in (11.3) do not satisfy (11.1). This confirms the maximality of $\mathsf{q}^m D_\lambda^{(k)}$, proving the first assertion.

We now prove the second assertion. By definition, $\mathsf{q}^{-m} \mathbb{D}_\lambda^{(k)}$ is generated by a vector of $\widetilde{\mathfrak{h}}$-weight $\lambda - k\Lambda_0 - m\delta$. Moreover, it appears as a subquotient of

$$^\theta L(w(-\lambda + k\Lambda_0 + m\delta)),$$

where $w \in W_{\mathrm{af}}$ is chosen such that $w(-\lambda + k\Lambda_0) \in P_{\mathrm{af}}^+$. Again by Proposition 11.1, any $\widetilde{\mathfrak{h}}$-weight in the W_{af}-orbit of $-\lambda + k\Lambda_0 + m\delta$ is quotiented out. Therefore, all $\widetilde{\mathfrak{h}}$-weights of $\mathsf{q}^m (\mathbb{D}_\lambda^{(k)})^\vee$ satisfy (11.2).

It remains to prove the maximality of $\mathbb{D}_\lambda^{(k)}$. By Theorem 5.4, the projective cover $Q_{\lambda-k\Lambda_0}$ of $\mathbb{D}_\lambda^{(k)}$ in $\mathfrak{B}$ admits a $\mathbb{D}^{(k)}$-filtration. Since $Q_{\lambda-k\Lambda_0}$ has a simple head, any grading shift of $\mathbb{D}_\mu^{(k)}$ appearing in $Q_{\lambda-k\Lambda_0}$ that extends $\mathbb{D}_\lambda^{(k)}$ must correspond to an extension by a one-dimensional module $\mathbb{C}_{\mu-k\Lambda_0}$.

For each $\mu \in P$, we have

$$(Q_{\lambda-k\Lambda_0} : \mathbb{D}_\mu^{(k)})_{q^{-1}} = \operatorname{gdim} \hom_{\mathfrak{B}}\left(Q_{\lambda-k\Lambda_0}, (D_{-\mu}^{(k)})^*\right)$$
$$= \operatorname{gdim} \hom_{\mathfrak{h}\oplus\mathbb{C}d}\left(\mathbb{C}_\lambda, (D_{-\mu}^{(k)})^*\right)$$

by Corollary 9.6.

Thus, $\mathbb{C}_{\mu+s\delta}$ can extend $\mathbb{D}_\lambda^{(k)}$ only if $\mathbb{C}_\lambda$ appears in $\mathsf{q}^s(D_{-\mu}^{(k)})^*$. By the first assertion, this implies

$$-\lambda + k\Lambda_0 \blacktriangleleft -\mu - s\delta + k\Lambda_0.$$

Hence, any $\widetilde{\mathfrak{h}}$-weight occurring in a potential one-dimensional extension as a $\widetilde{\mathfrak{b}}$-module violates (11.2). This establishes the maximality of $\mathbb{D}_\lambda^{(k)}$ and completes the proof of the second assertion.

The proof is complete. $\qquad\square$

Corollary 11.5. *For each $k \in \mathbb{Z}_{>0}$, $\lambda \in P_+$, and $m \in \mathbb{Z}$, we have:*

1. *The module $\mathsf{q}^m W_\lambda^{(k)}$ is the largest $\widetilde{\mathfrak{g}}_{\geq 0}$-module generated by a $\widetilde{\mathfrak{h}}$-weight of the form $\lambda_- + k\Lambda_0 + m\delta$, whose $\widetilde{\mathfrak{h}}$-weights Λ satisfy*

$$\Lambda \blacktriangleleft \lambda + k\Lambda_0 + m\delta.$$

2. *The module $\mathsf{q}^m(\mathbb{W}_\lambda^{(k)})^\vee$ is the largest $\widetilde{\mathfrak{g}}_{\geq 0}$-module cogenerated by a $\widetilde{\mathfrak{h}}$-weight of the form $-\lambda + k\Lambda_0 + m\delta$, whose $\widetilde{\mathfrak{h}}$-weights Λ satisfy*

$$\Lambda \blacktriangleleft -\lambda + k\Lambda_0 + m\delta.$$

Proof. Since any $\widetilde{\mathfrak{g}}_{\geq 0}$-module extension restricts to a $\widetilde{\mathfrak{b}}$-module extension, we may restrict the module structures and reduce the assertions to Proposition 11.4.

The first assertion follows from Proposition 11.4(1), using the identification $W_\lambda^{(k)} = D_{\lambda_-}^{(k)}$. The second assertion follows from Proposition 2.10(1)(3) and Proposition 11.4(2). $\qquad\square$

Remark 11.6. Proposition 11.4 shows that the collection $\{\mathsf{q}^m D_\lambda^{(k)}\}_{\lambda,m}$ forms the set of standard modules, and $\{\mathsf{q}^m(\mathbb{D}_\lambda^{(k)})^\vee\}_{\lambda,m}$ forms the set of costandard modules, with respect to the partial order $\blacktriangleleft$, in the sense of highest weight theory [17] (without the finiteness assumption on the poset). An axiomatic treatment of such a setting is given in [7].

The same description applies to the pair $\{\mathsf{q}^m W_\lambda^{(k)}\}_{\lambda,m}$ and $\{\mathsf{q}^m(\mathbb{W}_\lambda^{(k)})^\vee\}_{\lambda,m}$ by Corollary 11.5. Thus, Theorem 5.4 establishes that the collection of modules under consideration satisfies the axioms required for a highest weight category with respect to $\blacktriangleleft$.

Appendix: Higher level q-Cauchy kernel identities

In this appendix, we derive several numerical identities from the main results of the paper. We adopt the same conventions as in the main body (see §2).

In addition, we define gch_x and gch_y as variants of gch in which the $\mathfrak{h}$-character e^λ is recorded independently as x^λ (for gch_x) and y^λ (for gch_y), while the q-grading is recorded identically in both cases.

Throughout this appendix, we understand all equalities to hold formally.

Proposition A.1. *For each $k \in \mathbb{Z}_{>0}$, we have the following identity:*

$$\sum_{\lambda \in P^+} (\mathrm{gch}_x V_\lambda^*) \cdot \mathrm{gch}_y P_\lambda = \sum_{\lambda \in P^+} (\mathrm{gch}_x W_{-w_0\lambda}^{(k)}) \cdot \mathrm{gch}_y \mathbb{W}_\lambda^{(k)}.$$

Proof. By the numerical identities

$$\lim_{k \to \infty} \mathrm{gch}_x W_{-w_0\lambda}^{(k)} = \mathrm{gch}_x V_\lambda^*, \qquad \lim_{k \to \infty} \mathrm{gch}_y \mathbb{W}_\lambda^{(k)} = \mathrm{gch}_y P_\lambda,$$

as observed in Remark 10.3, it suffices to show that the right-hand side is independent of k. This follows from (9.2), as:

$$
\begin{aligned}
\sum_{\lambda \in P^+} \mathrm{gch}_x W_{-w_0\lambda}^{(k)} \cdot \mathrm{gch}_y \mathbb{W}_\lambda^{(k)} &= \sum_{\lambda,\mu \in P^+} (W_{-w_0\lambda}^{(k)} : W_{-w_0\mu}^{(k+1)})_q \cdot \mathrm{gch}_x W_{-w_0\mu}^{(k+1)} \cdot \mathrm{gch}_y \mathbb{W}_\lambda^{(k)} \\
&= \sum_{\lambda,\mu \in P^+} (\mathbb{W}_{-w_0\mu}^{(k+1)} : \mathbb{W}_{-w_0\lambda}^{(k)})_q \cdot \mathrm{gch}_x W_{-w_0\mu}^{(k+1)} \cdot \mathrm{gch}_y \mathbb{W}_\lambda^{(k)} \\
&= \sum_{\lambda,\mu \in P^+} (\mathbb{W}_\mu^{(k+1)} : \mathbb{W}_\lambda^{(k)})_q \cdot \mathrm{gch}_x W_{-w_0\mu}^{(k+1)} \cdot \mathrm{gch}_y \mathbb{W}_\lambda^{(k)} \\
&= \sum_{\mu \in P^+} \mathrm{gch}_x W_{-w_0\mu}^{(k+1)} \cdot \mathrm{gch}_y \mathbb{W}_\mu^{(k+1)},
\end{aligned}
$$

as desired. $\qquad\square$

Proposition A.2. *For each $k \in \mathbb{Z}_{>0}$, we have the following identity:*

$$\sum_{\lambda \in P} \mathrm{ch}_x \mathbb{C}_{-\lambda} \cdot \mathrm{gch}_y Q_\lambda = \sum_{\lambda \in P} (\mathrm{gch}_x D_{-\lambda}^{(k)}) \cdot \mathrm{gch}_y \mathbb{D}_\lambda^{(k)}.$$

Proof. By Theorem 9.1, an argument analogous to the proof of Proposition A.1 shows that

$$\sum_{\lambda \in P} (\mathrm{gch}_x D_{-\lambda}^{(k+1)}) \cdot \mathrm{gch}_y \mathbb{D}_\lambda^{(k+1)} = \sum_{\lambda \in P} (\mathrm{gch}_x D_{-\lambda}^{(k)}) \cdot \mathrm{gch}_y \mathbb{D}_\lambda^{(k)}.$$

The degree-zero part with respect to the d-grading (i.e., the coefficient of q^0) is shared by both sides and corresponds to the characters of

$$D_\lambda := U(\mathfrak{b}) \cdot \mathbf{v}_\lambda \subset V(\lambda), \qquad \mathring{D}_\lambda := \frac{D_\lambda}{\sum_{\mu \in W\lambda,\, D_\lambda \supsetneq D_\mu} D_\mu}.$$

We have

$$\lim_{k \to \infty} \mathrm{gch}_x D_\lambda^{(k)} = \mathrm{gch}_x D_\lambda$$

since $(\lambda + k\Lambda_0) \in WP_{\mathrm{af}}^+$ for $k \gg 0$ and fixed $\lambda \in P$.

Similarly, we have

$$\lim_{k\to\infty} \mathrm{gch}_y \mathbb{D}_\lambda^{(k)} = \mathrm{gch}_y\left(U(\widetilde{\mathfrak{b}}) \otimes_{U(\mathfrak{b}+\widetilde{\mathfrak{h}})} \mathring{D}_\lambda\right),$$

because applying $s_0(\lambda + k\Lambda_0)$ increases the d-degree indefinitely as $k \to \infty$, and thus the quotient by d-degree $\neq 0$ parts in $Q_{\lambda+k\Lambda_0}$ disappears in the limit. It follows that

$$\mathrm{gch}_y U(\widetilde{\mathfrak{b}}/(\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{h}})) \cdot \mathrm{gch}_y \mathring{D}_\lambda = \lim_{k\to\infty} \mathrm{gch}_y \mathbb{D}_\lambda^{(k)} \qquad (\lambda \in P),$$

and hence,

$$\mathrm{gch}_y U(\widetilde{\mathfrak{b}}/(\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{h}})) \cdot \sum_{\lambda\in P}(\mathrm{gch}_x D_{-\lambda}) \cdot \mathrm{gch}_y \mathring{D}_\lambda = \sum_{\lambda\in P}(\mathrm{gch}_x D_{-\lambda}^{(k)}) \cdot \mathrm{gch}_y \mathbb{D}_\lambda^{(k)}.$$

By [62, Theorem 1.13], we have the identity:

$$\sum_{\lambda\in P}(\mathrm{gch}_x D_{-\lambda}) \cdot \mathrm{gch}_y \mathring{D}_\lambda = \mathrm{ch}_{x,y}\,\mathbb{C}[B],$$

where $\mathrm{ch}_{x,y}$ denotes the character as an $(\mathfrak{h} \oplus \mathfrak{h})$-module. We also have

$$\mathrm{ch}_{x,y}\,\mathbb{C}[B] = \sum_{\lambda\in P} \mathrm{ch}_x\,\mathbb{C}_{-\lambda} \cdot \mathrm{ch}_y\left(U(\mathfrak{b}) \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda\right)$$

by direct computation.

Putting everything together, we obtain:

$$\begin{aligned}
\sum_{\lambda\in P}(\mathrm{gch}_x D_{-\lambda}^{(k)}) \cdot \mathrm{gch}_y \mathbb{D}_\lambda^{(k)} &= \mathrm{gch}_y U(\widetilde{\mathfrak{b}}/(\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{h}})) \cdot \sum_{\lambda\in P}(\mathrm{gch}_x D_{-\lambda}) \cdot \mathrm{gch}_y \mathring{D}_\lambda \\
&= \mathrm{gch}_y U(\widetilde{\mathfrak{b}}/(\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{h}})) \cdot \sum_{\lambda\in P} \mathrm{ch}_x\,\mathbb{C}_{-\lambda} \cdot \mathrm{ch}_y(U(\mathfrak{n}) \otimes \mathbb{C}_\lambda) \\
&= \sum_{\lambda\in P} \mathrm{ch}_x\,\mathbb{C}_{-\lambda} \cdot \mathrm{gch}_y Q_\lambda,
\end{aligned}$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let G be the connected, simply connected, simple algebraic group over $\mathbb{C}$ whose Lie algebra is $\mathfrak{g}$, and let N and H denote the connected closed subgroups of G whose Lie algebras are $\mathfrak{n}$ and $\mathfrak{h}$, respectively.

Let $\widetilde{G}$ denote the (pro-)algebraic group whose Lie algebra is $[\widetilde{\mathfrak{g}}_{\geq 0}, \widetilde{\mathfrak{g}}_{\geq 0}]$, and let $\mathbb{C}[\widetilde{G}]$ be its coordinate ring. Let $\widetilde{B}$ denote its closed subgroup scheme with Lie algebra $\widetilde{\mathfrak{b}} \cap [\widetilde{\mathfrak{g}}_{\geq 0}, \widetilde{\mathfrak{g}}_{\geq 0}]$, and let $\mathbb{C}[\widetilde{B}]$ denote its coordinate ring.

Both $\widetilde{G}$ and $\widetilde{B}$ admit actions of $B \times B$ as well as of $\mathbb{C}^\times$, the latter recording the grading. Accordingly, we define the graded character $\mathrm{gch}_{x,y}$, which records the left H-action via $x^\bullet$, the right H-action via $y^\bullet$, and the common grading via the variable q.

The restricted dual $(\cdot)^\vee$ in Theorem A.3 is understood as the restricted dual taken with respect to the structure of an $(H \times H \times \mathbb{G}_m)$-module—note that this differs from the restricted dual considered in §1.3.

Theorem A.3. *For each $k \in \mathbb{Z}_{>0}$, we have the following identities:*

$$\mathrm{gch}_{x,y}\, \mathbb{C}[\widetilde{G}]^\vee = \sum_{\lambda \in P^+} (\mathrm{gch}_x\, W^{(k)}_{-w_0\lambda}) \cdot \mathrm{gch}_y\, \mathbb{W}^{(k)}_\lambda,$$

$$\mathrm{gch}_{x,y}\, \mathbb{C}[\widetilde{B}]^\vee = \sum_{\lambda \in P} (\mathrm{gch}_x\, D^{(k)}_{-\lambda}) \cdot \mathrm{gch}_y\, \mathbb{D}^{(k)}_\lambda.$$

Remark A.4. When $k = 1$, Theorem A.3 recovers the identities established in [19] and [22, 20], respectively. As explained in Remark 10.2, an identity involving theta functions arises when $\mathfrak{g} = \mathfrak{sl}(2)$ and $k = 2$.

Proof of Theorem A.3. We set

$$\widetilde{G}_+ := \ker\big(\widetilde{G} \xrightarrow{\mathrm{ev}_0} G\big),$$

where ev_0 denotes the evaluation map at $z = 0$. We then have

$$\mathbb{C}[\widetilde{G}] \cong \mathbb{C}[G] \otimes \mathbb{C}[\widetilde{G}_+], \qquad \mathbb{C}[\widetilde{B}] \cong \mathbb{C}[H \cdot N] \otimes \mathbb{C}[\widetilde{G}_+].$$

By the algebraic Peter–Weyl theorem, we have

$$\mathbb{C}[\widetilde{G}] \cong \bigoplus_{\lambda \in P^+} V_\lambda \boxtimes \big(V_\lambda^* \otimes \mathbb{C}[\widetilde{G}_+]\big).$$

Moreover, for each $\lambda \in P^+$, we have an isomorphism of right $\widetilde{\mathfrak{g}}_{\geq 0}$-modules:

$$P_\lambda \xrightarrow{\sim} V_\lambda \otimes \mathbb{C}[\widetilde{G}_+]^\vee,$$

compatible with the left G-action. Taking the character as an $(H \times H \times \mathbb{G}_m)$-module, we obtain

$$\mathrm{gch}_{x,y}\, \mathbb{C}[\widetilde{G}]^\vee = \sum_{\lambda \in P^+} \big(\mathrm{gch}_x\, V_\lambda^*\big) \cdot \mathrm{gch}_y\, P_\lambda.$$

Thus, Proposition A.1 establishes the first identity.

Similarly, we have

$$\mathbb{C}[\widetilde{B}]^\vee \cong \bigoplus_{\lambda \in P} \mathbb{C}_{-\lambda} \boxtimes \big(\mathbb{C}_\lambda \otimes \mathbb{C}[N \cdot \widetilde{G}_+]^\vee\big),$$

and, for each $\lambda \in P$, an isomorphism of right $\widetilde{\mathfrak{b}}$-modules:

$$Q_\lambda \xrightarrow{\sim} \mathbb{C}_\lambda \otimes \mathbb{C}[N \cdot \widetilde{G}_+]^\vee,$$

compatible with the left H-action. Taking the character again as an $(H \times H \times \mathbb{G}_m)$-module, we obtain

$$\mathrm{gch}_{x,y}\, \mathbb{C}[\widetilde{B}]^\vee = \sum_{\lambda \in P} \big(\mathrm{gch}_x\, \mathbb{C}_{-\lambda}\big) \cdot \mathrm{gch}_y\, Q_\lambda.$$

Hence, Proposition A.2 yields the second identity. $\qquad\square$

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