A geometric realization of Catalan functions*

Syu KATO†

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Abstract

We construct a smooth projective algebraic variety $\mathfrak{X}_\Psi$ that compactifies the total space of an equivariant vector subbundle of the cotangent bundle of the flag variety for $GL(n)$ (determined by a root ideal Ψ). The variety $\mathfrak{X}_\Psi$ carries a natural family of line bundles whose spaces of global sections give rise to the symmetric functions known as Catalan functions, defined by Chen-Haiman [PhD thesis, University of California, Berkeley (2010)] and studied in Blasiak-Morse-Pun-Summers [J. Amer. Math. Soc. (2019), Invent. Math. (2024)]. Analyzing the geometry of $\mathfrak{X}_\Psi$, we prove the vanishing conjectures of Chen-Haiman and (the tame case of that of) Blasiak-Morse-Pun, as well as the monotonicity conjectures of Shimozono-Weyman [Electronic J. Combin. (2000)].

Introduction

In search of a better understanding of the internal structure of Macdonald polynomials [26] after Haiman's solution [14] of the Macdonald positivity conjecture, LaPointeLascouxMorse [23] proposed the concept of k-Schur functions. These functions have been shown to represent Schubert classes of affine Grassmannians [21], and hence play a role in the study of the quantum cohomology of the flag variety X associated with $G = GL(n, \mathbb{C})$ [34, 22]. However, the precise relationship with Macdonald polynomials, as well as their connection to computations in quantum cohomology, is not yet fully understood.

ChenHaiman [8] made remarkable conjectures about the internal structure of k-Schur functions and their generalizations, sometimes called Catalan functions, through a geometric interpretation in terms of certain vector bundles on the flag variety X. Their conjectures include, as special cases, a conjectural resolution of a question posed by Broer [5, 3.16] (in type A) and a conjecture of ShimozonoWeyman [35]. Although the numerical part of their conjectures has been established by BlasiakMorsePunSummers [3, 2], the cohomological vanishing component, further refined in [2], remains open. These conjectures lie at the heart of the geometric framework of [8], and also underpin the logic of the monotonicity conjectures in [35, §2.10]. In this light, the vanishing assertions may be viewed as completing a conceptual framework whose structure has gradually emerged through decades of work by ChenHaiman, ShimozonoWeyman, and others.

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†Department of Mathematics, Kyoto University, Oiwake Kita-Shirakawa Sakyo Kyoto 606-8502 JAPAN `E-mail:syuchan@math.kyoto-u.ac.jp`

In this paper, we define and study a smooth projective variety $\mathfrak{X}_\Psi$, which compactifies the G-equivariant vector subbundle $T^*_\Psi X \subset T^* X$ introduced in [8]. To state our results more precisely, we fix notation as follows: Let Ψ denote a Dyck path of size n, corresponding to a root ideal in type A_{n-1} [6]. Then, the above $T^*_\Psi X$ is specified by Ψ. Let $\mathtt{Par}$ denote the set of partitions of length at most n. The set $\mathtt{Par}$ parametrizes the irreducible polynomial representations of G up to isomorphism. For each $\lambda \in \mathtt{Par}$, let $V(\lambda)$ denote the corresponding representation, whose character is the Schur polynomial s_λ. Encoding the $\mathbb{C}^\times$-weights as powers of q, we consider the graded character $\mathrm{gch}\, V$ of a rational $(G \times \mathbb{C}^\times)$-module V. For a $(G \times \mathbb{C}^\times)$-module M, let $M^\vee$ denote its restricted dual, i.e., the direct sum of the duals of $\mathbb{C}^\times$-isotypic components.

The Catalan symmetric function associated to a Dyck path Ψ of size n and $\lambda \in \mathtt{Par}$ is defined as:

$$H(\Psi;\lambda) := \sum_{\mu \in \mathtt{Par}, m \in \mathbb{Z}} q^m s_\mu \cdot \dim \mathrm{Hom}_{G \times \mathbb{C}^\times}(V(\mu) \boxtimes \mathbb{C}_{m\delta}, H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda))^\vee), \tag{0.1}$$

where $H(\Psi;\lambda) = H(\Psi;\lambda;w_0)$ in [3, (2.2)]. We remark that the sum in (0.1) is finite, while we have

$$\dim H^0(T^*_\Psi X, \mathcal{O}_{T_\Psi X}(\lambda)) = \infty$$

in general. Most of the irreducible rational representations of G appearing in $H^0(T^*_\Psi X, \mathcal{O}_{T_\Psi X}(\lambda))$ are therefore not captured by (0.1); they are precisely the rational but non-polynomial representations of G.

Our main results are summarized below.

Theorem A ($\doteq$ Theorems 3.9, 5.1, and 4.1)**.** *There exists a smooth projective algebraic variety $\mathfrak{X}_\Psi$ equipped with a $(G\times\mathbb{C}^\times)$-action which satisfies the following properties:*

1. *There is an open embedding $T^*_\Psi X \hookrightarrow \mathfrak{X}_\Psi$;*
2. *For each $\lambda \in \mathtt{Par}$, there exists a $(G \times \mathbb{C}^\times)$-equivariant line bundle $\mathcal{O}_{\mathfrak{X}_\Psi}(\lambda)$ on $\mathfrak{X}_\Psi$ such that:*

$$H^{>0}(\mathfrak{X}_\Psi, \mathcal{O}_{\mathfrak{X}_\Psi}(\lambda)) = 0,$$
$$\mathrm{gch}\, H^0(\mathfrak{X}_\Psi, \mathcal{O}_{\mathfrak{X}_\Psi}(\lambda))^\vee = [H(\Psi;\lambda)]_{q \mapsto q^{-1}}.$$

3. *There exists a $(G \times \mathbb{C}^\times)$-equivariant effective Cartier divisor ∂ supported on $X_\Psi \setminus T^*_\Psi X$ such that*

$$H^{>0}(\mathfrak{X}_\Psi, \mathcal{O}_{\mathfrak{X}_\Psi}(\lambda + m\partial)) = 0 \quad \textit{for all } \lambda \in \mathtt{Par},\ m \geq 0.$$

In particular, we have

$$H^{>0}(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda)) = \varinjlim_m H^{>0}(\mathfrak{X}_\Psi, \mathcal{O}_{\mathfrak{X}_\Psi}(\lambda + m\partial)) = 0.$$

A parabolic analogue of this vanishing result also holds; see Corollary 5.4.

Part (3) of Theorem A resolves the vanishing conjecture of Chen-Haiman [8, Conjecture 5.4.3(2)]. Combined with [2, Theorem 2.18], this establishes [8, Conjecture 5.4.3] in full generality. Since this conjecture answers a question of Broer [5, 3.16] (in type A) and generalizes that of ShimozonoWeyman [35, §2.4], our result settles these as well (see Remark 5.2). When Ψ is maximal (so that $T^*_\Psi X = T^*X$), the variety $\mathcal{X}_\Psi$ recovers the smooth resolution [31, 28] of Lusztig's compactification [25] of the nilpotent cone of $\mathfrak{gl}(n, \mathbb{C})$. For reference, we note in Remark 5.5 that our proof generalizes to positive characteristic.

As a corollary of Theorem A, we find:

Corollary B ($\doteq$ Lemma 5.6)**.** *There exists an action of $GL(n, \mathbb{C}[\![z]\!]) \rtimes \mathbb{G}_m$ on X_Ψ that makes*

$$H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda))^\vee \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathcal{X}_\Psi, \mathcal{O}_{\mathcal{X}_\Psi}(\lambda))^\vee, \qquad \lambda \in \mathtt{Par}$$

into a quotient as (graded) representations of $\mathfrak{gl}(n, \mathbb{C}[z])$.

A local chart analysis of $\mathcal{X}_\Psi$ further leads to the following result:

Theorem C ($\doteq$ Theorem 5.8)**.** *For each $\lambda \in \mathtt{Par}$, the space $H^0(\mathcal{X}_\Psi, \mathcal{O}_{\mathcal{X}_\Psi}(\lambda))$ has a simple head as a (graded) $\mathfrak{gl}(n, \mathbb{C}[z])$-module.*

As an additional consequence of our construction, we have:

Corollary D ($\doteq$ Corollary 5.21)**.** *Let $\Psi' \subset \Psi$ be an inclusion of Dyck paths that yields $T^*_{\Psi'} X \subset T^*_\Psi X$. For each $\lambda \in \mathtt{Par}$, the restriction map*

$$H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda)) \longrightarrow H^0(T^*_{\Psi'} X, \mathcal{O}_{T^*_{\Psi'} X}(\lambda))$$

is surjective.

Corollary 5.21 establish [35, Conjecture 12] and its generalizations as their module-theoretic upgrades.

The organization of this paper is as follows: In Section 1, we fix notation and recall basic facts. In Section 2, we present a new expression for the rotation theorem from [2]. In Section 3, we construct the variety $\mathcal{X}_\Psi$ (Theorem 3.9) and work out an explicit example (Example 3.10). In Section 4, we establish parts (1) and (2) of Theorem A. In Section 5, we explore consequences of our construction, including:

- part (3) of Theorem A (Section 5.1),
- Corollary B and Theorem C (Section 5.2), and
- Corollary D (Section 5.4).

Since the proof of Theorem C is technically involved, we devote Section 5.3 and its three subsubsections to this purpose.

A previous version of this paper claimed the full proofs of two conjectures of Blasiak-Morse-Pun. We retract the general case of [3, Conjecture 3.4(ii)], retaining only the tame case (Theorem 5.1) due to a gap in the original proof. In contrast, we make [3, Conjecture 3.4(iii)] explicit as Corollary 5.3.

The varieties introduced here serve as natural geometric counterparts of the Catalan functions. A natural direction for future research is to place these constructions within the context of topological field theories and geometric realizations of Macdonald polynomials arising from $G = GL(n)$. We hope to answer these questions in the sequel.

1 Preliminaries

We work over the field $\mathbb{C}$ of complex numbers. By a *variety*, we mean a separated, integral, normal scheme of finite type over $\mathbb{C}$. We often identify a variety $\mathcal{X}$ with its set of $\mathbb{C}$-points $\mathcal{X}(\mathbb{C})$ when the topology and scheme structure are clear from context. In particular, the algebraic groups $\mathbb{G}_m$ and $\mathbb{G}_a$ denote the multiplicative group $\mathbb{C}^\times$ and the additive group $\mathbb{C}$, respectively.

For a $\mathbb{C}$-vector space V, let $S^\bullet V = \bigoplus_{i\geq 0} S^i V$ denote its symmetric algebra. Let $\mathtt{L}$ be a free abelian monoid. A $\mathtt{L}$-graded vector space V is a $\mathbb{C}$-vector space V equipped with a direct sum decomposition $V = \bigoplus_{a\in\mathtt{L}} V_a$ such that $\dim V_a < \infty$ for each $a \in \mathtt{L}$. For a $\mathtt{L}$-graded vector space $V = \bigoplus_{a\in\mathtt{L}} V_a$, we set

$$V^\vee := \bigoplus_{a\in\mathtt{L}} V_a^*.$$

A $\mathtt{L}$-graded ring R is a unital $\mathbb{C}$-algebra that is a $\mathtt{L}$-graded $\mathbb{C}$-vector space such that $\mathbb{C}\cdot 1 = R_0$ and $R_a \cdot R_{a'} \subset R_{a+a'}$ $(a, a' \in \mathtt{L})$.

If R is commutative, then we define

$$\mathrm{Proj}_{\mathtt{L}}\, R := \Big(\mathrm{Spec}\, R \setminus \mathrm{irr}\Big)/(\mathbb{G}_m)^{\mathrm{rank}\,\mathtt{L}}, \tag{1.1}$$

where $\mathrm{irr} \subset \mathrm{Spec}\, R$ denotes the closed subscheme consisting of points where the $(\mathbb{G}_m)^{\mathrm{rank}\,\mathtt{L}}$-action is not free.

For a representation M, we define its head to be its largest semisimple quotient module.

For general background, we refer the reader to Kumar [20] and Chriss-Ginzburg [9].

1.1 Algebraic Groups

We fix an integer $n > 0$ and define the algebraic group

$$G = \mathbb{C}^\times \mathrm{Id} \cdot SL(n) = GL(n) \subset M_n \cong \mathbb{C}^{n^2}.$$

We also define a (pro-)algebraic group $\mathbf{G} = \mathbb{C}^\times \mathrm{Id} \cdot SL(n, \mathbb{C}[\![z]\!])$ over $\mathbb{C}$. We also consider the group

$$G(\!(z)\!) := \mathbb{C}^\times \mathrm{Id} \cdot SL(n, \mathbb{C}(\!(z)\!)),$$

regarded as a topological group.

Let $E_{ij} \in M_n$ $(1 \leq i, j \leq n)$ be the matrix unit. Let $T \subset G$ be the diagonal torus and let $B \subset G$ (resp. $B^- \subset G$) be the upper (resp. the lower) triangular part of G. The group $N := [B, B] \subset B$ is the group of upper unitriangular matrices. We have the evaluation map

$$\mathtt{ev}_0 : \mathbf{G} \longrightarrow G \qquad z \mapsto 0.$$

We set $\mathbf{B} := \mathtt{ev}_0^{-1}(B)$.

For each $1 \leq i < n$, let $P_i \subset G$ be the (algebraic) subgroup generated by B and $\mathrm{Id} + \mathbb{C}E_{i+1,i}$, and let $\mathbf{P}_i \subset \mathbf{G}$ be the (proalgebraic) subgroup generated by $\mathbf{B}$ and $\mathrm{Id} + \mathbb{C}E_{i+1,i}$. We set $\mathbf{P}_0$ as the (pro)algebraic group generated by $\mathbf{B}$ and $\mathrm{Id} + \mathbb{C}z^{-1}E_{1,n}$ inside $G(\!(z)\!)$. Observe that there is a loop rotation $\mathbb{G}_m$-action (denoted $\mathbb{G}_m^{\mathrm{rot}}$) on each of $\mathbf{B}$, $\mathbf{P}_i$, and $\mathbf{G}$.

We denote by $\widehat{\mathbf{B}}, \widehat{\mathbf{P}}_i$, and $\widehat{\mathbf{G}}$ the semidirect products of $\mathbf{B}, \mathbf{P}_i$, and $\mathbf{G}$ with $\mathbb{G}_m^{\mathrm{rot}}$, respectively. In addition, the group $G(\!(z)\!)$ admits a central extension by $\mathbb{C}^\times$, that induces a trivial central extension $\widetilde{\mathbf{P}}_i$ $(0 \le i < n)$ of $\widehat{\mathbf{P}}_i$ by $\mathbb{G}_m$ (which we denote by $\mathbb{G}_m^{\mathrm{c}}$). We define the extended torus

$$\widehat{T} := T \times \mathbb{G}_m^{\mathrm{rot}} \times \{1\} \subset T \times \mathbb{G}_m^{\mathrm{rot}} \times \mathbb{G}_m^{\mathrm{c}} =: \widetilde{T},$$

so that $\widetilde{\mathbf{B}} := \widehat{\mathbf{B}} \times \mathbb{G}_m^{\mathrm{c}}$ contains $\widetilde{T}$, and $\widehat{\mathbf{B}} \cap \widetilde{T} = \widehat{T}$. We also set $\widetilde{\mathbf{G}} := \widehat{\mathbf{G}} \times \mathbb{G}_m^{\mathrm{c}} \supset \widetilde{\mathbf{B}}, \widehat{\mathbf{G}}$ such that $\widetilde{\mathbf{B}} \cap \widehat{\mathbf{G}} = \widehat{\mathbf{B}}$. We have $\widetilde{\mathbf{P}}_i \cap \widetilde{\mathbf{P}}_j = \widetilde{\mathbf{B}}$ when $i \neq j$. For each $0 \le i < n$, we have the unique $\widetilde{T}$-stable algebraic subgroup of $\widetilde{\mathbf{P}}_i$ isomorphic to $SL(2)$, which we denote by $SL(2,i)$. We sometimes denote the Lie algebra of an algebraic group by the corresponding German small letters.

For each $0 \le i < n$, we define a homomorphism $u_i : \mathbb{G}_a \to \widetilde{\mathbf{B}}$ by

$$u_i(x) := \mathrm{Id} + xE_i \in \widetilde{\mathbf{B}}, \quad \text{where} \quad x \in \mathbb{C} \quad \text{and} \quad E_i := \begin{cases} E_{i,i+1} & (i \neq 0) \\ zE_{n,1} & (i = 0) \end{cases}.$$

We define

$$\widetilde{G}(\!(z)\!) := \mathbb{G}_m^{\mathrm{rot}} \ltimes G(\!(z)\!) \ltimes \mathbb{G}_m^{\mathrm{c}}$$

as a group. Let $\widetilde{\mathbf{G}}^- \subset \widetilde{G}(\!(z)\!)$ be the subgroup generated by $(\widetilde{T} \cdot G)$ and $\mathrm{Id} + \mathbb{C}z^{-1}E_{1,n}$. Note that the groups $\widetilde{G}(\!(z)\!)$ and $\widetilde{\mathbf{G}}^-$ are not algebraic.

For $1 \le i \le n$, we have an algebraic character $\epsilon_i : T \to \mathbb{G}_m$ that extracts the i-th (diagonal) entry of T. We set $\mathsf{P} := \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i$. Consider its subsets

$$\mathtt{Comp} := \sum_{i=1} \mathbb{Z}_{\ge 0}\epsilon_i, \quad \text{and} \quad \mathsf{P}^+ := \{\sum_{i=1}^n \lambda_i \epsilon_i \in \mathsf{P} \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n\}.$$

For $\lambda = \sum_{i=1}^n \lambda_i \epsilon_i \in \mathsf{P}$, we set $|\lambda| := \sum_{i=1}^n \lambda_i \in \mathbb{Z}$. The permutations of indices define $\mathfrak{S}_n$-actions on P and $\mathtt{Comp}$.

We define $\mathtt{Par} := (\mathsf{P}^+ \cap \mathtt{Comp})$ and identify it with the set of partitions with its length at most n. The semi-group $\mathtt{Par}$ is generated by

$$\varpi_i := \epsilon_1 + \cdots + \epsilon_i \qquad 1 \le i \le n.$$

We write $\lambda \gg 0$ for $\lambda \in \mathtt{Par}$ to indicate that all coefficients in the $\{\varpi_i\}$-expansion of λ are sufficiently large.

Let $\wp$ and δ denote the degree one character of $\mathbb{G}_m^{\mathrm{c}}$ and $\mathbb{G}_m^{\mathrm{rot}}$ extended to $\widetilde{T}$ trivially, respectively. We may regard ϖ_i as a character of $\widetilde{T}$ through the projection to T. We refer this as the standard lift of ϖ_i. We define an alternative (non-standard) lift of ϖ_i to $\widetilde{T}$ by setting

$$\Lambda_i := \begin{cases} \varpi_i + \wp & \text{if } 1 \le i < n, \\ \varpi_n + \wp & \text{if } i = 0. \end{cases} \tag{1.2}$$

Extending by linearity defines a *non-standard lift* of a character of T to $\widetilde{T}$.

We set $\mathtt{I}_{\mathrm{af}} := \{0, 1, \ldots, (n-1)\}$ and $\mathtt{I} := \{1, 2, \ldots, (n-1)\}$. We frequently identify the index 0 with n in the sequel, and hence $\{\varpi_i\}_i$ is indexed by $\mathtt{I}_{\mathrm{af}}$.

Note that $\{\varpi_i\}_{i\in \mathtt{I}_{\mathrm{af}}}$ and $\{\Lambda_i\}_{i\in \mathtt{I}_{\mathrm{af}}}$ correspond to each other by restriction. We define the affine weight lattice and its subset of dominant weights as follows:

$$\mathsf{P}_{\mathrm{af}} := \bigoplus_{i=1}^{n} \mathbb{Z}\varpi_i \oplus \mathbb{Z}\wp \oplus \mathbb{Z}\delta, \quad \text{and} \quad \mathsf{P}_{\mathrm{af}}^{+} := \left(\sum_{i\in \mathtt{I}_{\mathrm{af}}} \mathbb{Z}_{\geq 0}\Lambda_i\right) + \mathbb{Z}\varpi_n + \mathbb{Z}\delta \subset \mathsf{P}_{\mathrm{af}}.$$

The set P_{af} is the character group of $\widetilde{T}$.

The set of positive roots Δ^+ of G is $\Delta^+ := \{\epsilon_i - \epsilon_j\}_{1\leq i<j\leq n} \subset \mathsf{P}$. We set $\alpha_i := (\epsilon_i - \epsilon_{i+1})$ for $1 \leq i < n$, and define $\alpha_0 := -\vartheta + \delta$, where $\vartheta := \epsilon_1 - \epsilon_n$. We define a bilinear form on P_{af} as:

$$\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}, \qquad \wp, \delta \in \mathrm{Rad}\,\langle \bullet, \bullet \rangle.$$

Let $\mathfrak{n} := \mathrm{Lie}\, N \subset M_n$. For $\alpha = (\epsilon_i - \epsilon_j) \in \Delta^+$, we set

$$\mathfrak{g}_\alpha := \mathbb{C}E_{ij} \subset \mathfrak{n} \subset M_n.$$

The root lattice $\mathsf{Q} \subset \mathsf{P}$ is defined as $\mathsf{Q} := \sum_{\beta\in\Delta^+} \mathbb{Z}\beta$. The permutation $\mathfrak{S}_n$-action on P restricts to Q, and we set

$$\widetilde{\mathfrak{S}}_n := \mathfrak{S}_n \ltimes \mathsf{Q}.$$

The embedding $\mathfrak{S}_n \subset G$ via permutation matrices naturally extends to an embedding $\widetilde{\mathfrak{S}}_n \hookrightarrow G(\!(z)\!)$ given by

$$\mathsf{Q} \ni \sum_{i=1}^{n} \mu_i \epsilon_i \mapsto z^\mu := \begin{pmatrix} z^{\mu_1} & 0 & \cdots & 0 \\ 0 & z^{\mu_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z^{\mu_n} \end{pmatrix} \in G(\!(z)\!) \qquad \sum_{i=1}^{n} \mu_i = 0.$$

The group $\widetilde{\mathfrak{S}}_n$ is generated by $\{s_i\}_{i\in \mathtt{I}_{\mathrm{af}}}$, where

$$s_i = \begin{cases} (i, i+1) & (1 \leq i < n) \\ (1, n) \cdot z^{-\vartheta} & (i = 0) \end{cases}.$$

We have $s_i \in \widetilde{\mathbf{P}}_i$ for each $i \in \mathtt{I}_{\mathrm{af}}$. We have an action of $\widetilde{\mathfrak{S}}_n$ on P_{af} given by

$$s_i(\Lambda) := \Lambda - (\langle \alpha_i, \Lambda \rangle + \delta_{i0}\Lambda(K))\alpha_i \quad\ i \in \mathtt{I}_{\mathrm{af}},$$

where $K \in \mathrm{Hom}(\mathsf{P}_{\mathrm{af}}, \mathbb{Z})$ is defined as

$$\varpi_i(K) = 0 \quad\ (i \in \mathtt{I}_{\mathrm{af}}), \qquad \delta(K) = 0, \quad \text{and} \quad \wp(K) = 1.$$

Elements in the $\widetilde{\mathfrak{S}}_n$-orbit of $\{\alpha_i\}_{i\in \mathtt{I}_{\mathrm{af}}} \subset \mathsf{P}_{\mathrm{af}}$ are called affine roots. If an affine root is contained in the non-negative integer span of $\{\alpha_i\}_{i\in \mathtt{I}_{\mathrm{af}}}$, then we call it a positive affine root. Note that the affine Dynkin diagram automorphism of type $\mathsf{A}_{n-1}^{(1)}$ acts on the set of affine roots and positive affine roots by the linear transform that shifts the index uniformly (modulo n). This induces an automorphism of $\widetilde{G}(\!(z)\!)$ that fixes scalar matrices.

Every $w \in \widetilde{\mathfrak{S}}_n$ can be written as a product

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell} \qquad i_1, \ldots, i_\ell \in \mathtt{I}_{\mathrm{af}}. \tag{1.3}$$

Let $\mathbf{i} := (i_1, i_2, \ldots, i_\ell)$ be the sequence of indices appearing in (1.3). If the length ℓ of $\mathbf{i}$ is minimal among all such expressions in (1.3), then we call $\mathbf{i}$ a reduced expression of w and call ℓ the length of w.

We define the (strong) Bruhat order on $\widetilde{\mathfrak{S}}_n$ by setting $w < v$ if a reduced expression of w appears as an ordered subword of a reduced expression of v. We denote the length of $w \in \widetilde{\mathfrak{S}}_n$ by $\ell(w)$. Let $w_0 \in \mathfrak{S}_n$ denote the longest element, i.e. $w_0(i) = n - i + 1$ for $1 \le i \le n$.

1.2 Recollection on root ideals

Definition 1.1 (Root ideals)**.** A subset $\Psi \subset \Delta^+$ is called a root ideal if and only if

$$(\Psi + \Delta^+) \cap \Delta^+ \subset \Psi.$$

Equivalently, Ψ is a root ideal if for every $(\epsilon_i - \epsilon_j) \in \Psi$, we also have $(\epsilon_i - \epsilon_j), (\epsilon_i - \epsilon_j) \in \Psi$ for all $i < i$ and $j < j$. For a root ideal $\Psi \subset \Delta^+$, we define

$$\mathfrak{n}(\Psi) := \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \subset \mathfrak{n}.$$

We denote by $|\Psi|$ the cardinality of Ψ, which equals $\dim \mathfrak{n}(\Psi)$.

For better intuition, it might be helpful to consult the diagram in Example 1.4.

Definition 1.2. For a root ideal $\Psi \subset \Delta^+$ and $1 \le i < n$, we set

$$d_i(\Psi) := \#\{i \le j \le n \mid E_{ij} \notin \mathfrak{n}(\Psi)\}, \quad \text{and} \quad e_i(\Psi) := i + d_i(\Psi)$$

and define

$$\mathtt{I}(\Psi) := \{1 \le i < n \mid e_i(\Psi) \le n, d_i(\Psi) \le d_{i+1}(\Psi)\}.$$

We set $\ell(\Psi) := |\mathtt{I}(\Psi)|$. We denote the increasing rearrangement of $\{e_i(\Psi) \mid i \in \mathtt{I}(\Psi)\}$ by

$$\{e_i(\Psi)\}_{i \in \mathtt{I}(\Psi)} = \{\mathtt{e}_1(\Psi) < \mathtt{e}_2(\Psi) < \cdots < \mathtt{e}_\ell(\Psi)\},$$

where $\ell = \ell(\Psi)$, and set $\mathtt{e}_{\ell+1}(\Psi) = e_{n+1}(\Psi) := (n+1)$. For each $1 \le j \le \ell$, there exists a unique $i \in \mathtt{I}(\Psi)$ such that $\mathtt{e}_j(\Psi) = e_i(\Psi)$ and set $\mathtt{i}_j(\Psi) := i$. By convention, we set $\mathtt{i}_0(\Psi) = 0$ and $e_0(\Psi) = 1$. For $e_1(\Psi) \le k \le n$, we set

$$\mathtt{h}_k(\Psi) := \mathtt{i}_j(\Psi), \qquad \text{where} \qquad \mathtt{e}_j(\Psi) \le k < \mathtt{e}_{j+1}(\Psi). \tag{1.4}$$

By convention, we set $\mathtt{h}_{d_1(\Psi)}(\Psi)$, or equivalently $\mathtt{h}_{e_1(\Psi)-1}(\Psi)$, equal to zero.

Definition 1.3 (Ψ-tame elements)**.** Let $\Psi \subset \Delta^+$ be a root ideal. We say that $w \in \mathfrak{S}_n$ is Ψ-tame if $ws_i < w$ for each $d_1(\Psi) < i < n$. We set w_0^Ψ to be the longest element in

$$\mathfrak{S}_{n-d_1(\Psi)} \cong \left\langle s_{e_1(\Psi)}, s_{e_1(\Psi)+1}, \ldots, s_{(n-1)} \right\rangle \subset \mathfrak{S}_n.$$

Example 1.4. Assume that $n = 6$, and

$$\Psi = \{\epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_1 - \epsilon_5, \epsilon_1 - \epsilon_6, \epsilon_2 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_2 - \epsilon_5, \epsilon_2 - \epsilon_6, \epsilon_3 - \epsilon_6\}.$$

We have $d_1(\Psi) = 2, d_2(\Psi) = 1, d_3(\Psi) = 3, d_4(\Psi) = 3, d_5(\Psi) = 2, d_6(\Psi) = 1$, and hence

$$e_1(\Psi) = 3, e_2(\Psi) = 3, e_3(\Psi) = 6, e_4(\Psi) = 7,$$

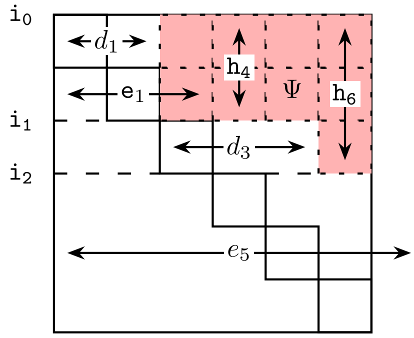
and therefore

$$\mathtt{e}_1(\Psi) = 3, \mathtt{e}_2(\Psi) = 6, \ell(\Psi) = 2.$$

We have $e_5(\Psi) = e_6(\Psi) = 7 > n = 6$, and these do not contribute to $\mathtt{e}_\bullet(\Psi), \ell(\Psi)$, and $\mathtt{i}_\bullet(\Psi)$. We have $\mathtt{i}_0(\Psi) = 0$ by convention, and $\mathtt{i}_1(\Psi) = 2, \mathtt{i}_2(\Psi) = 3$ from the above. Thus, we have $\mathtt{I}(\Psi) = \{\mathtt{i}_1(\Psi), \mathtt{i}_2(\Psi)\} = \{2, 3\}$. For $3 = e_1(\Psi) \leq k \leq n$, we have

$$\mathtt{h}_3(\Psi) = \mathtt{i}_1(\Psi) = 2, \mathtt{h}_4(\Psi) = \mathtt{i}_1(\Psi) = 2, \mathtt{h}_5(\Psi) = \mathtt{i}_1(\Psi) = 2, \mathtt{h}_6(\Psi) = \mathtt{i}_2(\Psi) = 3$$

from $\mathtt{e}_1(\Psi) = e_2(\Psi) = 3$ and $\mathtt{e}_2(\Psi) = e_3(\Psi) = 6$. This situation is illustrated as follows:



We note that the red-shaded boxes represent the elements of Ψ.

Let us summarize basic properties of our invariants associated to the root ideal Ψ.

Lemma 1.5 (Cellini [6] §3)**.** *For a root ideal $\Psi \subset \Delta^+$, the subspace $\mathfrak{n}(\Psi) \subset \mathfrak{n}$ is B-stable. In addition, every B-stable subspace of $\mathfrak{n}$ arises in this way.* □

Remark 1.6. As shown in [32, §4], the set of B-stable ideals in $\mathfrak{n}$equivalently, root idealsis in natural bijection with the set of Dyck paths of size n.

Lemma 1.7. *For a root ideal $\Psi \subset \Delta^+$ and $1 \leq i < n$, we have*

$$d_i(\Psi) \leq d_{i+1}(\Psi) + 1, \quad \textit{and} \quad i < e_i(\Psi) \leq e_{i+1}(\Psi) \leq (n+1).$$

In addition, we have $\mathtt{i}_{j-1}(\Psi) < \mathtt{i}_j(\Psi)$ and $\mathtt{i}_j(\Psi) < \mathtt{e}_j(\Psi)$ for $1 \leq j \leq \ell(\Psi)$.

Proof. Straightforward. □

Lemma 1.8. *Let $\Psi \subset \Delta^+$ be a root ideal. Let $1 \leq s \leq \ell(\Psi)$ and let $\mathtt{e}_s(\Psi) \leq j < \mathtt{e}_{s+1}(\Psi)$. Then, we have $E_{ij} \in \mathfrak{n}(\Psi)$ if and only if $1 \leq i \leq \mathtt{i}_s(\Psi) = \mathtt{h}_j(\Psi)$.*

Proof. The equality $\mathtt{i}_s(\Psi) = \mathtt{h}_j(\Psi)$ is by definition. For $1 \leq i \leq \mathtt{i}_s(\Psi)$, we have $e_i(\Psi) \leq \mathtt{e}_s(\Psi)$ by the monotonicity of $e_\bullet(\Psi)$ (Lemma 1.7). Hence, the if direction follows. For $i > \mathtt{i}_s(\Psi)$, we have $e_i(\Psi) > \mathtt{e}_s(\Psi)$ since $\mathtt{i}_s(\Psi)$ is the largest index such that $e_\bullet(\Psi) = \mathtt{e}_s(\Psi)$. This implies the only if part of the assertion, completing the proof. □

Lemma 1.9. *For a root ideal $\Psi \subset \Delta^+$ and $e_1(\Psi) \le k \le n$, we have*

$$\mathtt{h}_{k-1}(\Psi) \le \mathtt{h}_k(\Psi) < k.$$

In addition, we have

$$|\Psi| = \sum_{k=e_1(\Psi)}^{n} \mathtt{h}_k(\Psi). \tag{1.5}$$

Proof. Since $\mathtt{i}_\bullet(\Psi)$ is non-decreasing, we find $\mathtt{h}_{k-1}(\Psi) \le \mathtt{h}_k(\Psi)$. By Lemma 1.8, we have

$$\mathtt{h}_k(\Psi) = |\{i \mid E_{ik} \in \mathfrak{n}(\Psi)\}|.$$

It follows that $\mathtt{h}_k(\Psi) < k$ and $|\Psi| = \sum_{k=e_1(\Psi)}^{n} \mathtt{h}_k(\Psi)$ as required. $\square$

1.3 Representations

For a finite-dimensional rational representation V of T, we define its character as

$$\operatorname{ch} V := \sum_{\lambda \in \mathsf{P}} e^{\lambda} \cdot \dim \operatorname{Hom}_T(\mathbb{C}_\lambda, V).$$

In particular, the character of a rational representation of G or P_i can be defined by restriction to T. For a rational representation V of $\widetilde{T}$, we define

$$\operatorname{gch} V := \sum_{\lambda \in \mathsf{P}, m \in \mathbb{Z}} q^m e^{\lambda} \cdot \dim \operatorname{Hom}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathbb{C}_{\lambda + m\delta}, V).$$

For two rational $\widetilde{T}$-representations V and V', we write $\operatorname{gch} V \le \operatorname{gch} V$ to mean that the inequality holds coefficientwise, i.e.,

$$\dim \operatorname{Hom}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathbb{C}_{\lambda + m\delta}, V) \le \dim \operatorname{Hom}_{T \times \mathbb{G}_m^{\mathrm{rot}}}(\mathbb{C}_{\lambda + m\delta}, V') \qquad \lambda \in \mathsf{P}, m \in \mathbb{Z}.$$

A rational representation of $\widetilde{\mathbf{B}}$ (resp. $\widetilde{\mathbf{P}}_i$ for $i \in \mathtt{I}_{\mathrm{af}}$) is a representation V of $\widetilde{\mathbf{B}}$ (resp. $\widetilde{\mathbf{P}}_i$), where the group action factors through a finite-dimensional quotient, yielding a rational representation of an algebraic group.

For each $\lambda \in \mathsf{P}^+$, let $V(\lambda)$ denote the irreducible finite-dimensional G-module generated by a B-eigenvector $\mathbf{v}_\lambda$ of T-weight λ. By the natural $\mathfrak{S}_n$-action on $V(\lambda)$, we have a T-eigenvector $\mathbf{v}_{w\lambda} \in V(\lambda)$ of weight $w\lambda \in \mathsf{P}$ for each $w \in \mathfrak{S}_n$. For each $\Lambda \in \mathsf{P}^+_{\mathrm{af}}$, we have an integrable highest weight module $L(\Lambda)$ of $\widetilde{G}(\!(z)\!)$ generated by a $\widetilde{\mathbf{B}}$-eigenvector $\mathbf{v}_\Lambda$ of $\widetilde{T}$-weight Λ. The natural $\widetilde{\mathfrak{S}}_n$-action on $L(\Lambda)$ yields a $\widetilde{T}$-eigenvector $\mathbf{v}_{w\Lambda} \in L(\Lambda)$ of weight $w\Lambda$ for each $w \in \widetilde{\mathfrak{S}}_n$.

For $\lambda \in \mathsf{P}^+$ and $w \in \mathfrak{S}_n$, we define a Demazure module of $V(\lambda)$ by

$$V_w(\lambda) := \operatorname{Span} \langle B \mathbf{v}_{w\lambda} \rangle \subset V(\lambda).$$

Similarly, for $\Lambda \in \mathsf{P}^+_{\mathrm{af}}$ and $w \in \widetilde{\mathfrak{S}}_n$, we define a Demazure module of $L(\Lambda)$ by

$$L_w(\Lambda) := \operatorname{Span} \left\langle \widetilde{\mathbf{B}} \mathbf{v}_{w\Lambda} \right\rangle \subset L(\Lambda).$$

1.4 Flag varieties and Demazure functors

We set $X := G/B$ and call it the flag manifold of G. For each $\lambda \in \mathsf{P}$, we define $\mathcal{O}_X(\lambda)$ to be the G-equivariant line bundle on X whose fiber at the point $B/B \in X$ is $\mathbb{C}_{-\lambda}$. We set $X(w) := \overline{BwB/B} \subset X$ for each $w \in \mathfrak{S}_n$ and call it the Schubert subvariety of X attached to w. The restriction of $\mathcal{O}_X(\lambda)$ to $X(w)$ is denoted by $\mathcal{O}_{X(w)}(\lambda)$.

Using Lemma 1.5, we define a $(G \times \mathbb{G}_m)$-equivariant vector subbundle

$$T^*_\Psi X := G \times^B \mathfrak{n}(\Psi) \subset G \times^B \mathfrak{n} \cong T^* X$$

for a root ideal $\Psi \subset \Delta^+$, where $\mathbb{G}_m$ acts by fiberwise scalar dilation. Let $\pi_\Psi : T^*_\Psi X \to X$ be the projection map. We set

$$T^*_\Psi X(w) := \pi_\Psi^{-1}(X(w)) \qquad w \in \mathfrak{S}_n.$$

We may denote the restriction of π_Ψ to $T^*_\Psi X(w)$ by the same symbol, by slight abuse of notation.

For a sequence $\mathbf{i} := (i_1, i_2, \ldots, i_\ell)$ of elements of $\mathtt{I}_{\mathrm{af}}$, we define the following $\widetilde{\mathbf{B}}$-schemes:

$$\widetilde{X}(\mathbf{i}) := \widetilde{\mathbf{P}}_{i_1} \times^{\widetilde{\mathbf{B}}} \widetilde{\mathbf{P}}_{i_2} \times^{\widetilde{\mathbf{B}}} \cdots \times^{\widetilde{\mathbf{B}}} \widetilde{\mathbf{P}}_{i_\ell} \quad \text{and} \quad X(\mathbf{i}) := \widetilde{X}(\mathbf{i})/\widetilde{\mathbf{B}}. \tag{1.6}$$

By convention, we set $X(\emptyset) = \mathrm{pt}$.

Lemma 1.10 (Kumar [20] §7.1)**.** *Let $\mathbf{i} := (i_1, i_2, \ldots, i_\ell)$ be a sequence of elements of $\mathtt{I}_{\mathrm{af}}$. It holds:*

1. *Let $\mathbf{i}^\flat$ be the sequence obtained by forgetting the last element i_ℓ in $\mathbf{i}$. Then, $X(\mathbf{i})$ is a $\mathbb{P}^1$-fibration over $X(\mathbf{i}^\flat)$ whose fiber is isomorphic to $\widetilde{\mathbf{P}}_{i_\ell}/\widetilde{\mathbf{B}}$;*
2. *Let $1 \le j_1 < j_2 < \cdots < j_m \le \ell$. We set $\mathbf{i}' := (i_{j_1}, i_{j_2}, \ldots, i_{j_m})$. Then there is a $\widetilde{\mathbf{B}}$-equivariant embedding $X(\mathbf{i}') \hookrightarrow X(\mathbf{i})$ induced by the group homomorphism*
$$\prod_{t=1}^m \widetilde{\mathbf{P}}_{j_t} \ni (g_{j_t}) \mapsto (g_j) \in \prod_{j=1}^\ell \widetilde{\mathbf{P}}_j,$$
where $g_j = 1 \in \widetilde{\mathbf{B}}$ for all $j \notin \{j_1, \ldots, j_m\}$. □

For any rational $\widetilde{\mathbf{B}}$-module M, we have a vector bundle

$$\mathcal{E}_{\mathbf{i}}(M) := \mathcal{X}(\mathbf{i}) \times^{\widetilde{\mathbf{B}}} M^\vee \longrightarrow X(\mathbf{i}).$$

In case $M \cong \mathbb{C}_\Lambda$ for a $\widetilde{T}$-weight Λ, we set $\mathcal{O}_{\mathbf{i}}(\Lambda) := \mathcal{E}_{\mathbf{i}}(\mathbb{C}_\Lambda)$. By Lemma 1.10 (2), the restriction of $\mathcal{E}_{\mathbf{i}}(M)$ to $X(\mathbf{i}')$ is identified with $\mathcal{E}_{\mathbf{i}'}(M)$ as a $\widetilde{\mathbf{B}}$-equivariant vector bundle.

Definition 1.11 (Demazure functors)**.** The (covariant) functor assigning a rational $\widetilde{\mathbf{B}}$-module M to a rational $\widetilde{\mathbf{B}}$-module $\Gamma(X(\mathbf{i}), \mathcal{E}_{\mathbf{i}}(M))^\vee$ is called the Demazure functor associated with the sequence $\mathbf{i}$, and is denoted by $\mathcal{D}_{\mathbf{i}}$. In particular, $\mathcal{D}_i$ denotes the Demazure functor corresponding to $i \in \mathtt{I}_{\mathrm{af}}$. We also define its contragredient variant by

$$\mathcal{D}_i^\dagger(\bullet) := \mathcal{D}_i(\bullet^\vee)^\vee.$$

If a sequence $\mathbf{i}$ in $\mathtt{I}_{\mathrm{af}}$ is a concatenation of $\mathbf{i}_1$ followed by $\mathbf{i}_2$, then the corresponding Demazure functors satisfy $\mathcal{D}_{\mathbf{i}} \cong \mathcal{D}_{\mathbf{i}_1} \circ \mathcal{D}_{\mathbf{i}_2}$ by repeated applications of Lemma 1.10(1).

Definition 1.12. Let $\mathtt{L}$ be a free abelian monoid, and let R be a $\mathtt{L}$-graded $\mathbb{C}$-algebra. We say that R is $\widetilde{\mathbf{B}}$-equivariant if the following conditions are satisfied:

- For each $a \in \mathtt{L}$, the graded component R_a admits a rational $\widetilde{\mathbf{B}}$-action;
- The multiplication maps $R_a \otimes R_b \to R_{a+b}$ are $\widetilde{\mathbf{B}}$-equivariant;
- $R_0 = \mathbb{C}$ is the trivial $\widetilde{\mathbf{B}}$-module.

Lemma 1.13. *Let* $\mathtt{L}$ *be a free abelian monoid, and let* R *be a* $\widetilde{\mathbf{B}}$*-equivariant* $\mathtt{L}$*-graded* $\mathbb{C}$*-algebra. Then* $\mathcal{D}_i^{\dagger}(R)$ *naturally acquires the structure of a* $\widetilde{\mathbf{B}}$*-equivariant* $\mathtt{L}$*-graded* $\mathbb{C}$*-algebra for each* $i \in \mathtt{I}_{\mathrm{af}}$*. Moreover, the following hold:*

- $\mathcal{D}_i^{\dagger}(R)$ *is commutative if* R *is commutative;*
- $\mathcal{D}_i^{\dagger}(R)$ *is integral if* R *is integral;*
- $\mathcal{D}_i^{\dagger}(R)$ *is integrally closed if* R *is integrally closed.*

Proof. The $\widetilde{\mathbf{B}}$-equivariant algebra R induces a $\mathtt{L}$-graded $\widetilde{\mathbf{P}}_i$-equivariant sheaf of algebras $\mathcal{E}_i(R)$ over $X(i) = \widetilde{X}(i)/\widetilde{\mathbf{B}} = \widetilde{\mathbf{P}}_i/\widetilde{\mathbf{B}}$. Therefore, its global sections form a $\mathtt{L}$-graded algebra equipped with a degreewise rational $\widetilde{\mathbf{P}}_i$-action compatible with multiplication. The degree zero part of $\mathcal{D}_i^{\dagger}(R)$ is given by

$$\mathbb{C} = \Gamma(X(i), \mathcal{O}_{X(i)}) = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}).$$

If R is commutative, then $\mathcal{E}_i(R)$ is a sheaf of commutative algebras, and hence $\mathcal{D}_i^{\dagger}(R)$ is also commutative.

Note that the ring $R \otimes_{\mathbb{C}} \mathbb{C}[t]$ is integral if R is integral. In addition, this ring is integrally closed if R is integrally closed, which can be verified inductively by examining the coefficients of t starting from the lowest degree term. For each $x \in \mathbb{P}^1$, there exists an affine open neighborhood U_x with local coordinate t_x such that $\Gamma(U_x, \mathcal{E}_i(R)) \cong R \otimes_{R_0} \mathbb{C}[t_x]$. Since we have $\mathbb{P}^1 = \bigcup_{x \in \mathbb{P}^1} U_x$, we find

$$\mathcal{D}_i^{\dagger}(R) = \Gamma(\mathbb{P}^1, \mathcal{E}_i(R^{\vee})) = \bigcap_{x \in \mathbb{P}^1} R \otimes_{\mathbb{C}} \mathbb{C}[t_x].$$

It follows that $\mathcal{D}_i^{\dagger}(R)$ is integral (resp. integrally closed) if R is (since the intersection of integrally closed domains with the same field of fractions is again integrally closed). This completes the proof. $\square$

Theorem 1.14 (Joseph [16])**.** *For each* $i \in \mathtt{I}_{\mathrm{af}}$*, it holds:*

1. *There exists a natural transformation* $\mathrm{Id} \to \mathcal{D}_i$*;*
2. *We have an isomorphism of functors* $\mathcal{D}_i \to \mathcal{D}_i \circ \mathcal{D}_i$*;*
3. *For a rational* $\widetilde{\mathbf{P}}_i$*-module* M*, we have an isomorphism of functors*

$$\mathcal{D}_i(M \otimes \bullet) \cong M \otimes \mathcal{D}_i(\bullet);$$

4. *Let $w \in \widetilde{\mathfrak{S}}_n$ admit two reduced expressions $\mathbf{i}$ and $\mathbf{i}'$ connected by a sequence of braid relations. Then, we have an isomorphism $\mathcal{D}_{\mathbf{i}} \cong \mathcal{D}_{\mathbf{i}'}$ of functors.*

Moreover, $\mathcal{D}_i$ maps finite-dimensional rational $\widetilde{\mathbf{B}}$-modules to finite-dimensional rational $\widetilde{\mathbf{P}}_i$-modules, which may be regarded as $\widetilde{\mathbf{B}}$-modules via restriction.

By Theorem 1.14(4), we set $\mathcal{D}_w := \mathcal{D}_{\mathbf{i}}$ for a reduced expression $\mathbf{i}$ of $w \in \widetilde{\mathfrak{S}}_n$. We have a natural transformation $\mathcal{D}_w \to \mathcal{D}_v$ when $w < v$ in $\widetilde{\mathfrak{S}}_n$ by Theorem 1.14(1).

Theorem 1.15 (Demazure character formula, see e.g. [20])**.** *The following hold:*

1. *Let $\lambda \in \mathsf{P}^+$ and $w \in \mathfrak{S}_n$, with $\mathbf{i}$ a reduced expression of w. We have*

$$H^m(X(w), \mathcal{O}_{X(w)}(\lambda))^* \cong H^m(X(\mathbf{i}), \mathcal{O}_{\mathbf{i}}(\lambda))^* \cong \mathbb{L}^{-m}\mathcal{D}_{\mathbf{i}}(\mathbb{C}_\lambda) \cong \begin{cases} V_w(\lambda) & (m = 0) \\ 0 & (m \neq 0) \end{cases};$$

2. *Let $\Lambda \in \mathsf{P}^+_{\mathrm{af}}$ and let $\mathbf{i}$ be a sequence of elements of $\mathtt{I}_{\mathrm{af}}$. Then, there exists $w \in \widetilde{\mathfrak{S}}_n$ such that*

$$H^m(X(\mathbf{i}), \mathcal{O}_{\mathbf{i}}(\Lambda))^* \cong \mathbb{L}^{-m}\mathcal{D}_{\mathbf{i}}(\mathbb{C}_\Lambda) \cong \begin{cases} L_w(\Lambda) & (m = 0) \\ 0 & (m \neq 0) \end{cases};$$

3. *The line bundle $\mathcal{O}_{\mathbf{i}}(\Lambda)$ on $X(\mathbf{i})$ is base-point-free for $\Lambda \in \mathsf{P}^+_{\mathrm{af}}$.*

Proof. The first two assertions are special cases of [20, 8.1.26 Corollary], and the last follows from the second together with [20, 7.1.15 Proposition]. □

1.5 Affine Demazure modules

For each $\lambda \in \mathsf{P}$ and $k \in \mathbb{Z}_{>0}$, there exists $w \in \widetilde{\mathfrak{S}}_n$ such that

$$\lambda + k\Lambda_0 = w\Lambda \in \mathsf{P}^+_{\mathrm{af}}, \tag{1.7}$$

as guaranteed by [18, Corollary 10.1]. We set

$$D^{(k)}_\lambda := \mathcal{D}_w(\mathbb{C}_\Lambda) \equiv L_w(\Lambda) \subset L(\Lambda)$$

and call it the Demazure module (of level k). They are finite-dimensional rational $\widetilde{\mathbf{B}}$-modules, and are independent of the choice of w in (1.7).

Definition 1.16. Let $k \in \mathbb{Z}_{>0}$. A finite-dimensional $\widetilde{\mathbf{B}}$-module M is said to be $D^{(k)}$-filtered if it admits a finite filtration whose associated graded module is the direct sum of Demazure modules of level k.

Theorem 1.17 (Joseph [17], see also [30, 19])**.** *For each $\lambda \in \mathsf{P}$ and $k \in \mathbb{Z}_{>0}$, it holds:*

1. *For each $i \in \mathtt{I}_{\mathrm{af}}$, the module $D^{(k)}_\lambda \otimes \mathbb{C}_{\Lambda_i}$ admits a $D^{(k+1)}$-filtration;*

2. *For a $D^{(k)}$-filtered module M and $i \in \mathtt{I}_{\mathrm{af}}$, we have $\mathbb{L}^{<0}\mathcal{D}_i(M) = 0$ and $\mathcal{D}_i(M)$ is $D^{(k)}$-filtered.*

Proof. The first assertion is a special case of [17, 5.22 Theorem] ([30, Remark 4.15] for the $n = 2$ case, and [19] for another proof). In light of the first assertion, the second follows by repeated application of Theorem 1.15(2) to the short exact sequences arising from the $D^{(k)}$-filtration. $\square$

Corollary 1.18 (Demazure module branching). *Let $k \in \mathbb{Z}_{>0}$ and $w \in \widetilde{\mathfrak{S}}_n$. For a $D^{(k)}$-filtered module M and $m \in \mathbb{Z}_{\geq 0}$, we have $\mathbb{L}^{<0}\mathcal{D}_w(\mathbb{C}_{m\Lambda_i} \otimes M) = 0$, and the $\widetilde{\mathbf{B}}$-module $\mathcal{D}_w(\mathbb{C}_{m\Lambda_i} \otimes M)$ is $D^{(m+k)}$-filtered. In addition, we have*

$$\mathbb{C}_{m\Lambda_i} \otimes M \subset \mathcal{D}_w(\mathbb{C}_{m\Lambda_i} \otimes M). \tag{1.8}$$

Proof. Consider a finite-dimensional $\widetilde{T}$-semisimple $\widetilde{\mathbf{B}}$-module N that fits into a short exact sequence

$$0 \to N_1 \to N \to N_2 \to 0$$

such that N_2 is a Demazure module and $N_1 \subset \mathcal{D}_w(N_1)$. Applying the Leray spectral sequence to $\mathbb{L}^{\bullet}\mathcal{D}_{\mathbf{i}}$ for a reduced expression $\mathbf{i}$ of w, we deduce

$$\mathbb{L}^{<0}\mathcal{D}_w(N_2) = 0$$

by Theorem 1.17 2). We have a commutative diagram of short exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[r] & N_1 \arrow[r] \arrow[d, hook] & N \arrow[r] \arrow[d] & N_2 \arrow[r] \arrow[d, "\imath"] & 0 . \\
0 \arrow[r, equal] & \mathbb{L}^{-1}\mathcal{D}_w(N_2) \arrow[r] & \mathcal{D}_w(N_1) \arrow[r] & \mathcal{D}_w(N) \arrow[r] & \mathcal{D}_w(N_2) \arrow[r] & 0
\end{tikzcd} \tag{1.9}$$

The map $\imath$ is injective by Theorem 1.15 2) and the inclusion relations of Demazure modules. Thus, the five lemma implies $N \subset \mathcal{D}_w(N)$.

If we have $\mathbb{L}^{<0}\mathcal{D}_w(N_1) = 0$ in addition, then we have $\mathbb{L}^{<0}\mathcal{D}_w(N) = 0$ by the long exact sequence associated to the bottom row of (1.9).

Therefore, we apply Theorem 1.17(1) m-times to obtain a $D^{(m+k)}$-filtration on $\mathbb{C}_{m\Lambda_i} \otimes M$. Then, all the assertions follow by induction on the length of the filtration by Demazure modules via the above discussion. $\square$

Proposition 1.19 (Joseph, see also [19] Lemma 4.1). *For each $\lambda \in \mathtt{Comp}$ and $k \in \mathbb{Z}_{>0}$, we have*

$$\operatorname{gch} D^{(k)}_{\lambda} \in \mathbb{Z}[q][X_1, \ldots, X_n],$$

with $X_i := e^{\epsilon_i}$ for $1 \leq i \leq n$.

Proof. Let λ_+ be the unique element in $(\mathfrak{S}_n\lambda \cap \mathsf{P}^+)$, and we set $\lambda_- := w_0\lambda_+$. By the comparison of defining equations of $D^{(\bullet)}_{\lambda_-}$ ([16, 3.5], see [12, Theorem 1] or [19, Proof of Lemma 4.1] for explicit equations), we find that $D^{(k)}_{\lambda_-}$ is a quotient of $D^{(1)}_{\lambda_-}$. Moreover, $D^{(1)}_{\lambda_-}$ is the local Weyl module whose highest weight is λ_+ by [7, Corollary 1.5.1]. We have

$$[D^{(1)}_{\lambda_-} : V_\mu] \neq 0 \quad \Rightarrow \quad \lambda_+ - \mu \in \sum_{i \in \mathtt{I}} \mathbb{Z}_{\geq 0}\alpha_i \tag{1.10}$$

by [7, Definition 1.2.1]. It follows that $\mu \in \mathsf{P}^+$ (and $\lambda \in \mathtt{Comp}$) implies $\mu \in \mathtt{Par}$ under the situation of (1.10). Observe that $e^\mu \in \mathbb{Z}[X_1, \dots, X_n]$ ($\mu \in \mathsf{P}$) if and only if $\mu \in \mathtt{Comp}$. Therefore, we obtain

$$\mathrm{gch}\, D^{(1)}_{\lambda_-} \in \mathbb{Z}[q][X_1, \dots, X_n]^{\mathfrak{S}_n}.$$

Taking Theorem 1.15(2) into account, we deduce $D^{(k)}_\lambda \subset D^{(k)}_{\lambda_-}$ as the inclusion relation of Demazure modules. Hence, we have

$$\mathrm{gch}\, D^{(1)}_{\lambda_-} \geq \mathrm{gch}\, D^{(k)}_{\lambda_-} \geq \mathrm{gch}\, D^{(k)}_{\lambda}.$$

This completes the proof. □

2 An interpretation of the rotation theorem

Keep the setting of § 1. For each $\lambda \in \mathtt{Par}$, we define

$$m_i(\lambda) := \begin{cases} \lambda_i - \lambda_{i+1} & \text{for } 1 \leq i < n, \\ \lambda_n & \text{for } i = n. \end{cases} \tag{2.1}$$

For each $1 \leq i < n$ and $1 \leq e \leq n$, we define the composition functors as:

$$\mathcal{C}_{i,e} := (\mathcal{D}_{i-1} \circ \mathcal{D}_{i-2} \circ \cdots \circ \mathcal{D}_e)$$
$$\mathcal{C}_{i,e}(\lambda)(\bullet) := (\mathcal{D}_{i-1} \circ \mathcal{D}_{i-2} \circ \cdots \circ \mathcal{D}_e)\,(\mathbb{C}_{m_e(\lambda)\Lambda_e} \otimes \bullet).$$

Here, the composition of $\mathcal{D}_\bullet$'s is taken from $(i-1)$ down to e (modulo n). When $i \leq e$, we interpret the sequence modulo n as wrapping around from 0 to $n-1$, and the total number of terms is $(i+n-e)$.

For a root ideal $\Psi \subset \Delta^+$ and $1 \leq j \leq \ell(\Psi)$, we define

$$\mathcal{C}^\Psi_j(\lambda)(\bullet) := \Big(\mathcal{C}_{\mathtt{i}_j(\Psi),\mathtt{e}_j(\Psi)}(\lambda) \circ \mathcal{C}_{\mathtt{i}_j(\Psi),\mathtt{e}_j(\Psi)+1}(\lambda) \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j(\Psi),\mathtt{e}_{(j+1)}(\Psi)-1}(\lambda)\Big)\,(\bullet).$$

We set $\lambda(\Psi) := \sum_{j=1}^{d_1(\Psi)} m_j(\lambda)\Lambda_j$. Using these, we define

$$N^\Psi_w(\lambda) := \mathcal{D}_w\left(\mathbb{C}_{\lambda(\Psi)} \otimes (\mathcal{C}^\Psi_1(\lambda) \circ \mathcal{C}^\Psi_2(\lambda) \circ \cdots \mathcal{C}^\Psi_{\ell(\Psi)}(\lambda))(\mathbb{C})\right), \quad \text{and} \tag{2.2}$$

$$M^\Psi_w(\lambda) := \mathcal{D}_w\Big(\mathbb{C}_{m_1(\lambda)\Lambda_1} \otimes (\mathcal{C}_{1,e_1(\Psi)}(\mathbb{C}_{m_2(\lambda)\Lambda_2} \otimes \mathcal{C}_{2,e_2(\Psi)}(\mathbb{C}_{m_3(\lambda)\Lambda_3} \otimes \mathcal{C}_{3,e_3(\Psi)}(\cdots(\mathbb{C}_{m_{n-1}(\lambda)\Lambda_{n-1}} \otimes \mathcal{C}_{n-1,e_{n-1}(\Psi)}(\mathbb{C}_{m_n(\lambda)\Lambda_n})\cdots)\Big) \tag{2.3}$$

for each $w \in \mathfrak{S}_n$.

Proposition 2.1. *Let $\Psi \subset \Delta^+$ be a root ideal, $w \in \mathfrak{S}_n$, and $\lambda \in \mathtt{Par}$. We have the following vanishing of the total homology complex associated to* (2.3)*:*

$$\mathbb{L}^{<0}\Big(\mathcal{D}_w\Big(\mathbb{C}_{m_1(\lambda)\Lambda_1} \otimes (\mathcal{C}_{1,e_1(\Psi)}(\mathbb{C}_{m_2(\lambda)\Lambda_2} \otimes \mathcal{C}_{2,e_2(\Psi)}(\mathbb{C}_{m_3(\lambda)\Lambda_3} \otimes \mathcal{C}_{3,e_3(\Psi)}(\cdots(\mathbb{C}_{m_{n-1}(\lambda)\Lambda_{n-1}} \otimes \mathcal{C}_{n-1,e_{n-1}(\Psi)}(\mathbb{C}_{m_n(\lambda)\Lambda_n})\cdots)\Big)\Big) = 0.$$

Proof. There is a Leray spectral sequence of the form

$$\mathbb{L}^r \mathcal{D}_i(\mathbb{C}_{m\Lambda_i} \otimes \mathbb{L}^s \mathcal{D}_w(M)) \Rightarrow \mathbb{L}^{s+r}(\mathcal{D}_i \circ (\mathbb{C}_{m\Lambda_i} \otimes \mathcal{D}_w))(M),$$

for $i \in \mathtt{I}_{\mathrm{af}}$, $m \in \mathbb{Z}_{\geq 0}$, and $w \in \widetilde{\mathfrak{S}}_n$, and M is any $D^{(k)}$-filtered $\widetilde{\mathbf{B}}$-equivariant graded module for some $k \in \mathbb{Z}_{>0}$.

By Corollary 1.18, this spectral sequence degenerates at the E_2-stage, and all negative-degree terms vanish. Hence, we apply this starting from the rightmost factors in (2.3), and proceed inductively to conclude the vanishing of all negative derived functors. □

Proposition 2.2. *Let $\Psi \subset \Delta^+$ be a root ideal, $w \in \mathfrak{S}_n$, and $\lambda \in \mathtt{Par}$. We have the following vanishing of the total homology complex associated to* (2.2)*:*

$$\mathbb{L}^{<0}\left(\mathcal{D}_w\left(\mathbb{C}_{\lambda(\Psi)} \otimes (\mathcal{C}_1^{\Psi}(\lambda) \circ \mathcal{C}_2^{\Psi}(\lambda) \circ \cdots \mathcal{C}_{\ell(\Psi)}^{\Psi}(\lambda))(\mathbb{C})\right)\right) = 0.$$

Moreover, for each $e_1(\Psi) \leq k \leq n$, we have

$$\mathbb{L}^{<0}\big(\mathcal{C}_{\mathtt{h}_k,k}(\lambda) \circ \mathcal{C}_{\mathtt{h}_{k+1},k+1}(\lambda) \circ \cdots \mathcal{C}_{\mathtt{h}_n,n}(\lambda)\big)(\mathbb{C}) = 0.$$

Proof. The assertions follow from the Leray spectral sequence applied to repeated applications of Corollary 1.18. □

Theorem 2.3 (Blasiak-Morse-Pun [2] Theorem 2.3)**.** *Let $\Psi \subset \Delta^+$ be a root ideal, and suppose $w \in \mathfrak{S}_n$ is Ψ-tame. Then, we have*

$$H(\Psi; \lambda; w) = \left[\operatorname{gch} M_w^{\Psi}(\lambda)\right]_{q \mapsto q^{-1}} \qquad \lambda \in \mathtt{Par},$$

where $H(\Psi; \lambda; w)$ is defined as a natural generalization of (0.1) *in [2, (2.2)].*

Remark 2.4. The automorphism Φ in [2, (2.4)] is a lift of the affine Dynkin diagram automorphism (of type $\mathsf{A}_{\ell-1}^{(1)}$) that satisfies

$$\Phi \circ \pi_i = \pi_{i+1} \circ \Phi \qquad 0 \leq i < \ell,$$

where π_i ($1 \leq i < \ell$) is the Demazure operator (the graded character counterpart of the functor $\mathcal{D}_i$) borrowed from [2, (2.1)] (see also [20, 8.2.7]), and $\pi_0 = \pi_\ell$ is defined here for the first time. Thus, by moving all occurrences of Φ to the right, we rewrite the right-hand side of [2, (2.5)] as follows:

$$\operatorname{gch} M_w^{\Psi}(\lambda) = \operatorname{gch} \mathcal{D}_w\Big(\mathbb{C}_{m_1(\lambda)\Lambda_1} \otimes \cdots (\mathbb{C}_{m_{n-1}(\lambda)\Lambda_{n-1}} \otimes \mathcal{C}_{n-1,e_{n-1}(\Psi)}(\mathbb{C}_{m_n(\lambda)\Lambda_n}) \cdots)\Big)$$

obtained from (2.3) and Proposition 2.1 (up to substitution $q \mapsto q^{-1}$). We note that in [2, (2.4)], $\Phi(x_\ell) = qx_1$, which under our convention corresponds to $\Phi(X_n) = q^{-1}X_1$ with $n = \ell$, that explains the substitution $q \mapsto q^{-1}$.

Lemma 2.5. *Let $1 \leq i < e \leq n$. If $\mathcal{D}_j(M) \cong M$ for all $0 < j < i$, then $\mathcal{D}_j(\mathcal{C}_{i,e}(M)) \cong \mathcal{C}_{i,e}(M)$ for all $0 \leq j < i$.*

Proof. Let v' and v be the longest elements in

$$\langle s_{i-1}, s_{i-2}, \ldots, s_1\rangle \subset \langle s_{i-1}, s_{i-2}, \ldots, s_0\rangle \subset \widetilde{\mathfrak{S}}_n,$$

respectively, i.e. v' is the longest element in $\mathfrak{S}_i$ and v is the longest element of $\mathfrak{S}_{i+1}$. By assumption, we have $\mathcal{D}_{v'}(M) \cong M$, and hence $\mathcal{C}_{i,e}(\mathcal{D}_{v'}(M)) \cong \mathcal{C}_{i,e}(M)$. We have $vs_j < v$ for $0 \le j < i$. Moreover, we have

$$\begin{aligned}\mathcal{C}_{i,e} \circ \mathcal{D}_{v'} &\cong (\mathcal{D}_{i-1} \circ \cdots \circ \mathcal{D}_1 \circ \mathcal{D}_0) \circ (\mathcal{D}_{n-1} \circ \cdots \circ \mathcal{D}_e) \circ \mathcal{D}_{v'} \\ &\cong \mathcal{D}_v \circ (\mathcal{D}_{n-1} \circ \cdots \circ \mathcal{D}_e) \cong \mathcal{D}_v \circ \mathcal{C}_{i,e}\end{aligned}$$

by inspection using Theorem 1.14(2,4). We have $s_j v < v$ for $0 \le j < i$. Thus, we deduce $\mathcal{D}_j \circ \mathcal{D}_v \cong \mathcal{D}_v$ by Theorem 1.14(2,4). The claim follows. □

Corollary 2.6. *Let $1 \le i < e \le e' \le n$. If $\mathcal{D}_j(M) \cong M$ for all $0 \le j < i$ and $e' < j < n$, then $\mathcal{D}_j(\mathcal{C}_{i,e}(M)) \cong \mathcal{C}_{i,e}(M)$ for all $0 \le j < i$ and $e' \le j < n$.*

Proof. Since we have automorphisms of affine Dynkin diagram of type $\mathsf{A}^{(1)}_{n-1}$ given by the cyclic rotation of indices of simple roots, we simply add $(n - e')$ to all the indices (modulo n) to deduce the result from Lemma 2.5. □

Lemma 2.7. *Let $1 \le i < e \le n$. For each $e \le j < n$ or $0 \le j < i - 1$, we have*

$$\mathcal{D}_j \circ \mathcal{C}_{i,e} \cong \mathcal{C}_{i,e} \circ \mathcal{D}_{j+1}.$$

Proof. By the isomorphism of functors

$$\mathcal{D}_j \circ \mathcal{D}_{j+1} \circ \mathcal{D}_j \cong \mathcal{D}_{j+1} \circ \mathcal{D}_j \circ \mathcal{D}_{j+1},$$

that is a special case of Theorem 1.14(4), the assertion reduces to checking that $\mathcal{D}_{i-1}, \ldots, \mathcal{D}_{j+2}$ commutes with $\mathcal{D}_j$, and $\mathcal{D}_{j-1}, \ldots, \mathcal{D}_e$ commutes with $\mathcal{D}_{j+1}$. □

Corollary 2.8. *Let $1 \le i < e < n$. For each $e \le e' < n$ or $0 \le e' < i - 1$, we have*

$$\mathcal{C}_{i-1,e'} \circ \mathcal{C}_{i,e} \cong \mathcal{C}_{i,e} \circ \mathcal{C}_{i,e'+1}.$$

Proof. Apply Lemma 2.7 to $\mathcal{C}_{i-1,e'} \circ \mathcal{C}_{i,e} \equiv \mathcal{D}_{i-2} \circ \cdots \circ \mathcal{D}_{e'} \circ \mathcal{C}_{i,e}$ repeatedly to deduce

$$\mathcal{D}_{i-2} \circ \cdots \circ \mathcal{D}_{e'} \circ \mathcal{C}_{i,e} \cong \mathcal{C}_{i,e} \circ \mathcal{D}_{i-1} \circ \cdots \circ \mathcal{D}_{e'+1},$$

that is equivalent to the assertion. □

Proposition 2.9. *Let $\Psi \subset \Delta^+$ be a root ideal. Assume that $w \in \mathfrak{S}_n$ is Ψ-tame. Then $N^\Psi_w(\lambda) \cong M^\Psi_w(\lambda)$ for all $\lambda \in$ `Par`.*

Example 2.10. We illustrate the arguments in the proof of Proposition 2.9 in the setting of Example 1.4 with $\lambda = \varpi_n$. We need to transform

$$\mathcal{C}_{2,3} \circ \mathcal{C}_{2,4} \circ \mathcal{C}_{2,5} \circ \mathcal{C}_{3,6} = (\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5)(\mathcal{D}_2\mathcal{D}_1\mathcal{D}_0)$$

into

$$(\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_2\mathcal{D}_1\mathcal{D}_0)(\mathcal{D}_3\mathcal{D}_2\mathcal{D}_1)(\mathcal{D}_4\mathcal{D}_3\mathcal{D}_2\mathcal{D}_1) \tag{2.4}$$

by applying a character $\mathbb{C}_{\Lambda_n}$ from the RHS, and let $\mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5$ act freely from the LHS. Observe that (2.4) simplifies to

$$(\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_2\mathcal{D}_1\mathcal{D}_0) \tag{2.5}$$

as $\mathcal{D}_i(\mathbb{C}_{\Lambda_n}) = \mathbb{C}_{\Lambda_n}$ for $i \neq 0$. Here, we have

$$\mathcal{D}_i(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3) = (\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)\mathcal{D}_{i+1} \qquad i = 3, 4, 5, 0.$$

This transforms (2.5) into

$$(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4)(\mathcal{D}_2\mathcal{D}_1\mathcal{D}_0). \tag{2.6}$$

For $i = 3, 4, 5$, we have

$$\mathcal{D}_i(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4) = (\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4)\mathcal{D}_{i+2}.$$

Hence, applying the left actions of $\mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5$ to (2.6), we obtain the identity

$$(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4\mathcal{D}_3)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5\mathcal{D}_4)(\mathcal{D}_1\mathcal{D}_0\mathcal{D}_5)(\mathcal{D}_2\mathcal{D}_1\mathcal{D}_0) = \mathcal{C}_{2,3} \circ \mathcal{C}_{2,4} \circ \mathcal{C}_{2,5} \circ \mathcal{C}_{3,6}.$$

Proof of Proposition 2.9. By Theorem 1.14, we find an isomorphism $\mathbb{L}^\bullet\mathcal{D}_i(\mathbb{C}_{\Lambda_j} \otimes \bullet) \cong \mathbb{C}_{\Lambda_j} \otimes \mathbb{L}^\bullet\mathcal{D}_i(\bullet)$ for distinct $i, j \in \mathtt{I}_{\mathrm{af}}$. In addition, we have $\mathbb{L}^\bullet\mathcal{D}_i(\mathbb{C}) \cong \mathbb{C}$. Using these two facts repeatedly, we shift the character twists to the left and discard $\mathcal{D}_i$'s with trivial effects repeatedly to obtain

$$M_w^\Psi(\lambda) \cong \mathcal{D}_w\Big(\mathbb{C}_{\mu_0} \otimes (\mathcal{C}_{1,e_1(\Psi)}(\mathbb{C}_{\mu_1} \otimes \mathcal{C}_{2,e_2(\Psi)}(\cdots(\mathbb{C}_{\mu_{r-1}} \otimes \mathcal{C}_{r,e_r(\Psi)}(\mathbb{C}_{\mu_r})\cdots)\Big), \tag{2.7}$$

where $r = \max\{1 \le s < n \mid s + d(\Psi)_s \le n\} = \mathtt{i}_{\ell(\Psi)}(\Psi)$, and

$$\mu_i = \sum_{j=e_i(\Psi)}^{e_{i+1}(\Psi)-1} m_j(\lambda)\Lambda_j.$$

For $0 \le j \le \ell(\Psi)$, the value $e_k(\Psi)$ is constant for $\mathtt{i}_j(\Psi) < k \le \mathtt{i}_{j+1}(\Psi)$. In particular, we have $\mu_i \neq 0$ $(i \ge 1)$ only if $i \in \mathtt{I}(\Psi)$.

In the below (during this proof), we drop Ψ from the notation of numbers presented by the typesetting fonts (i.e. $\mathtt{i}$ and $\mathtt{e}$). We discard $\mathbb{C}_{\mu_i}$ with $\mu_i \equiv 0$ in (2.7). Then, we inductively transform the sequence of terms

$$\mathbb{C}_{\mu_{\mathtt{i}_{(j-1)}}} \otimes (\mathcal{C}_{\mathtt{i}_{(j-1)}+1,\mathtt{e}_j} \circ \mathcal{C}_{\mathtt{i}_{(j-1)}+2,\mathtt{e}_j} \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j})(\mathbb{C}_{\mu_{\mathtt{i}_j}} \otimes \bullet), \tag{2.8}$$

that is a part of (2.7), into

$$\mathbb{C}_{\mu_{\mathtt{i}_{(j-1)}}} \otimes (\mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j} \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j+1} \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{i}_j-1})(\mathbb{C}_{\mu_{\mathtt{i}_j}} \otimes \bullet) \tag{2.9}$$

for each $1 \le j \le \ell(\Psi)$ by assuming that we can freely apply

$$\mathcal{D}_{\mathtt{i}_{(j-1)}-1}, \mathcal{D}_{\mathtt{i}_{(j-1)}-2}, \ldots, \mathcal{D}_{\mathtt{e}_j} \tag{2.10}$$

to (2.8) from the LHS without affecting the total output of (2.7). For the initial case $j = 1$, the functors in (2.10) arises from $\mathcal{D}_w$ since we have $ws_i < w$ for $e_1(\Psi) \le i < n$, that implies $\mathcal{D}_w \cong \mathcal{D}_w \circ \mathcal{D}_i$ for $e_1(\Psi) \le i < n$ (and we have $\mathtt{i}_0 = 0$ by convention).

Note that each of the terms in (2.10) commute with $\mathbb{C}_{\mu_{\mathtt{i}_{(j-1)}}}$ since we have

$$\mathtt{i}_{(j-1)} - 1 < \mathtt{i}_{(j-1)} + d_{\mathtt{i}_{(j-1)}}(\Psi) = \mathtt{e}_{(j-1)} < \mathtt{e}_j,$$

by Lemma 1.7. In particular, we can add each of (2.10) freely in front of $\mathcal{C}_{\mathtt{i}_{(j-1)+1},\mathtt{e}_j}$ in (2.8). Applying Lemma 2.7 repeatedly, this is the same as adding each of

$$\mathcal{D}_{\mathtt{i}_j-1}, \mathcal{D}_{\mathtt{i}_j-2}, \ldots, \mathcal{D}_{\mathtt{e}_j+\mathtt{i}_j-\mathtt{i}_{(j-1)}}. \tag{2.11}$$

freely just after $\mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j}$ in (2.8) without affecting the output. Hence, we can freely insert

$$\mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j+\mathtt{i}_j-\mathtt{i}_{(j-1)}}, \ldots, \mathcal{C}_{\mathtt{i}_j,\mathtt{i}_j-1} \tag{2.12}$$

just after $\mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j}$ in addition to (2.11).

By using Corollary 2.8 repeatedly, we have

$$\mathcal{C}_{\mathtt{i}_{(j-1)}+1,\mathtt{e}_j} \circ \mathcal{C}_{\mathtt{i}_{(j-1)}+2,\mathtt{e}_j} \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j} \cong \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j} \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j+1} \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j+\mathtt{i}_j-\mathtt{i}_{(j-1)}-1}.$$

Combining this with (2.12), we obtain the desired composition form as in (2.9). Here the product of $\mathcal{C}$'s in (2.9) gives a reduced expression of the longest element of

$$\left\langle s_{\mathtt{i}_j-1}, \ldots, s_0, \ldots, s_{\mathtt{e}_j} \right\rangle \subset \widetilde{\mathfrak{S}}_n. \tag{2.13}$$

Thus, we can add each of

$$\mathcal{D}_{\mathtt{i}_j-1}, \mathcal{D}_{\mathtt{i}_j-2}, \ldots, \mathcal{D}_{\mathtt{e}_{(j+1)}}$$

just after $\mathcal{C}_{\mathtt{i}_j,\mathtt{i}_j-1}$ in (2.9) without modifying the output. This allows us to proceed with the induction on j. Hence, we can replace every (2.8) in (2.7) into (2.9) inductively.

The terms

$$\mathcal{C}_{\mathtt{i}_j,\mathtt{e}_{(j+1)}}, \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_{(j+1)}+1}, \cdots, \mathcal{C}_{\mathtt{i}_j,\mathtt{i}_j-1} \tag{2.14}$$

in (2.7) commute with $\mathbb{C}_{\mu_{\mathtt{i}_j}}$, and can be moved to the component (2.9) for j replaced with $(j+1)$. Each of (2.14) is the composition of Demazure functors corresponding to the simple reflections listed in (2.13), with j replaced by $(j+1)$ (as $\mathtt{i}_j < \mathtt{i}_{(j+1)}$). Thus, we can delete them making use of the expression (2.9) for j replaced with $(j+1)$ when $j < \ell(\Psi)$ and $\mathcal{D}_i(\mathbb{C}_{m_k\Lambda_k}) = \mathbb{C}_{m_k\Lambda_k}$ for $1 \le i < \mathtt{i}_{\ell(\Psi)}(\Psi)$ and $\mathtt{e}_{\ell(\Psi)} \le k \le n$ when $j = \ell(\Psi)$ coming from Theorem 1.14(3).

This procedure further replaces (2.8) in (2.7) with

$$\mathbb{C}_{\mu_{\mathtt{i}_{(j-1)}}} \otimes (\mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j} \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_j+1} \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j,\mathtt{e}_{(j+1)}-1})(\mathbb{C}_{\mu_{\mathtt{i}_j}} \otimes \bullet).$$

This is identical to the definition of $\mathcal{C}^{\Psi}_j(\lambda)$, tensored by $\mathbb{C}_{\mu_{\mathtt{i}_{(j-1)}}}$.

This completes the transformation from $M^{\Psi}_w(\lambda)$ to $N^{\Psi}_w(\lambda)$, and establishes the proposition. □

For $e_1(\Psi) \le k \le n$, we have unique $1 \le j \le \ell(\Psi)$ such that $\mathtt{e}_j(\Psi) \le k < \mathtt{e}_{j+1}(\Psi)$ by the monotonicity of $\mathtt{e}_\bullet$. We define

$$\begin{aligned} N^{\Psi}(\lambda; k) &:= \Big(\big(\mathcal{C}_{\mathtt{i}_j(\Psi),k}(\lambda) \circ \cdots \circ \mathcal{C}_{\mathtt{i}_j(\Psi),\mathtt{e}_{j+1}(\Psi)-1}(\lambda) \big) \circ \big(\mathcal{C}^{\Psi}_{j+1}(\lambda) \circ \cdots \mathcal{C}^{\Psi}_{\ell(\Psi)}(\lambda) \big)(\mathbb{C}) \Big) \\ &= \big(\mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda) \circ \mathcal{C}_{\mathtt{h}_{k+1}(\Psi),k+1}(\lambda) \circ \cdots \mathcal{C}_{\mathtt{h}_n(\Psi),n}(\lambda) \big)(\mathbb{C}), \end{aligned}$$

where the two expressions are the same by examining (1.4).

Lemma 2.11. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $\lambda \in \mathtt{Par}$. For each $e_1(\Psi) \le k \le n$, the $\widetilde{\mathbf{B}}$-module $N^{\Psi}(\lambda; k)$ is invariant under $\mathcal{D}_k, \ldots, \mathcal{D}_{n-1}, \mathcal{D}_0, \ldots, \mathcal{D}_{\mathtt{h}_k(\Psi)-1}$.*

Proof. We proceed by downward induction on k, starting from $k = n$. If $k = n$, then the functor $\mathcal{D}_i$ $(1 \le i < n)$ acts trivially on $\mathbb{C}_{m_n(\lambda)\Lambda_n}$, and hence the assertion holds by Lemma 2.5 for $e = n$.

We assume the assertion for all larger values of k. We have $\mathtt{i}_j(\Psi) < \mathtt{i}_{j+1}(\Psi)$ and $\mathtt{e}_j(\Psi) < \mathtt{e}_{j+1}(\Psi)$ by Lemma 1.7. By the induction hypothesis,

$$\Big(\mathcal{C}_{\mathtt{h}_{k+1}(\Psi),k+1}(\lambda) \circ \cdots \circ \mathcal{C}_{\mathtt{h}_n(\Psi),n}(\lambda)\Big)(\mathbb{C})$$

is invariant under the application of $\mathcal{D}_{k+1}, \ldots, \mathcal{D}_{n-1}, \mathcal{D}_0, \ldots, \mathcal{D}_{\mathtt{h}_k(\Psi)-1}$. We have $\mathcal{D}_j(\mathbb{C}_{m_k(\lambda)\Lambda_k} \otimes \bullet) \cong \mathbb{C}_{m_k(\lambda)\Lambda_k} \otimes \mathcal{D}_j(\bullet)$ for $j \not\equiv k \mod n$ by $\langle \alpha_j, \Lambda_k \rangle = 0$ and Theorem 1.14(3). Hence, Corollary 2.6 implies the assertion from the induction hypothesis. This allows us to proceed with the induction. This completes the proof. □

Lemma 2.12. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $w \in \mathfrak{S}_n$. We have*

$$N_w^\Psi(\lambda) \cong N_{ws_i}^\Psi(\lambda) \qquad \lambda \in \mathtt{Par}, e_1(\Psi) \le i < n.$$

Proof. By Lemma 2.11, $N^\Psi(\lambda; e_1(\Psi))$ is invariant under all $\mathcal{D}_j$ with $e_1(\Psi) \le j < n$. Therefore, the claim follows from

$$\langle \alpha_j, \sum_{i=1}^{d_1(\Psi)} m_i(\lambda)\Lambda_i \rangle = 0, \quad \text{for } e_1(\Psi) \le j < n,$$

together with Theorem 1.14(3). □

3 Construction of the variety $\mathcal{X}_\Psi$

Keep the setting of the previous section.

Lemma 3.1. *Let $\Psi \subset \Delta^+$ be a root ideal. Let $w \in \mathfrak{S}_n$ and $e_1(\Psi) \le k \le n$. Then, the* $\mathtt{Par}$*-graded vector spaces*

$$\bigoplus_{\lambda \in \mathtt{Par}} N_w^\Psi(\lambda)^* \quad \textit{and} \quad \bigoplus_{\lambda \in \mathtt{Par}} N^\Psi(\lambda; k)^*, \tag{3.1}$$

acquire the structure of commutative $\mathbb{C}$-algebras with $\widetilde{\mathbf{B}}$-actions that respect the grading. In addition, they are integral and integrally closed.

Proof. The character twists appearing in the construction of the modules $N_w^\Psi(\lambda)$ and $N^\Psi(\lambda; k)$ are additive with respect to the monoid structure of $\mathtt{Par}$. Therefore, we may apply Lemma 1.13 repeatedly to deduce the desired properties. □

For a root ideal $\Psi \subset \Delta^+$ and $w \in \mathfrak{S}_n$, we define a $\widetilde{\mathbf{B}}$-scheme

$$\mathcal{X}_\Psi(w) := \mathrm{Proj}_{\mathtt{Par}} \bigoplus_{\lambda \in \mathtt{Par}} N_w^\Psi(\lambda)^*$$

as a multigraded Proj over $\mathbb{C}$ following the construction in (1.1). Similarly, for $e_1(\Psi) \le r \le n$, we define a $\widetilde{\mathbf{B}}$-scheme

$$X_\Psi(k) := \mathrm{Proj}_{\mathtt{Par}} \bigoplus_{\lambda \in \mathtt{Par}} N^\Psi(\lambda; k)^*$$

In view of Lemma 3.1, both of the $\mathfrak{X}_\Psi(w)$ and $X_\Psi(k)$ are integral and normal schemes.

For a more explicit illustration, we refer to an example calculation for $n = 4$ about the construction of $\mathfrak{X}_\Psi(w_0)$ in Example 3.10 in the end of this section.

Corollary 3.2. *Let $\Psi \subset \Delta^+$ be a root ideal. Then, there exist natural $\widetilde{\mathbf{B}}$-equivariant morphisms*

$$\mathfrak{X}_\Psi(w_0^\Psi) \longrightarrow \prod_{k=e_1(\Psi)}^{n} \mathbb{P}(N^\Psi_{w_0^\Psi}(\varpi_k)) \hookrightarrow \prod_{k\in \mathtt{I}_{\mathrm{af}}} \mathbb{P}(L(\Lambda_k)), \tag{3.2}$$

where the second map is the closed embedding.

Proof. For each $e_1(\Psi) \le k \le n$, the module $N^\Psi_{w_0^\Psi}(\varpi_k)$ is the (dual of the) space of the global sections of the line bundle $\mathcal{O}_{\mathbf{i}'}(\Lambda_k)$, where $\mathbf{i}'$ is the sequence in $\mathtt{I}_{\mathrm{af}}$ read out from (2.2) as the index of $\mathcal{D}$'s in the definition of $N^\Psi_e(\varpi_k)$ (thanks to Lemma 2.12) until the character twist by $\mathbb{C}_{\Lambda_k}$. Let $\mathbf{i}$ be the sequence $\mathbf{i}'$ corresponding to the case $k = n$. Since $\mathbf{i}$ is an initial subsequence of $\mathbf{i}$, we obtain a map $f : X(\mathbf{i}) \to X(\mathbf{i}')$ by repeated applications of Lemma 1.10(1). By Theorem 1.15(3), the line bundle $\mathcal{O}_{\mathbf{i}'}(\Lambda_k)$ is base-point-free over $X(\mathbf{i}')$. Hence, $f^*\mathcal{O}_{\mathbf{i}'}(\Lambda_k)$ is base-point-free on $X(\mathbf{i})$. Here $X(\mathbf{i})$ maps onto $X_\Psi(w_0^\Psi)$ and $f^*\mathcal{O}_{\mathbf{i}'}(\Lambda_k)$ is the pullback of a line bundle on $\mathfrak{X}_\Psi(w_0^\Psi)$ by the definition of $\mathcal{D}_{\mathbf{i}}$ and (2.2). Therefore, we find an induced map

$$\mathfrak{X}_\Psi(w_0^\Psi) \longrightarrow \mathbb{P}(N^\Psi_{w_0^\Psi}(\varpi_k)).$$

Taking the product of these morphisms gives the first map in (3.2).

In view of Theorem 1.15(2), we have $N^\Psi_{w_0^\Psi}(\varpi_k) \subset L(\Lambda_k)$ for $1 \le k \le n$. Moreover, we have $N^\Psi_{w_0^\Psi}(\varpi_k) = \mathbb{C}\mathbf{v}_{\Lambda_k}$ for $1 \le k \le d_1(\Psi)$. This yields the second embedding by sending all points to $[\mathbf{v}_{\Lambda_k}] \in \mathbb{P}(L(\Lambda_k))$ for $1 \le k \le d_1(\Psi)$. $\square$

Lemma 3.3. *Let $\Psi \subset \Delta^+$ be a root ideal. Let $w \in \mathfrak{S}_n$ and $e_1(\Psi) \le k \le n$. Then, we have closed embeddings $X_\Psi(k) \subset \mathfrak{X}_\Psi(w) \subset \mathfrak{X}_\Psi(w_0)$ of $\widetilde{\mathbf{B}}$-schemes. In particular, we have $X_\Psi(e_1(\Psi)) = \mathfrak{X}_\Psi(w_0^\Psi)$.*

Proof. In view of Lemma 2.12, we have an identification of the coordinate rings of $X_\Psi(e_1(\Psi)) = \mathfrak{X}_\Psi(w_0^\Psi)$ as $N^\Psi_e(\lambda) = N^\Psi(\lambda; e_1(\Psi))$ for each $\lambda \in \mathtt{Par}$. The remaining inclusions follow from surjections of homogeneous coordinate rings that are guaranteed by repeated applications of Corollary 1.18. $\square$

The graded components of the ring (3.1) define $\widetilde{\mathbf{B}}$-equivariant line bundles $\mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)$ on $\mathfrak{X}_\Psi(w)$ and $\mathcal{O}_{X_\Psi(k)}(\lambda)$ on $X_\Psi(k)$ for each $\lambda \in \mathtt{Par}$, extended to $\lambda \in \mathsf{P}$ by taking the duals and tensor products. We define two subgroups of $\widetilde{G}(\!(z)\!)$ as

$$\widetilde{\mathbf{P}}(k) := \left\langle \widetilde{\mathbf{P}}_i \mid k \le i \le n \text{ or } 1 \le i < \mathtt{h}_k(\Psi) \right\rangle \quad \text{and}$$
$$G(k) := \langle SL(2,i) \mid k \le i \le n \text{ or } 1 \le i < \mathtt{h}_k(\Psi) \rangle\,,$$

where $e_1(\Psi) \le k \le n$. By convention, we set $\widetilde{\mathbf{P}}(n+1) := \widetilde{\mathbf{G}}$.

Lemma 3.4. *Let $\Psi \subset \Delta^+$ be a root ideal. For each $e_1(\Psi) \le k \le n$, we have $G(k) \cong SL(\mathtt{h}_k(\Psi) + n - k + 1)$, $\widetilde{\mathbf{P}}(k) = G(k) \cdot \widetilde{\mathbf{B}}$, and the group $\widetilde{\mathbf{P}}(k)$ is a proalgebraic group. In addition, we have a split quotient map*

$$\widetilde{\mathbf{P}}(k) \longrightarrow G(k).$$

Proof. We make use of the Dynkin diagram automorphism of type $\mathsf{A}^{(1)}_{n-1}$, which permutes the subgroups $SL(2,i)$ corresponding to $i \in \mathtt{I}_{\mathrm{af}}$. We apply the cyclic shift to the simple roots $\pm\alpha_k, \dots, \pm\alpha_{\mathtt{h}_k(\Psi)}$ of $G(k)$ by uniformly adding $(n-k)$ to the index (modulo n). Since $\mathtt{h}_k(\Psi) < k$, the corresponding one-parameter subgroups generate $SL(\mathtt{h}_k(\Psi)+n-k+1)$ inside $G \subset \widetilde{\mathbf{G}}$. The rotation of each $\widetilde{\mathbf{P}}_i$ $(k \le i < \mathtt{h}_k(\Psi))$ is also contained by $\widetilde{\mathbf{G}}$. Hence, it generates a closed subgroup of $\widetilde{\mathbf{G}}$, that is proalgebraic. Here the rotation of $\widetilde{\mathbf{P}}_i$ has its image $P_{i+n-k} \subset G$ that generates $SL(\mathtt{h}_k(\Psi)+n-k+1)$. It follows that $\widetilde{\mathbf{P}}(k) = G(k) \cdot \widetilde{\mathbf{B}}$. Moreover, we conclude that $z \mapsto 0$ (after cyclic shift) yields the desired split quotient map of proalgebraic groups. $\square$

Lemma 3.5. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $e_1(\Psi) \le k < n$. Then, the algebraic subgroup*

$$G(k) \cap \widetilde{\mathbf{P}}(k+1) \subset G(k),$$

is a maximal proper parabolic subgroup such that the natural map

$$G(k)/(G(k) \cap \widetilde{\mathbf{P}}(k+1)) \longrightarrow \widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1)) \tag{3.3}$$

induced by the inclusion is an isomorphism. In addition, the resulting variety (3.3) *is isomorphic to the projective space of dimension* $(\mathtt{h}_k(\Psi) + n - k)$.

Proof. The Iwahori subgroup $\widetilde{\mathbf{B}}$ is stable by the Dynkin diagram automorphism of type $\mathsf{A}^{(1)}_{n-1}$. It follows that $(G(k) \cap \widetilde{\mathbf{B}})$ must contain a Borel subgroup of $G(k)$ corresponding to the positive affine roots. Thus, $(G(k) \cap \widetilde{\mathbf{P}}(k+1))$ is a parabolic subgroup of $G(k)$. Now the natural inclusion induces a map (3.3), that is injective. By $\widetilde{\mathbf{P}}(k) = G(k) \cdot \widetilde{\mathbf{B}}$, we conclude the map is also surjective. By the comparison the definition using $\mathtt{h}_k(\Psi) \le \mathtt{h}_{k+1}(\Psi)$, we have

$$SL(2,i) \subset G(k) \cap G(k+1) \quad \text{ if and only if } \quad k < i \le n \text{ or } 1 \le i < \mathtt{h}_k(\Psi).$$

Thus, $(G(k) \cap \widetilde{\mathbf{P}}(k+1))$ is the maximal proper parabolic subgroup of $G(k) \cong SL(\mathtt{h}_k + n - k + 1)$ whose Levi component is $SL(\mathtt{h}_k + n - k)$. This implies the last assertion, concluding the proof. $\square$

Lemma 3.6. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $e_1(\Psi) \le k < n$. Suppose M is a $\widetilde{\mathbf{P}}(k+1)$-module whose restriction to $G(k+1)$ is a rational representation. Then, we have*

$$\mathcal{C}_{\mathtt{h}_k,k}(\lambda)(M) \cong H^0(\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k+1) \cap \widetilde{\mathbf{P}}(k)), \mathcal{F}(M))^\vee,$$

where $\mathcal{F}(M)$ is the vector bundle on

$$\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1)) \cong \mathbb{P}^{\mathtt{h}_k(\Psi)+n-k}$$

induced by the $(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1))$-module structure of $M^\vee$.

Proof. Let $\mathbf{i} := \{k, k+1, \ldots, \mathtt{h}_k - 1\}$. Then, we have a map

$$\pi_{\mathbf{i}} : X(\mathbf{i}) \longrightarrow \widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k+1) \cap \widetilde{\mathbf{P}}(k)) \cong \mathbb{P}^{\mathtt{h}_k(\Psi)+n-k}.$$

We have $\pi_{\mathbf{i}}^* \mathcal{F}(M) \cong \mathcal{E}_{\mathbf{i}}(M)$. Hence, we have

$$H^0(X(\mathbf{i}), \mathcal{E}_{\mathbf{i}}(M)) \cong \mathcal{C}_{\mathtt{h}_k,k}(\lambda)(M)^\vee.$$

We have

$$\mathbb{R}^\bullet(\pi_{\mathbf{i}})_* \pi_{\mathbf{i}}^* \mathcal{F}(M) \cong (\mathbb{R}^\bullet(\pi_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})}) \otimes_{\mathcal{O}_{\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k+1)\cap\widetilde{\mathbf{P}}(k))}} \mathcal{F}(M)$$

by the projection formula. Since we know

$$\mathbb{R}^{>0}(\pi_{\mathbf{i}})_* \mathcal{O}_{X(\mathbf{i})} = 0$$

from [20, 8.2.2 Theorem (c) and A.24], the Leray spectral sequence

$$H^q(\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k+1) \cap \widetilde{\mathbf{P}}(k)), \mathbb{R}^p(\pi_{\mathbf{i}})_* \pi_{\mathbf{i}}^* \mathcal{F}(M)) \Rightarrow H^{q+p}(X(\mathbf{i}), \pi_{\mathbf{i}}^* \mathcal{F}(M))$$

degenerates at the E_1-stage. This implies

$$\mathcal{C}_{\mathtt{h}_k,k}(\lambda)(M)^\vee = H^0(X(\mathbf{i}), \pi_{\mathbf{i}}^* \mathcal{F}(M)) \cong H^0(\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k+1) \cap \widetilde{\mathbf{P}}(k)), \mathcal{F}(M))$$

as desired. □

Proposition 3.7. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $e_1(\Psi) \le k < n$. The variety $X_\Psi(k)$ is a $\widetilde{\mathbf{P}}(k)$-equivariant $X_\Psi(k+1)$-fibration over the base space $\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1))$.*

Proof. We have a natural map

$$\bigoplus_{\lambda \in \mathtt{Par}} N^\Psi(\lambda; k)^* \longrightarrow \bigoplus_{\lambda \in \mathtt{Par}} N^\Psi(\lambda; k+1)^* \otimes \mathbb{C}_{-m_k(\lambda)\Lambda_k}$$

offered by Corollary 1.18, that is surjective. By Lemma 3.6, the LHS is the global section of the sheaf $\mathcal{A}$ of algebras over $\widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1))$ arising from the RHS. The sheaf of algebras $\mathcal{A}$ defines $X_\Psi(k+1)$ through taking $\mathtt{Proj}_{\mathtt{Par}}$ over each fiber. Thus, we obtain a map

$$\pi_k : \widetilde{\mathbf{P}}(k) \times^{(\widetilde{\mathbf{P}}(k)\cap\widetilde{\mathbf{P}}(k+1))} X_\Psi(k+1) \longrightarrow X_\Psi(k).$$

The above surjection implies that the fiber $X_\Psi(k+1)$ embeds into $X_\Psi(k)$ as a closed subscheme. Here we consider the weights of the form $c\varpi_k$ for $c \in \mathbb{Z}_{>0}$. We have $m_j(c\varpi_k) = 0$ for $j > k$ and $m_k(c\varpi_k) = c$. Since $\mathcal{D}_j(\mathbb{C}) \cong \mathbb{C}$ for all $j \in \mathtt{I}_{\mathrm{af}}$, the construction of $N^\Psi(c\varpi_k)$ yields

$$N^\Psi(c\varpi_k; k+1) = \mathbb{C} \qquad c \in \mathbb{Z}_{\ge 0}.$$

Therefore, the $\widetilde{\mathbf{P}}(k)$-equivariant morphism

$$\psi : X(k) \longrightarrow \mathbb{P}(L(\Lambda_k))$$

induced by $\mathcal{O}_{X(k)}(\varpi_k)$, that exists in view of Theorem 1.15(3), sends the fiber $X_\Psi(k+1)$ over the point $(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1))/(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1))$ to

$$\mathrm{pt} = \mathtt{Proj}_{\mathbb{Z}_{\geq 0}} \bigoplus_{c \geq 0} \mathbb{C}_{-c\Lambda_k} \subset \mathbb{P}(N^\Psi(\varpi_k; k)),$$

that is a $\widetilde{\mathbf{P}}(k+1)$-fixed point. Since the unique $\widetilde{\mathbf{P}}(k+1)$-eigenvector in $L(\Lambda_k)$ (up to scalar) is $\mathbf{v}_{\Lambda_k}$, we find that the image of $X_\Psi(k+1)$ is $[\mathbf{v}_{\Lambda_k}]$. It follows that

$$\mathrm{Im}\, \psi \cong \widetilde{\mathbf{P}}(k)/(\widetilde{\mathbf{P}}(k) \cap \widetilde{\mathbf{P}}(k+1)) \cong \mathbb{P}^{\mathtt{h}_k(\Psi)+n-k}$$

as topological space. Since $\mathbb{P}^{\mathtt{h}_k(\Psi)+n-k}$ is $\widetilde{\mathbf{P}}(k)$-homogeneous, we find that $X_\Psi(k)$ admits a $\widetilde{\mathbf{P}}(k)$-equivariant fiber bundle structure over $\mathbb{P}^{\mathtt{h}_k(\Psi)+n-k}$ with its fiber $X_\Psi(k+1)$. Thus, we find that the map π_k is locally an isomorphism. Now the $\widetilde{\mathbf{P}}(k)$-action makes π_k into a $\widetilde{\mathbf{P}}(k)$-equivariant isomorphism as required. □

Corollary 3.8. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $e_1(\Psi) \leq k < n$. The variety $X_\Psi(k)$ is smooth and*

$$\dim \mathcal{X}_\Psi(w_0^\Psi) = |\Psi| + \ell(w_0^\Psi). \tag{3.4}$$

Proof. By Proposition 3.7, the algebraic variety $X_\Psi(k)$ is a successive projective space bundle of dimensions $\{\mathtt{h}_j + n - j\}_{j=k}^n$. Thus, it must be smooth. Here w_0^Ψ is a longest element of $\mathfrak{S}_{n-d_1(\Psi)}$ that has length

$$\ell(w_0^\Psi) = \sum_{i=0}^{n-d_1(\Psi)-1} i = \frac{(n-d_1(\Psi))(n-d_1(\Psi)-1)}{2}.$$

Now we compute the dimension as

$$\begin{aligned}\dim \mathcal{X}_\Psi(w_0^\Psi) = \dim X_\Psi(e_1(\Psi)) = \sum_{k=e_1(\Psi)}^{n} (\mathtt{h}_k + n - k) = \sum_{k=e_1(\Psi)}^{n} \mathtt{h}_k + \sum_{i=0}^{n-d_1(\Psi)-1} i \\ = |\Psi| + \frac{(n-d_1(\Psi))(n-d_1(\Psi)-1)}{2} = |\Psi| + \ell(w_0^\Psi),\end{aligned}$$

where we used (1.5) in the first equality of the second line, as required. □

Theorem 3.9. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $w \in \mathfrak{S}_n$ be a Ψ-tame element. There exists a G-equivariant closed embedding*

$$\mathcal{X}_\Psi(w_0) \cong G \times^Q \mathcal{X}_\Psi(w_0^\Psi) \hookrightarrow \prod_{i=1}^n \mathbb{P}(L(\Lambda_i)), \tag{3.5}$$

where $Q \subset G$ is the parabolic subgroup generated by P_i $(e_1(\Psi) \leq i < n)$. We have

$$\dim \mathcal{X}_\Psi(w) = \ell(w) + |\Psi|. \tag{3.6}$$

The variety $\mathcal{X}_\Psi(w)$ is smooth if and only if the Schubert variety $X(w)$ is smooth.

Proof. By (2.2) and Lemma 3.3, the projective coordinate rings of $\mathcal{X}_\Psi(w_0)$ and $\mathcal{X}_\Psi(w_0^\Psi)$, denoted by $R_\Psi(w_0)$ and $R_\Psi(w_0^\Psi)$ respectively, satisfy

$$R_\Psi(w_0) \cong \mathcal{D}_{w_0}^\dagger(R_\Psi(w_0^\Psi)).$$

The functor $\mathcal{D}_{w_0}^\dagger$ transforms the $\widetilde{\mathbf{B}}$-equivariant $\mathtt{Par}$-graded algebra $R_\Psi(w_0^\Psi)$ into the space of global sections of a $\widetilde{\mathbf{G}}$-equivariant $\mathtt{Par}$-graded sheaf $\mathcal{R}$ on $\widetilde{\mathbf{G}}/\widetilde{\mathbf{B}} \cong G/B$. This sheaf has fiber $R_\Psi(w_0^\Psi)^\vee$ over B/B. Lemma 2.12 asserts that $R_\Psi(w_0)$ is stable under the action of $\widetilde{\mathbf{P}}_i$ for $e_1(\Psi) \le i < n$. Therefore, the parabolic subgroup Q acts on $R_\Psi(w_0^\Psi)$ as a rational representation, and we may form a $\widetilde{\mathbf{G}}$-equivariant $\mathtt{Par}$-graded sheaf of algebras on G/Q that carries $R_\Psi(w_0)^\vee$ as its fiber over Q/Q. Let $\pi_Q : X = G/B \to G/Q$ be a natural projection. Then, we have $\mathcal{R} \cong \pi_Q^*\mathcal{R}'$ and

$$\mathbb{R}^\bullet(\pi_Q)_*\mathcal{R} \cong \left(\mathbb{R}^\bullet(\pi_Q)_*\mathcal{O}_X\right) \otimes_{\mathcal{O}_{G/Q}} \mathcal{R}' \cong \mathcal{R}',$$

where the first isomorphism is the projection formula, and the second isomorphism follows from $\mathbb{R}^\bullet(\pi_Q)_*\mathcal{O}_X \cong \mathcal{O}_{G/Q}$, that in turn follows from

$$H^\bullet(Q/B, \mathcal{O}_{Q/B}) \cong \mathbb{C}.$$

It follows that

$$R_\Psi(w_0) \cong H^0(X, \mathcal{R}) \cong H^0(G/Q, \mathcal{R}'),$$

and that induces a surjective $\widetilde{\mathbf{G}}$-equivariant morphism

$$f : G \times^Q \mathcal{X}_\Psi(w_0^\Psi) \longrightarrow \mathcal{X}_\Psi(w_0).$$

The image of $\mathcal{X}_\Psi(w_0^\Psi)$ under the map (3.2) composed with the projection to $\prod_{i=1}^{d_1(\Psi)} \mathbb{P}(L(\Lambda_i))$ is a Q-fixed point

$$\{[\mathbf{v}_{\Lambda_i}]\}_{i=1}^{d_1(\Psi)} \in \prod_{i=1}^{d_1(\Psi)} \mathbb{P}(L(\Lambda_i)).$$

Each $\mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\varpi_i)$ $(1 \le i \le d_1(\Psi))$ induces a map

$$\mathcal{X}_\Psi(w_0) \longrightarrow \mathbb{P}(N_{w_0}^\Psi(\varpi_i)) \subset \mathbb{P}(L(\Lambda_i))$$

that sends $\mathcal{X}_\Psi(w_0^\Psi)$ to $\{[\mathbf{v}_{\Lambda_i}]\}_{i=1}^{d_1(\Psi)}$ since we have

$$(\mathcal{C}_1^\Psi(\varpi_i) \circ \cdots \circ \mathcal{C}_{\ell(\Psi)}^\Psi(\varpi_i))(\mathbb{C}) \cong \mathbb{C}$$

for $1 \le i \le d_1(\Psi)$. Thus, we find a $\widetilde{\mathbf{G}}$-equivariant map

$$\mathsf{pr} : \mathcal{X}_\Psi(w_0) \longrightarrow \prod_{i=1}^{d_1(\Psi)} \mathbb{P}(L(\Lambda_i)),$$

whose image is G/Q and whose fiber contains $\mathcal{X}_\Psi(w_0^\Psi)$. From this, we conclude that f is in fact a $\widetilde{\mathbf{G}}$-equivariant isomorphism. Using this isomorphism, the embedding (3.2) extends naturally to (3.5) via the G-action. Since $\mathcal{X}_\Psi(w_0^\Psi)$

is smooth and projective, so is $\mathfrak{X}_\Psi(w_0)$. Hence the embedding (3.5) must be closed. This yields the first part of the assertion.

Since w is Ψ-tame, we have $w = vw_0^\Psi$ for some $v \in \mathfrak{S}_n$ such that $\ell(w) = \ell(v) + \ell(w_0^\Psi)$. Let $v = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ be a reduced expression of v, that we record as $\mathbf{i}$. We set

$$Y(w) := \overline{BwQ/Q} \subset G/Q.$$

The variety $Y(w)$ is normal of dimension $\ell(v)$, equipped with a resolution

$$X(\mathbf{i}) \xrightarrow{f} Y(w) \quad \text{such that} \quad \mathcal{O}_{Y(w)} \stackrel{\cong}{\to} f_*\mathcal{O}_{X(\mathbf{i})} \tag{3.7}$$

([20, 8.2.2 Theorem (c) and A.24]). Since $\mathsf{pr}^{-1}(Y(w))$ is a locally trivial fibration over $Y(w)$ with fiber $\mathfrak{X}_\Psi(w_0^\Psi)$, we have

$$\dim \mathsf{pr}^{-1}(Y(w)) = \dim \mathfrak{X}_\Psi(w_0^\Psi) + \ell(v) = \ell(v) + \ell(w_0^\Psi) + |\Psi| = \ell(w) + |\Psi|.$$

It is smooth if and only if $Y(w)$ is smooth. Taking into account the locally trivial Q/B-fibration structure of $G/B \to G/Q$, it is also equivalent to $X(w)$ being smooth.

By Lemma 3.3, $\mathfrak{X}_\Psi(w)$ is a closed subvariety of $\mathfrak{X}_\Psi(w_0)$. Hence, the image of the map

$$\widetilde{X}(\mathbf{i}) \times^{\widetilde{\mathbf{B}}} X_\Psi(w_0^\Psi) \cong X(\mathbf{i}) \times_{Y(w)} \mathsf{pr}^{-1}(Y(w)) \longrightarrow \mathsf{pr}^{-1}(Y(w)) \subset \mathfrak{X}_\Psi(w_0) \tag{3.8}$$

is identified with $\mathfrak{X}_\Psi(w)$ as a set of points. This completes the proof. □

Example 3.10 ($n = 4$). We illustrate the construction of $\mathfrak{X}_\Psi(w_0)$ in the case $G = GL(4, \mathbb{C})$ with the root ideal

$$\Psi = \{\epsilon_1 - \epsilon_3, \epsilon_1 - \epsilon_4, \epsilon_2 - \epsilon_4\}.$$

In this case, we have $e_1(\Psi) = 3$ and

$$\mathtt{h}_2(\Psi) = 0, \mathtt{h}_3(\Psi) = 1, \mathtt{h}_4(\Psi) = 2.$$

We have vectors labelled by T-weights

$$V^{(4)} := \mathbb{C}\mathbf{v}_{1111} \oplus \mathbb{C}\mathbf{v}_{2110} \oplus \mathbb{C}\mathbf{v}_{1210} \subset L(\Lambda_4),$$

where $\mathbf{v}_{1111} := \mathbf{v}_{\Lambda_4} \in L(\Lambda_4)$ is the highest weight vector, and $\mathbf{v}_{2110}, \mathbf{v}_{1210} \in L(\Lambda_4)$ are unique vectors with their $\widetilde{T}$-weights $(\Lambda_4 - \alpha_0)$ and $(\Lambda_4 - \alpha_0 - \alpha_1)$, respectively. The vectors $\mathbf{v}_{2110}$ and $\mathbf{v}_{1210}$ have degree -1, while $\mathbf{v}_{1111}$ has degree 0. We have

$$\mathbb{P}^2 \cong X_\Psi(4) = \mathbb{P}(V^{(4)}) \subset \mathbb{P}(L(\Lambda_4)).$$

This variety is preserved under the action of

$$\widetilde{\mathbf{P}}(4) = \left\langle SL(2,0), SL(2,1), \widetilde{\mathbf{B}} \right\rangle \subset \widetilde{G}(\!(z)\!).$$

We additionally have vectors labelled by T-weights

$$V^{(3)} := \mathbb{C}\mathbf{v}_{1110} \oplus \mathbb{C}\mathbf{v}_{1101} \oplus \mathbb{C}\mathbf{v}_{2100} \subset L(\Lambda_3),$$

where $\mathbf{v}_{1110} := \mathbf{v}_{\Lambda_3} \in L(\Lambda_3)$ is the highest weight vector, and $\mathbf{v}_{1101}$ and $\mathbf{v}_{2100}$ are unique vector with their $\widetilde{T}$-weights $(\Lambda_3 - \alpha_3)$ and $(\Lambda_3 - \alpha_3 - \alpha_0)$, respectively. The vector $\mathbf{v}_{2100}$ has degree -1, while $\mathbf{v}_{1110}$ and $\mathbf{v}_{1101}$ have degree 0. The space $\mathbb{P}(V^{(3)}) \subset \mathbb{P}(L(\Lambda_3))$ is stable under the action of $\widetilde{\mathbf{P}}(3)$, that is generated by $SL(2,3), SL(2,0)$, and $\widetilde{\mathbf{B}}$. Define

$$G(3) := \left\langle SL(2,3), SL(2,0), \widetilde{\mathbf{B}} \right\rangle \subset \widetilde{G}(\!(z)\!).$$

We have $G(3) \cong SL(3) \subset \widetilde{G}(\!(z)\!)$. A parabolic subgroup $P(3) \subset G(3)$ that contains $SL(2,0)$ stabilizes $V^{(4)}$ since $V^{(4)}$ admits $SL(2,0)$-action and $\widetilde{\mathbf{B}}$-action. It follows that we have

$$X_\Psi(3) = G(3) \times^{P(3)} \mathbb{P}(V^{(4)}) \cong G(3)([\mathbf{v}_{1110}] \times \mathbb{P}(V^{(4)})) \subset \mathbb{P}(L(\Lambda_3)) \times \mathbb{P}(L(\Lambda_4)).$$

This is a $\mathbb{P}(V^{(4)})$-fibration over $\mathbb{P}(V^{(3)})$. To describe the $G(3)$-orbit $G(3)\mathbb{P}(V^{(4)})$, we must extend $V^{(4)}$ to include three additional vectors of degrees -1 and -2: namely, $\mathbf{v}_{2101}, \mathbf{v}_{1201}$ (degree -1), and $\mathbf{v}_{2200}$ (degree -2). Note that they are also labelled by T-weights, which uniquely determine the corresponding weight vectors in each graded component of $L(\Lambda_4)$. Define the extended space

$$W^{(4)} := V^{(4)} \oplus \mathbb{C}\mathbf{v}_{2101} \oplus \mathbb{C}\mathbf{v}_{1201} \oplus \mathbb{C}\mathbf{v}_{2200},$$

which is stable by the actions of $SL(2,3), SL(2,0)$, and $\widetilde{\mathbf{B}}$. We have

$$\begin{aligned} X_\Psi(3) = &\{ \begin{pmatrix} x^{(3)}_{1110} \\ x^{(3)}_{1101} \\ \color{red}{x^{(3)}_{2100}} \end{pmatrix} // \begin{pmatrix} \color{red}{x^{(4)}_{1210}} \\ \color{red}{x^{(4)}_{1201}} \\ \color{blue}{x^{(4)}_{2200}} \end{pmatrix}, x^{(3)}_{1110}\color{red}{x^{(4)}_{2101}} + x^{(3)}_{1101}\color{red}{x^{(4)}_{2110}} + \color{red}{x^{(3)}_{2100}}x^{(4)}_{1111} = 0\} \\ &\subset \mathbb{P}(V^{(3)}) \times \mathbb{P}(W^{(4)}) \cong \mathbb{P}^2 \times \mathbb{P}^5, \end{aligned}$$

where $x^{(3)}_\bullet$ and $x^{(4)}_\bullet$ are coefficients of the vectors $\mathbf{v}^{(3)}_\bullet \in V^{(3)}$ and $\mathbf{v}^{(4)}_\bullet \in W^{(4)}$ and the coloring pattern indicates degree 0 (black), 1 (red), and 2 (blue), respectively. The degree 0-part of $X_\Psi(3)$ is

$$\mathbb{P}^1 \cong \mathbb{P}(\mathbb{C}\mathbf{v}_{1110} \oplus \mathbb{C}\mathbf{v}_{1101}) \times \mathbb{P}(\mathbb{C}\mathbf{v}_{1111}) \subset \mathbb{P}(V^{(3)}) \times \mathbb{P}(W^{(4)})$$

since $\mathbb{P}(\mathbb{C}\mathbf{v}_{1111}) = \{\mathrm{pt}\}$. We consider a (locally closed) neighbourhood $\mathcal{U}^-$ of the $\widetilde{\mathbf{B}}$-fixed point $([\mathbf{v}_{1110}], [\mathbf{v}_{1111}]) \in \mathbb{P}^1$ along the negative degree direction. It is described by setting

$$x^{(3)}_{1101} = 0, x^{(3)}_{1110} = 1 = x^{(4)}_{1111},$$

where the parameters $x^{(4)}_{2110}, x^{(4)}_{1210}, x^{(4)}_{2101}$ can move freely. The variables $x^{(4)}_{2110}, x^{(4)}_{1210}, x^{(4)}_{2101}$ have degree 1 with their T-characters

$$-\epsilon_1 + \epsilon_4,\ -\epsilon_2 + \epsilon_4,\ -\epsilon_1 + \epsilon_3,$$

and we have a B-equivariant (degree preserving) identification

$$\mathcal{U}^- = \exp(\mathbb{C}E_{14}z^{-1} + \mathbb{C}E_{24}z^{-1} + \mathbb{C}E_{13}z^{-1})([\mathbf{v}_{1110}], [\mathbf{v}_{1111}]) \subset \mathbb{P}(V^{(3)}) \times \mathbb{P}(W^{(4)}).$$

Now $X_\Psi(3)$ is $SL(2,3)$-stable, and we have

$$T^*_\Psi X \cong G \times^B \mathcal{U}^- \subset G \times^{P_3} X_\Psi(3) \cong \mathfrak{X}_\Psi(w_0). \tag{3.9}$$

Note that $x^{(4)}_{1111} \neq 0$ automatically implies $(x^{(3)}_{1110}, x^{(3)}_{1101}) \neq (0,0)$ from

$$x^{(3)}_{1110}x^{(4)}_{2101} + x^{(3)}_{1101}x^{(4)}_{2110} + x^{(3)}_{2100}x^{(4)}_{1111} = 0,$$

which means $(\mathfrak{X}_\Psi(w_0) \setminus T^*_\Psi X) = (x^{(4)}_{1111})$.

4 Properties of the variety $\mathfrak{X}_\Psi(w)$

Keep the setting of the previous section.

Theorem 4.1. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $w \in \mathfrak{S}_n$ be Ψ-tame. For each $\lambda \in \mathtt{Par}$, we have:*

1. *$H^{>0}(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) = 0$;*
2. *$H^0(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda))^* \cong N^\Psi_w(\lambda)$ as $\widetilde{\mathbf{B}}$-modules;*
3. *the module $N^\Psi_w(\lambda)$ admits a $D^{(\lambda_1)}$-filtration, and $H^0(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda))$ admits an excellent filtration when regarded as a B-module in the sense of van der Kallen [36].*

Proof. We replace w with $w(w^\Psi_0)^{-1}$ by Lemma 2.12 to achieve

$$\ell(ww^\Psi_0) = \ell(w) + \ell(w^\Psi_0)$$

without altering the module $N^\Psi_w(\lambda)$ ($\lambda \in \mathtt{Par}$). Let $\mathbf{i}$ be the sequence in $\mathtt{I}_{\mathrm{af}}$ extracted from the definition of $N^\Psi_w(\lambda)$ ($\lambda \in \mathtt{Par}$) by fixing a reduced expression of w. By construction, the sequence $\mathbf{i}$ has length $\ell = \dim \mathfrak{X}_\Psi(w)$, as follows from Lemma 2.12 and the dimension formula (3.6).

By construction, we have a surjective $\widetilde{\mathbf{B}}$-equivariant morphism

$$\pi : X(\mathbf{i}) \longrightarrow \mathfrak{X}_\Psi(w)$$

of varieties. We have $\pi_* \mathcal{O}_{X(\mathbf{i})} = \mathcal{O}_{\mathfrak{X}_\Psi(w)}$ by Lemma 3.1.

Suppose, for contradiction, that $\mathbb{R}^k \pi_* \mathcal{O}_{X(\mathbf{i})} \neq 0$ for some $k > 0$. Then, we have

$$H^0(\mathfrak{X}_\Psi(w), (\mathbb{R}^k \pi_* \mathcal{O}_{X(\mathbf{i})}) \otimes_{\mathcal{O}_{\mathfrak{X}_\Psi(w)}} \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) \neq 0 \quad \lambda \gg 0.$$

For each $k' > 0$, Serre's vanishing theorem implies that

$$H^{k'}(\mathfrak{X}_\Psi(w), (\mathbb{R}^k \pi_* \mathcal{O}_{X(\mathbf{i})}) \otimes_{\mathcal{O}_{\mathfrak{X}_\Psi(w)}} \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) = 0 \quad \lambda \gg 0.$$

By the degeneration of the Leray spectral sequence

$$H^r(\mathfrak{X}_\Psi(w), (\mathbb{R}^p \pi_* \mathcal{O}_{X(\mathbf{i})}) \otimes_{\mathcal{O}_{\mathfrak{X}_\Psi(w)}} \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) \Rightarrow H^{p+r}(X(\mathbf{i}), \pi^* \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda))$$

at the E_1-stage, this implies

$$H^k(X(\mathbf{i}), \pi^* \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) \neq 0,$$

for sufficiently large λ, contradicting Proposition 2.2. Therefore, we must have $\mathbb{R}^{>0}\pi_* \mathcal{O}_{X(\mathbf{i})} = 0$.

By Proposition 2.2 and (2.2), we find

$$H^k(\mathfrak{X}_\Psi(w), \mathcal{O}_{X_\Psi(w)}(\lambda))^* \cong \begin{cases} N_w^\Psi(\lambda) & (k=0) \\ 0 & (k>0) \end{cases}.$$

for each $\lambda \in \mathtt{Par}$. This proves the first two assertions. The $\widetilde{\mathbf{B}}$-module $N_w^\Psi(\lambda)$ admits a $D^{(\lambda_1)}$-filtration by applying Corollary 1.18 repeatedly to the definition of $N_w^\Psi(\lambda)$. Taking into account the fact that $D_\lambda^{(k)}$ admits a $D^{(k+1)}$-filtration for each $k > 0$ (Theorem 1.17) and $D_\mu^{(k')}$ ($\mu \in \mathsf{P}$) is a Demazure module of G for $k' \gg 0$ (that can be read off from [16, 3.5] and [12, Theorem 1], cf. [19, Theorem B]). This completes the proof of the third assertion, and thus of the theorem. □

Corollary 4.2. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $w \in \mathfrak{S}_n$ be Ψ-tame. For each $\lambda \in \mathtt{Par}$, we have*

$$\mathrm{gch}\, H^0(X_\Psi(w), \mathcal{O}_{X_\Psi(w)}(\lambda))^* = \big[H(\Psi;\lambda;w)\big]_{q\mapsto q^{-1}}.$$

Proof. Combine Theorem 4.1 with Theorem 2.3. □

For each $i \in \mathtt{I}$, we have an embedding $V(\varpi_i) \subset L(\Lambda_i)$ of $\widetilde{\mathbf{G}}$-modules, that can be also understood to be the $\mathbb{G}_m$-fixed part of $L(\Lambda_i)$. We also have a $\widetilde{\mathbf{G}}$-module embedding $\mathbb{C} \cong V(0) \subset L(\Lambda_0)$. These induce a $\widetilde{\mathbf{G}}$-equivariant closed embedding

$$\prod_{i\in\mathtt{I}} \mathbb{P}(V(\varpi_i)) \hookrightarrow \prod_{i\in\mathtt{I}_{\mathrm{af}}} \mathbb{P}(L(\Lambda_i)).$$

Note that $L(\Lambda_i)$ is concentrated in degrees ≤ 0, so that the $\mathbb{G}_m$-action given by $t \mapsto \infty$ contracts general points of $\mathbb{P}(L(\Lambda_i))$ to the fixed point locus $\mathbb{P}(V(\varpi_i))$.

Lemma 4.3. *For a root ideal $\Psi \subset \Delta^+$ and a Ψ-tame $w \in \mathfrak{S}_n$, the intersection*

$$\mathfrak{X}_\Psi(w) \cap \prod_{j\in\mathtt{I}} \mathbb{P}(V(\varpi_i))$$

is isomorphic to the Schubert variety $X(w)$.

Proof. By the construction of $\mathfrak{X}_\Psi(w_0^\Psi)$ in the proof of Theorem 3.9, we find that the image of the composition map

$$f_j : \mathfrak{X}_\Psi(w_0^\Psi) \hookrightarrow \prod_{i=1}^n \mathbb{P}(L(\Lambda_i)) \longrightarrow \mathbb{P}(L(\Lambda_j)) \qquad 1 \le j \le n$$

satisfies $(\mathrm{Im}\, f_j \cap \mathbb{P}(V(\varpi_j))) = [\mathbf{v}_{\varpi_j}]$ for $1 \le j \le d_1(\Psi)$. We set

$$f_{\le k} := \prod_{j=e_1(\Psi)}^k f_j : X_\Psi(w_0^\Psi) \longrightarrow \prod_{i=e_1(\Psi)}^k \mathbb{P}(L(\Lambda_i))$$

for $e_1(\Psi) \le k \le n$.

We set $K := \langle SL(2,i) \mid e_1(\Psi) \le i < n \rangle \subset G$. By inspection, $K \cong SL(n - d_1(\Psi))$ is the largest semisimple algebraic subgroup of $\mathsf{Stab}_G(\{[\mathbf{v}_{\Lambda_j}]\}_{j=1}^{d_1(\Psi)})$ stable under the adjoint $\widetilde{T}$-action. We prove

$(\spadesuit)_k$ $K\{[\mathbf{v}_{\varpi_j}]\}_{j=e_1(\Psi)}^k = \left(\operatorname{Im} f_{\le k} \cap \prod_{j=e_1(\Psi)}^k \mathbb{P}(V(\varpi_j))\right)$

for $k \ge e_1(\Psi)$ by induction from the case $k = e_1(\Psi)$. The assertion $(\spadesuit)_{e_1(\Psi)}$ follows as the image is the projective space homogeneous under the action of $G(k)$, and its $\mathbb{G}_m$-attracting fixed points are homogeneous under the action of

$$G(e_1(\Psi)) \cap K \cong SL(n - d_1(\Psi)).$$

We assume $(\spadesuit)_k$, and we replace K with a smaller group

$$K(k+1) := K \cap G(e_1(\Psi)) \cap \cdots \cap G(k+1) = \langle SL(2,i) \mid k < i < n \rangle \subset G,$$

which acts on the fiber F of the map $\operatorname{Im} f_{\le k+1} \to \operatorname{Im} f_{\le k}$ along $\{[\mathbf{v}_{\Lambda_j}]\}_{j=e_1(\Psi)}^k \in \operatorname{Im} f_{\le k}$. Since F is $G(k+1)$-homogeneous by Proposition 3.7, its $\mathbb{G}_m$-attracting fixed point is homogeneous under the action of $K(k+1)$, that is isomorphic to $SL(n-k-1)$. This implies $(\spadesuit)_{k+1}$, and the induction proceeds.

Therefore, we have $(\spadesuit)_k$ holds for each $k \ge e_1(\Psi)$. In particular, we have

$$\mathfrak{X}_\Psi(w_0^\Psi) \cap \prod_{i \in \mathtt{I}} \mathbb{P}(V(\varpi_i)) = X(w_0^\Psi).$$

We have $\mathfrak{X}_\Psi(s_i w) = (\widetilde{\mathbf{P}}_i \cap G)\mathfrak{X}_\Psi(w)$ when $w \in \mathfrak{S}_n$ is Ψ-tame and $s_i w > w$ by (3.8). Since $(\widetilde{\mathbf{P}}_i \cap G)$ preserves $\prod_{j \in \mathtt{I}} \mathbb{P}(V(\varpi_j))$, we conclude

$$\mathfrak{X}_\Psi(s_i w) \cap \prod_{j \in \mathtt{I}} \mathbb{P}(V(\varpi_j)) = (\widetilde{\mathbf{P}}_i \cap G)(\mathfrak{X}_\Psi(w) \cap \prod_{j \in \mathtt{I}} \mathbb{P}(V(\varpi_j)))$$

in this case. Therefore, the assertion follows by induction on the length of w. $\square$

Theorem 4.4. *For a root ideal $\Psi \subset \Delta^+$, the $\mathbb{G}_m$-attracting set of $X = X(w_0) \subset \mathfrak{X}_\Psi(w_0)$ is open dense, and is isomorphic to $T^*_\Psi X$.*

Proof. Since $\mathfrak{X}_\Psi(w_0)$ is a connected smooth variety and $X \subset \mathfrak{X}_\Psi(w_0)$ is a connected component of its $\mathbb{G}_m$-fixed part, we find that the attracting locus $\mathring{X}_\Psi \subset \mathfrak{X}_\Psi(w_0)$ is identified with the intersection of the product of the attracting loci of the ambient spaces $\mathbb{P}(V(\varpi_i)) \subset \mathbb{P}(L(\Lambda_i))$ ($i \in \mathtt{I}_{\mathrm{af}}$) and the image of $\mathfrak{X}_\Psi(w_0)$ under the embedding (3.5). In particular, $\mathring{X}_\Psi$ is a Zariski open subset of $\mathfrak{X}_\Psi(w_0)$.

By Białnyki-Birula's theorem [1], we see that $\mathring{X}_\Psi$ is an affine bundle over X, that admits an action of $(G \times \mathbb{G}_m)$. By $X \cong G/B$, we take a base point $p = X(e) = B/B$. We have a direct sum decomposition

$$T_p X_\Psi(w_0) \cong T_p X \oplus E, \tag{4.1}$$

where $T_p X$ admits trivial $\mathbb{G}_m$-action and E has strictly negative $\mathbb{G}_m$-degree. Note that each direct summand of (4.1) is B-stable. In view of the fiber bundle structure of $\mathfrak{X}_\Psi(w_0)$, the $\widetilde{T}$-character of E is calculated from the tangent spaces of the projective spaces

$$G(k)/(G(k) \cap \widetilde{\mathbf{P}}(k+1)) \cong \mathbb{P}(\overline{G(k)\mathbf{v}_{\Lambda_k}}) \subset \mathbb{P}(L(\Lambda_k)) \quad e_1(\Psi) \le k \le n \tag{4.2}$$

borrowed from Proposition 3.7. Observe that

$$\overline{G(k)\mathbf{v}_{\Lambda_k}} \subset L(\Lambda_k)$$

defines a vector subspace, that is in fact a vector representation of $G(k) \cong SL(\mathtt{h}_k + n - k + 1)$. It follows that

$$T_{[\mathbf{v}_{\Lambda_k}]}\mathbb{P}(\overline{G(k)\mathbf{v}_{\Lambda_k}}) \cong \big(\bigoplus_{k<s\le n} \mathbb{C}_{\epsilon_s-\epsilon_k}\big) \oplus \big(\bigoplus_{1\le t<\mathtt{h}_k(\Psi)} \mathbb{C}_{\epsilon_t-\epsilon_k-\delta}\big) \tag{4.3}$$

by inspection. Let us denote by Π_k the set of $\widetilde{T}$-weights appearing in (4.3). We find that the $\widetilde{T}$-character contribution of E is precisely the $\widetilde{T}$-weights of $\mathfrak{n}(\Psi) \otimes \mathbb{C}_{-\delta}$ counted with multiplicities.

For each $\beta \in \Pi_k$ $(e_1(\Psi) \le k \le n)$, we have a $\widetilde{T}$-stable connected one-dimensional unipotent subgroup $U_\beta \subset \widetilde{G}(\!(z)\!)$ such that $\mathbb{C}_\beta \cong \mathrm{Lie}\, U_\beta$ as $\widetilde{T}$-modules. Since $U_\beta \subset G(k)$ for $\beta \in \Pi_k$, it preserves $X_\Psi(k)$. We have

$$U_\beta \mathbf{v}_{\Lambda_k} = \mathbf{v}_{\Lambda_k} \qquad \beta \in \Pi_{k'} \quad \text{for} \quad k' > k$$

since the $\widetilde{T}$-weight $(\beta + \Lambda_k)$-part of $L(\Lambda_k)$ is zero by inspection (e.g. $s_0\Lambda_k = \Lambda_k$ for $1 \le k < n$). Thus, we can apply U_γ's ($\gamma \in \Pi_{k'}$) to $p = \{[\mathbf{v}_{\Lambda_i}]\}_{i\in\mathtt{I}}$ from the case of $k' = n$ and then lowering k' consecutively to obtain a well-defined action map

$$\mathbb{A}^{\dim\, X_\Psi(k)} \cong \big(\prod_{\beta\in\Pi_{k+1}} U_\beta\big)\big(\prod_{\beta\in\Pi_{k+2}} U_\beta\big)\cdots\big(\prod_{\beta\in\Pi_n} U_\beta\big)p \hookrightarrow X_\Psi(k), \tag{4.4}$$

for each $d_1(\Psi) \le k < n$. By (4.3) and Proposition 3.7, we have

$$\sum_{k\le k'\le n} |\Pi_{k'}| = \dim\, X_\Psi(k).$$

Therefore, the image of (4.4) is an open subset of $X_\Psi(k)$. Gathering these gives rise to a $\widetilde{T}$-equivariant surjection

$$\mathrm{Lie}\, N^- \oplus (\mathfrak{n}(\Psi) \otimes \mathbb{C}_{-\delta}) \longrightarrow\!\!\!\!\!\rightarrow T_p\mathfrak{X}_\Psi(w_0)$$

by Lemma 1.8. The Lie algebras of the unipotent groups in the middle term of (4.4) generate a Lie subalgebra of $\mathfrak{g} \otimes \mathbb{C}[z^{-1}]$ that contains

$$\mathfrak{n}(\Psi) \otimes \mathbb{C}_{-\delta} \cong \mathfrak{n}(\Psi) \otimes \mathbb{C}z^{-1} \mod \mathfrak{g} \otimes z^{-2}\mathbb{C}[z^{-1}],$$

and any $\widetilde{T}$-module map $\mathfrak{g} \otimes z^{-2}\mathbb{C}[z^{-1}] \to T_p\mathfrak{X}_\Psi(w_0)$ is zero due to weight considerations. From this, we see that B acts on $(\mathfrak{n}(\Psi)\otimes\mathbb{C}_{-\delta})$ viewed as a subspace of

$$\mathfrak{g} \otimes \mathbb{C}z^{-1} \cong \mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}] \mod \mathfrak{g} \otimes z^{-2}\mathbb{C}[z^{-1}].$$

Thus, we have necessarily $E \cong \mathfrak{n}(\Psi) \otimes \mathbb{C}_{-\delta}$ as B-modules. Therefore, we conclude that

$$\mathring{X}_\Psi \cong G \times^B (\mathfrak{n}(\Psi) \otimes \mathbb{C}_{-\delta}) = T^*_\Psi X$$

as required. $\square$

Corollary 4.5 (Corollary of the proof of Theorem 4.4)**.** *Keep the setting of Theorem 4.4. The fiber of $T^*_\Psi X$ as a vector bundle on X injects into $\mathbb{P}(L(\Lambda_0))$ through the projection from the RHS of* (3.5)*.* $\square$

By the comparison with Lusztig [25], we find:

Corollary 4.6 (Ngô [31] and Mirković-Vybornov [28])**.** *The composition map*

$$\mathfrak{X}_{\Delta^+}(w_0) \hookrightarrow \prod_{i \in \mathtt{I}_{\mathrm{af}}} \mathbb{P}(L(\Lambda_i)) \to \mathbb{P}(L(\Lambda_0))$$

defines a resolution of a compactification of the nilpotent cone of $\mathfrak{gl}(n, \mathbb{C})$ *realized in the affine Grassmannian of* G. □

For each $\lambda \in \mathtt{Comp}$, let $\mathcal{O}_{T^*_\Psi X}(\lambda)$ be the restriction of $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ through Theorem 4.4.

Corollary 4.7. *For each* $\lambda \in \mathsf{P}$*, the restriction of* $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ *to* $T^*_\Psi X$ *is isomorphic to* $\pi^*_\Psi \mathcal{O}_X(\lambda)$.

Proof. Since both line bundles are G-homogeneous, it suffices to compare their restrictions to the fiber of $T^*_\Psi X$ over $B/B \in X$ as $\widetilde{B}$-equivariant line bundles. As a $\widetilde{B}$-equivariant line bundle on the affine space $\mathfrak{n}(\Psi)$ is completely determined by the character at the $\widetilde{T}$-fixed point, we conclude the result by the character comparison. □

Let us record the nef cone (see [24, Definition 1.4.1] for definition) of $\mathfrak{X}_\Psi(w_0)$:

Corollary 4.8. *For any nonempty root ideal* $\Psi \subset \Delta^+$*, we have* $\operatorname{Pic} \mathfrak{X}_\Psi(w_0) \cong \mathsf{P}$. *For each* $\lambda \in \mathsf{P}$*, the line bundle* $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ *is nef if and only if* $\lambda \in \mathtt{Par}$.

Proof. Proposition 3.7 and Theorem 3.9 imply that $\mathfrak{X}_\Psi(w_0)$ is a n-times successive projective space fibration realized as the projectifications of vector bundles. Here each $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\varpi_i)$ $(1 \le i \le n)$ yields a primitive ample line bundle of the fiber at the i-th step. Hence, we conclude that $\mathsf{P} \cong \operatorname{Pic} \mathfrak{X}_\Psi(w_0)$ by repeatedly applying [15, II Ex. 7.9].

The restriction of $\mathcal{O}_{X_\Psi(w_0)}(\lambda)$ $(\lambda \in \mathsf{P})$ to X is $\mathcal{O}_X(\lambda)$, and hence it is nef if and only if $\lambda_i - \lambda_{i+1} \ge 0$ for each $1 \le i < n$. Consider the subspace

$$Y := \mathbb{P}^{\mathtt{h}_n(\Psi)} \cong \mathbb{P}(\mathbb{C}\mathbf{v}_{\Lambda_0} \oplus \bigoplus_{t=1}^{\mathtt{h}_n(\Psi)} \mathbb{C}(z^{-1}E_{t,n})\mathbf{v}_{\Lambda_0}) \subset \overline{\mathfrak{n}(\Psi)} \subset \mathbb{P}(L(\Lambda_0))$$

arising from the fiber direction of $T^*_\Psi X$ at B/B (cf. proof of Theorem 4.4). Then, the restriction of $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ to Y is $\mathcal{O}(\lambda_n)$ as $m_n(\lambda) = \lambda_n$ by the construction of $N^\Psi_{w_0}(\lambda)$ in (2.2). Thus, the restriction of $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ to Y is nef only if $\lambda_n \ge 0$. Hence, $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ is nef only if $\lambda \in \mathtt{Par}$.

For each $1 \le i \le n$, the embedding (3.5) implies $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\varpi_i)$ is nef. Thus, $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)$ is nef if $\lambda \in \mathtt{Par}$ as required. □

Corollary 4.9. *For a root ideal* $\Psi \subset \Delta^+$ *and* $w \in \mathfrak{S}_n$ *that is* Ψ*-tame, the* $\mathbb{G}_m$*-attracting set of* $X(w) \subset \mathfrak{X}_\Psi(w)$ *is isomorphic to* $T^*_\Psi X(w)$.

Proof. Since $\mathfrak{X}_\Psi(w)$ is the restriction of (3.5) to $\overline{BwQ/Q} \subset G/Q$, the claim follows from Theorem 4.4. □

5 Consequences

Keep the setting of the previous section.

5.1 Vanishing theorems

Theorem 5.1. *Let $\Psi \subset \Delta^+$ be a root ideal. Then the line bundle $\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\varpi_n)$ defines an effective Cartier divisor D with*

$$\operatorname{supp} D = \mathfrak{X}_\Psi(w_0) \setminus T^*_\Psi X.$$

Moreover, we have

$$H^i(T^*_\Psi X(w), \mathcal{O}_{T^*_\Psi X(w)}(\lambda)) = \varinjlim_m H^i(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda + m\varpi_n)) \otimes \mathbb{C}_{m\varpi_n} \quad (5.1)$$

for each $i \in \mathbb{Z}$ and $\lambda \in \mathtt{Par}$ whenever w is Ψ-tame. In this case, we have

$$H^{>0}(T^*_\Psi X(w), \mathcal{O}_{T^*_\Psi X(w)}(\lambda)) = 0. \quad (5.2)$$

Proof. Let D be the divisor defined by the vanishing of the coordinate corresponding to $\mathbf{v}_{\Lambda_0}$ under the embedding

$$\mathfrak{X}_\Psi(w_0) \hookrightarrow \mathbb{P}(L(\Lambda_0)).$$

The divisor D is $\widetilde{T}$-stable and $T^*_\Psi X$ lies in the complement of the vanishing locus by the description of $\mathfrak{X}_\Psi(w_0)$ near the $\widetilde{T}$-fixed point $\{[\mathbf{v}_{\Lambda_i}]\}_{i \in \mathtt{I}_{\mathrm{af}}}$ in Theorem 4.4. Hence, we obtain

$$\operatorname{supp} D \cap T^*_\Psi X = \emptyset.$$

By Corollary 4.5, the boundary of the closure of a fiber of $T^*_\Psi X$ is contained in $\operatorname{supp} D$. Taking the G-action into account, we conclude

$$\operatorname{supp} D = \mathfrak{X}_\Psi(w_0) \setminus T^*_\Psi X,$$

that is the first assertion. It follows that the embedding $T^*_\Psi X \subset \mathfrak{X}_\Psi(w_0)$ is affine. Consequently, so is $T^*_\Psi X(w) \subset \mathfrak{X}_\Psi(w)$ by (3.5) and Corollary 4.9. In particular, we have

$$H^i(T^*_\Psi X(w), \pi^*_\Psi \mathcal{O}_{X(w)}(\lambda)) = H^i(\mathfrak{X}_\Psi(w), \jmath_* \pi^*_\Psi \mathcal{O}_{X(w)}(\lambda)) \qquad i \in \mathbb{Z},$$

where $\jmath : T^*_\Psi X(w) \hookrightarrow \mathfrak{X}_\Psi(w)$ is an inclusion. We have

$$\jmath_* \pi^*_\Psi \mathcal{O}_{X(w)} = \varinjlim_m \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda + m\varpi_n) \otimes \mathbb{C}_{m\varpi_n},$$

where the transition maps in the RHS are induced by multiplication by a homogeneous coordinate function that extracts the coefficient of $\mathbf{v}_{\Lambda_0}$ from the global sections of $\mathcal{O}(\varpi_n)$. From these, we conclude that

$$H^i(T^*_\Psi X(w), \pi^*_\Psi \mathcal{O}_{X(w)}(\lambda)) = \varinjlim_m H^i(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda + m\varpi_n)) \otimes \mathbb{C}_{m\varpi_n}$$

for each $i \in \mathbb{Z}$ by the commutation of the cohomology with direct limits ([15, III, Proposition 2.9]). This is the second assertion.

We combine the second assertion with Theorem 4.1 to conclude the third assertion. These complete the proof. $\square$

Remark 5.2. The vanishing result (5.2) establishes the tame case of the vanishing conjecture proposed by BlasiakMorsePun [2, Conjecture 3.4(ii)], which implies the vanishing conjectures of ChenHaiman [8, Conjecture 5.4.3(2)] and ShimozonoWeyman [35, Conjecture 5]. The result was previously known for strictly dominant λ [33, 27], or in certain special cases [4, 5, 13]. However, these earlier results do not fully encompass situations where $H(\Psi;\lambda;w_0)$ is a k-Schur polynomial [3], or where $\mathfrak{n}(\Psi)$ arises as the Lie algebra of the unipotent radical of a proper parabolic subgroup of G.

Corollary 5.3 (Conjecture 3.4 (iii) in Blasiak-Morse-Pun [2])**.** *Let $\Psi \subset \Delta^+$ be a root ideal, and let $w \in \mathfrak{S}_n$ be Ψ-tame. Then, the space*

$$H^0(T^*_\Psi X(w), \mathcal{O}_{T^*_\Psi X(w)}(\lambda)) \qquad \lambda \in \mathtt{Par}$$

admits an excellent filtration in the sense of van der Kallen [36].

Proof. By Theorem 4.1(3), each G-module appearing in the direct system on the right-hand side of (5.1) admits an excellent filtration in the sense of [36]. By [36, Corollary 1.8], the inductive limit of modules that admit excellent filtrations again admits an excellent filtration. ☐

Corollary 5.4 (Parabolic vanishing)**.** *Let $P \subset G$ be a parabolic subgroup that contains B, and let Ψ be a root idealsuch that $\mathfrak{n}(\Psi)$ is P-stable. We set*

$$T^*_\Psi X^P := G \times^P \mathfrak{n}(\Psi) \xrightarrow{\pi^P_\Psi} G/P =: X^P.$$

For $\lambda \in \mathtt{Par}$ such that $\langle \alpha_i, \lambda \rangle = 0$ when $P_i \subset P$, the line bundle $\mathcal{O}_X(\lambda)$ is isomorphic to the pullback of a line bundle $\mathcal{O}_{X^P}(\lambda)$ on X^P and we have

$$H^{>0}(T^*_\Psi X^P, (\pi^P_\Psi)^* \mathcal{O}_{X^P}(\lambda)) = 0.$$

Proof. Since P stabilizes $\mathfrak{n}(\Psi)$, we have a natural P/B-fibration

$$\eta : T^*_\Psi X = G \times^B \mathfrak{n}(\Psi) \longrightarrow G \times^P \mathfrak{n}(\Psi) = T^*_\Psi X^P.$$

The condition on λ guarantees that λ descends to a character of P, and hence $\mathcal{O}_X(\lambda)$ is a pullback of the G-equivariant line bundle $\mathcal{O}_{X^P}(\lambda)$ on G/P. In particular, $(\pi_\Psi)^*\mathcal{O}_X(\lambda)$ is trivial along the fiber of η. Since we have $H^i(P/B, \mathcal{O}_{P/B}) = \mathbb{C}^{\delta_{i,0}}$ by the Borel-Weil-Bott, the Leray spectral sequence

$$E_2^{q,p} := H^q(T^*_\Psi X^P, \mathbb{R}^p \eta_* (\pi_\Psi)^* \mathcal{O}_X(\lambda)) \Rightarrow H^{q+p}(T^*_\Psi X, (\pi_\Psi)^* \mathcal{O}_X(\lambda))$$

degenerates at the E_1-stage, as $E_1^{q,p} = 0$ for $p > 0$. This yields

$$E_2^{q,p} \cong H^q(T^*_\Psi X^P, \eta_*(\pi_\Psi)^* \mathcal{O}_X(\lambda)) = H^q(T^*_\Psi X^P, (\pi^P_\Psi)^* \mathcal{O}_{X^P}(\lambda)).$$

Therefore, (5.2) for $w = w_0$ implies the result. ☐

Remark 5.5. **(1)** Corollary 5.4 also admits a B-equivariant analogue, in the same sense as (5.2). **(2)** The results in §1.4 are valid in all characteristics. Those in §1.5 also hold in positive characteristic [17], except in the case $n = 2$, where the corresponding affine Lie algebra is not simply-laced. Therefore, all the results in §3, as well as Theorem 5.1 and Corollary 5.4, remain valid in arbitrary characteristic when $n \geq 3$. The exceptional case $n = 2$ in positive characteristicwhere the associated affine Lie algebra is not simply-lacedcan be treated separately by elementary methods and is left to the reader.

5.2 Simple head property

Lemma 5.6. *For a root ideal $\Psi \subset \Delta^+$, we have a natural infinitesimal action of $\mathfrak{gl}(n, \mathbb{C}[z])$ on $T^*_\Psi X$. This equips $H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda))^\vee$ ($\lambda \in \mathtt{Par}$) with the structure of a graded $\mathfrak{gl}(n, \mathbb{C}[z])$-module, and makes the map*

$$H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda))^\vee \longrightarrow\!\!\!\!\!\rightarrow H^0(X_\Psi(w_0), \mathcal{O}_{X_\Psi(w_0)}(\lambda))^*$$

into a graded $\mathfrak{gl}(n, \mathbb{C}[z])$-module surjection.

Proof. The action of $\widetilde{\mathbf{G}}$ on $X_\Psi(w_0)$ differentiates into the action of its Lie algebra, and this Lie algebra action is well-defined on an open subset $T^*_\Psi X$. Thus, we obtain the desired action and the quotient map. □

Remark 5.7. We caution that the $\mathfrak{gl}(n, \mathbb{C}[z])$-action on

$$H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\lambda)) \qquad \lambda \in \mathtt{Par}$$

is generally *not* compatible with the identification (5.1). This is parallel to the fact that the $\mathfrak{g}$-module map

$$H^0(X, \mathcal{O}_X(\lambda)) \hookrightarrow H^0(w_0 B w_0 B/B, \mathcal{O}_X(\lambda)) \qquad \lambda \in \mathtt{Par},$$

is not compatible with the character twists as $\mathfrak{b}$-modules.

Theorem 5.8. *For a root ideal $\Psi \subset \Delta^+$, the $\widetilde{\mathbf{G}}$-module*

$$H^0(\mathfrak{X}_\Psi(w_0), \mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)) \qquad \lambda \in \mathtt{Par}$$

has a simple head isomorphic to $H^0(X, \mathcal{O}_X(\lambda))$.

Since the proof of Theorem 5.8 is rather involved, we postpone its proof to §5.3 and discuss one consequence here:

Corollary 5.9. *Let $\Psi \subset \Delta^+$ be a root ideal, and let $w \in \mathfrak{S}_n$ be Ψ-tame. For each $\lambda \in \mathtt{Par}$, the $\widetilde{\mathbf{B}}$-module $H^0(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda))$ has a simple head.*

Proof. We employ the setting of the proof of Theorem 5.8. A reduced expression $\mathbf{i}'$ of w can be extended to a reduced expression $\mathbf{i}$ of w_0 by prepending additional simple reflections from $\mathtt{I}$. By repeated applications of Corollary 1.18 to the presentations in (2.2), we find a surjective $\widetilde{\mathbf{B}}$-module map

$$H^0(\mathfrak{X}_\Psi(w_0), \mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)) \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)). \tag{5.3}$$

By Theorem 5.8 and the PBW theorem, we find that the LHS of (5.3) has a simple head as a $\widetilde{\mathbf{B}}$-module (the lowest weight part of V^*_λ). Thus, so is the RHS of (5.3) as required. □

5.3 Proof of Theorem 5.8

The proof is divided into two steps: first, we enlarge the space $H^0(\mathfrak{X}_\Psi(w_0), \mathcal{O}(\lambda))$ via twisting by ϖ_n (§5.3.1); second, we analyze the limiting behavior using the representation theory of affine Lie algebras (§5.3.2). We combine both steps in §5.3.3.

We warn the reader that we use non-standard lifts of weights in P to P_{af} throughout the proof.

5.3.1 The first step

We have a short exact sequence of line bundles

$$0 \to \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda) \to \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda + \varpi_n) \otimes \mathbb{C}_{\varpi_n} \to \mathrm{coker} \to 0,$$

where coker is supported on $\partial = (\mathcal{X}_\Psi(w_0) \setminus T^*_\Psi X)$. This induces the short exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{X}_\Psi(w_0), \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda)) \to H^0(\mathcal{X}_\Psi(w_0), \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n} \\ \to H^0(\mathcal{X}_\Psi(w_0), \mathrm{coker}) \to 0, \end{aligned} \tag{5.4}$$

of graded G-modules by Theorem 4.1. We remark that ∂ is not $\widetilde{\mathbf{G}}$-stable, and hence this short exact sequence cannot be lifted into a short exact sequence of graded $\mathfrak{sl}(n, \mathbb{C}[z])$-modules.

Lemma 5.10. *Let Ψ be a root ideal and let $\lambda \in \mathtt{Par}$. We have a graded $\mathfrak{gl}(n, \mathbb{C}[z])$-module map*

$$\eta(\lambda) : H^0(\mathcal{X}_\Psi(w_0), \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n} \longrightarrow H^0(\mathcal{X}_\Psi(w_0), \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda)).$$

Proof. The effect of tensoring with $\mathbb{C}_{\Lambda_n}$ enlarges the output of the functor $\mathcal{D}_0$. Thus, a $\widetilde{\mathbf{B}}$-module M induces a $\widetilde{\mathbf{B}}$-module inclusion

$$\mathcal{D}_0(M) \hookrightarrow \mathcal{D}_0(M \otimes \mathbb{C}_{\Lambda_n}) \otimes \mathbb{C}_{-\Lambda_n}.$$

Since $\mathcal{D}_0$ is a covariant functor, a $\widetilde{\mathbf{B}}$-module map $M' \longrightarrow M$ yields

$$\mathcal{D}_0(M') \longrightarrow \mathcal{D}_0(M) \longrightarrow \mathcal{D}_0(M \otimes \mathbb{C}_{\Lambda_n}) \otimes \mathbb{C}_{-\Lambda_n}.$$

Therefore, a $\widetilde{\mathbf{B}}$-module map $M' \longrightarrow M$ yields

$$\mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda)(M') \longrightarrow \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + \varpi_n)(M) \otimes \mathbb{C}_{-\Lambda_n}$$

for each $e_1(\Psi) \le k \le n$; see (2.2) for the definition of $\mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda)$.

Now we compose them for $e_1(\Psi) \le k \le n$ to obtain a $\widetilde{\mathbf{B}}$-module map

$$H^0(\mathcal{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathcal{X}_\Psi(w_0^\Psi)}(\lambda))^\vee \longrightarrow H^0(\mathcal{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathcal{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n))^\vee \otimes \mathbb{C}_{-\varpi_n}.$$

Applying $\mathcal{D}_{w_0}$ and dualizing this yields the desired map $\eta(\lambda)$ as required. $\square$

We set $P(k) := (G(k) \cap \widetilde{\mathbf{P}}(k+1))$ for $e_1(\Psi) \le k \le n$ and $P(d_1(\Psi)) := Q$, and set $G(d_1(\Psi)) := G$ for notational simplicity. As seen in §3, $P(k)$ is a maximal parabolic subgroup of $G(k) \cong SL(\mathtt{h}_k + n - k + 1)$ for $e_1(\Psi) \le k \le n$. Let $U(k)$ be the unipotent radical of $P(k) \subset G(k)$ for $d_1(\Psi) \le k \le n$. Let $U^-(k)$ $(d_1(\Psi) \le k \le n)$ denote the opposite unipotent subgroup of $U(k)$ with respect to the respective reductive groups and its parabolic subgroups.

For $e_1(\Psi) \le k \le n$, set $\mathbf{i}_k := (\mathtt{h}_{k-1}(\Psi), \ldots, 0, \ldots, k)$. Let $\mathbf{i}'[k]$ be a sequence of $\mathtt{I}_{\mathrm{af}}$ that records a reduced expression of $w_0 w_0^\Psi$ followed by the sequence read from (2.2) until the tensor twist of $\mathbb{C}_{m_{k-1}\Lambda_k}$. We have a map

$$f : X(\mathbf{i}_k) \longrightarrow G(k)/P(k).$$

For a $P(k)\widetilde{\mathbf{B}}$-module M, we define $\mathcal{M}^\sharp := f_* \mathcal{E}_{\mathbf{i}}(M \otimes \mathbb{C}_{\Lambda_n})$. Let $\mathcal{M}$ be the $G(k)$-equivariant vector bundle on $G(k)/P(k)$ whose fibers at $P(k)/P(k)$ is $M^\vee$. We set $G_k^\flat := \langle SL(2,i) \mid 1 \le i < \mathtt{h}_k(\Psi) \rangle \subset G(k)$ and

$$P_k^\flat := (\langle SL(2,i) \mid 1 < i < \mathtt{h}_k(\Psi) \rangle \, \widetilde{\mathbf{B}} \cap SL(\mathtt{h}_k(\Psi), \mathbb{C}))$$

for each $e_1(\Psi) \le k \le n$.

Let $Y(k)$ to be the pullback of $(G_k^\flat / P_k^\flat)$ to $X_\Psi(k)$, and define

$$\mathring{Y}(k) := U^-(d_1(\Psi)) \times U^-(e_1(\Psi)) \times \cdots \times U^-(k-1) \times Y(k) \subset \mathfrak{X}_\Psi(w_0), \quad (5.5)$$

for each $e_1(\Psi) \le k \le n$. Note that $Y(k) \subset \partial$, and its codimension is

$$\mathrm{codim}_{G(k)/P(k)}(G_k^\flat / P_k^\flat) - 1 = (n-k).$$

Lemma 5.11. *For each $e_1(\Psi) \le k \le n$, we have*

$$Y(k) \subset \overline{U^-(k) Y(k+1)}, \quad (5.6)$$

which in turn implies that $\mathring{Y}(k) \subset \overline{\mathring{Y}(k+1)}$.

Proof. We prove (5.6). The case $k = n$ is trivial, and the case $k = (n-1)$ is $Y(k) \subset \partial$. It suffices to consider the image of $\mathfrak{X}_\Psi(w_0)$ under the map

$$\mathfrak{X}_\Psi(w_0) \longrightarrow \mathbb{P}(L(\Lambda_k)) \times \mathbb{P}(L(\Lambda_{k+1}))$$

induced from (3.2), and examine the inclusion. Let $A := (\mathrm{Id}_{k+1}, a\mathrm{Id}_{n-k-1}) \in G$ be a diagonal matrix with $a \in \mathbb{R}_{>1}$. We have an inclusion of the A-fixed parts of $Y(k) \subset \overline{U^-(k) Y(k+1)}$ by the case $k = (n-1)$ (c.f. Corollary 4.5). We apply the action of $GL(n-k) \subset G$ to obtain (5.6). The latter assertion follows from (5.6) by applying $U^-(l)$ for $d_1(\Psi) \le l < k$. This completes the proof. $\square$

Proposition 5.12. *In the setting of Lemma 5.10, we have*

$$\mathrm{gch}\ \ker \eta(\lambda) \le \mathrm{gch}\, H^0(\partial, \mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n}. \quad (5.7)$$

Proof. We adopt the notation and framework from the proof of Lemma 5.10. We consider

$$F := \mathrm{coker}((\mathcal{D}_0 \circ \cdots \circ \mathcal{D}_k)(M) \hookrightarrow (\mathcal{D}_0(\mathbb{C}_{\Lambda_n} \otimes (\mathcal{D}_{n-1} \circ \cdots \circ \mathcal{D}_k))(M) \otimes \mathbb{C}_{-\Lambda_n}).$$

We observe that F is a $\widetilde{\mathbf{B}}$-module equipped with an action of $P_k^\flat$. If we denote by $\mathcal{F}$ the $G_k^\flat$-equivariant vector bundle on $G_k^\flat / P_k^\flat$ whose fiber at $P_k^\flat / P_k^\flat$ is $F^\vee$, then we have a short exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}^\sharp \to \mathcal{F} \to 0. \quad (5.8)$$

In particular, the space

$$\mathrm{coker}(\mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda)(M) \longrightarrow \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + \varpi_n)(M) \otimes \mathbb{C}_{-\Lambda_n})^\vee$$

is identified with the space of rational sections of $\mathcal{M}$ having a simple pole along $(G_k^\flat / P_k^\flat)$.

In the following, we consider the case

$$M = \Gamma(X_\Psi(k+1), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda)) \subset \Gamma(X_\Psi(k+1), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n}$$

for $e_1(\Psi) \le k \le n$. We have

$$\Gamma(G(k)/P(k), \mathcal{M}^\sharp) \hookrightarrow \Gamma(X_\Psi(k), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n}. \tag{5.9}$$

The image of $\Gamma(G(k)/P(k), \mathcal{M}^\sharp)$ under (5.9) vanishes along general points of

$$U^-(k) \times Y(k+1) \subset \mathfrak{X}_\Psi(w_0).$$

The space $\Gamma(G(k)/P(k), \mathcal{F})$ inflates to a $\widetilde{T}$-equivariant vector bundle $\mathcal{F}(k)$ on $\mathring{Y}(k)$ by adjusting characters $\Lambda_1, \ldots, \Lambda_{k-1}$ appearing in (2.2). A non-zero element of $\Gamma(G(k)/P(k), \mathcal{F})$ is a non-zero along general points of $Y(k)$.

By construction, we have

$$H^0(X(\mathbf{i}'), \mathcal{E}_{\mathbf{i}'}(\Gamma(G(k)/P(k), \mathcal{F})) \otimes_{\mathcal{O}_{X(\mathbf{i}')}} \mathcal{L}) \subset \Gamma(\mathring{Y}(k), \mathcal{F}(k)), \tag{5.10}$$

where $\mathcal{L}$ is a line bundle that adjusts the twists by $m_i \Lambda_i$ $(1 \le i < k)$ that appears in the middle of $X(\mathbf{i}')$. Let f be a non-zero section of the RHS of (5.10). Then, f is non-zero on general points of $\mathring{Y}(k)$, since it specializes to a non-zero element of $\Gamma(G(k)/P(k), \mathcal{F})$ at a general fiber $Y(k)$. On the other hand, the lift of f to

$$\Gamma(U^-(d_1(\Psi)) \times U^-(e_1(\Psi)) \times \cdots \times U^-(k-1) \times X_\Psi(k), \mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n}$$

through (5.9) vanishes at general points of $\mathring{Y}(k+1)$ by construction.

A non-zero section in the LHS of (5.10) for $e_1(\Psi) \le k \le n$ is precisely a section of $H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n}$ that vanishes along $\mathring{Y}(k+1)$, but whose leading term in the local expansion along $\mathring{Y}(k)$ is generically non-zero. Thus, the LHS of (5.10) contributes distinctly to

$$H^0(\partial, \mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n}.$$

This concludes the proof of (5.7). □

Corollary 5.13. *The map $\eta(\lambda)$ in Lemma 5.10 is surjective.*

Proof. By construction, we have

$$\begin{aligned} \operatorname{gch} \ker \eta(\lambda) \ge{} & \operatorname{gch} H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n} \\ & - \operatorname{gch} H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda)), \end{aligned}$$

with an equality when the map $\eta(\lambda)$ is surjective. Since (5.4) implies

$$\begin{aligned} \operatorname{gch} H^0(\partial, \mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n} ={} & \operatorname{gch} H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n} \\ & - \operatorname{gch} H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda)). \end{aligned}$$

Thus, (5.7) yields an opposite inequality

$$\begin{aligned} \operatorname{gch} \ker \eta(\lambda) \le{} & \operatorname{gch} H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda + \varpi_n)) \otimes \mathbb{C}_{\varpi_n} \\ & - \operatorname{gch} H^0(\mathfrak{X}_\Psi(w_0^\Psi), \mathcal{O}_{\mathfrak{X}_\Psi(w_0^\Psi)}(\lambda)). \end{aligned}$$

This establishes the surjectivity of $\eta(\lambda)$, completing the proof. □

5.3.2 The second step

To apply more advanced representation-theoretic tools from affine Lie algebras, we introduce some additional objects beyond the material presented in the end of §1.3. Let $\widetilde{\mathfrak{g}} \subset \mathrm{Lie}\,\widetilde{G}(\!(z)\!)$ be the affine Kac-Moody Lie algebra of type $\mathsf{A}^{(1)}_{n-1}$, and $\widetilde{\mathfrak{b}} \subset \mathrm{Lie}\,\widetilde{\mathbf{B}}$ be its non-negative part [18, Chapter 6]. Let $\widetilde{\mathfrak{n}}^{-}$ be the lower (opposite) triangular part of $\widetilde{\mathfrak{g}}$ with respect to $\widetilde{\mathfrak{b}}$.

Remark 5.14. Real roots (resp. positive simple roots) of $\widetilde{\mathfrak{g}}$ correspond precisely to the affine roots (resp. $\{\alpha_i\}_{i\in\mathtt{I}_{\mathrm{af}}}$) in our convention, and $\mathfrak{sl}(2,i)$ $(i \in \mathtt{I}_{\mathrm{af}})$ contains both of the positive/negative Kac-Moody generators E_i/F_i. We have $\mathfrak{g}(k) \subset \widetilde{\mathfrak{g}}$ and $(\mathfrak{g}(k) \cap \widetilde{\mathfrak{b}}) \subset \mathfrak{p}(k)$ for each $d_1(\Psi) \le k \le n$.

Since Id commutes with $\mathfrak{sl}(n,\mathbb{C})$ in $\mathfrak{g}$, we may adjoin an additional factor $\mathbb{C}[z]\mathrm{Id}$ to $\mathfrak{sl}(n,\mathbb{C}[z])$ to track the eigenvalues of Id. Note that $z\mathbb{C}[z]\mathrm{Id}$ acts trivially.

Lemma 5.15. *We have a surjective inverse system of graded* $\mathfrak{gl}(n,\mathbb{C}[z])$*-modules* $\{L(\lambda + m\varpi_n)^\vee \otimes \mathbb{C}_{m\varpi_n}\}_{m\ge 0}$ *such that*

$$\varprojlim_m L(\lambda + m\varpi_n)^\vee \otimes \mathbb{C}_{m\varpi_n} \cong U(\mathfrak{sl}(n,\mathbb{C}[z])) \otimes_{U(\mathfrak{sl}(n))} V_\lambda^*. \tag{5.11}$$

In particular, (5.11) *has simple head* V_λ^* *as a graded* $\mathfrak{gl}(n,\mathbb{C}[z])$*-module.*

Proof. By the presentation of $L(\lambda)$ in [18, Proof of Lemma 10.1], we obtain the following description of the dual module:

$$L(\lambda)^\vee \cong \frac{U(\mathfrak{sl}(n,\mathbb{C}[z])) \otimes_{U(\mathfrak{sl}(n,\mathbb{C}))} V_\lambda^*}{(zE_{n,1})^{m_n+1}(1 \otimes \mathbf{v}_\lambda^\vee)},$$

where $\mathbf{v}_\lambda^\vee$ is the $\mathfrak{g}$-lowest weight vector of V_λ^*. Thus, we can form an inverse limit of surjective systems

$$\varprojlim_m L(\lambda + m\varpi_n)^\vee \otimes \mathbb{C}_{m\varpi_n}, \tag{5.12}$$

that is isomorphic to $(U(\mathfrak{sl}(n,\mathbb{C}[z])) \otimes_{U(\mathfrak{sl}(n))} V_\lambda^*)$ as graded $\mathfrak{gl}(n,\mathbb{C}[z])$-modules. Hence, we deduce (5.11). From its presentation, we see that

$$U(\mathfrak{sl}(n,\mathbb{C}[z])) \otimes_{U(\mathfrak{sl}(n))} V_\lambda^*$$

has simple head V_λ^* as graded $\mathfrak{gl}(n,\mathbb{C}[z])$-modules, completing the proof. □

For each $e_1(\Psi) \le k \le n$, we set $G^0(k) := \langle SL(2,j) \mid k \le j < n\rangle$, and let

$$P^0(k) := \big(\langle SL(2,j) \mid k < j < n\rangle\, \widetilde{\mathbf{B}} \cap G^0(k)\big) \subset G^0(k)$$

be its parabolic subgroup. We set

$$V(k) := \Big(\bigoplus_{j=1}^{\mathtt{h}_k(\Psi)} E_{j,k}\Big).$$

Remark 5.16. The subgroups $G^\flat(k)$ and $G^0(k)$ of $G(k)$ are distinct: the present argument takes place along the base $X \subset \mathfrak{X}_\Psi(w_0)$, whereas the argument in §5.3.1 concerns the boundary ∂.

Proposition 5.17. *Let Ψ be a root ideal and let $\lambda \in \mathtt{Par}$. For $e_1(\Psi) \le k \le n$ and $1 \le i < k$, we have an isomorphism*

$$\mathbb{C}_{\Lambda_i} \otimes \varinjlim_m \mathbb{C}_{-m\varpi_n} \otimes \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\bullet) \cong \varinjlim_m \mathbb{C}_{-m\varpi_n} \otimes \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\mathbb{C}_{\Lambda_i} \otimes \bullet)$$

of endofunctors on the category of $P^0(k)\widetilde{\mathbf{B}}$-modules.

Proof. Geometrically, the application of $\mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\bullet) \otimes \mathbb{C}_{-m\varpi_n}$ corresponds to considering the dual of the global sections of the associated vector bundle over $G(k)/P(k)$ by Lemma 3.6.

Taking the limit as $m \to \infty$ amounts to computing the global sections over an open subspace of $G(k)/P(k)$ that contributes to $T^*_\Psi X$, namely

$$Z(k) := G^0(k) \times^{P^0(k)} V(k) \subset G(k)/P(k). \tag{5.13}$$

In particular, we have isomorphisms

$$\varinjlim_m \mathbb{C}_{-m\varpi_n} \otimes \mathcal{C}^\Psi_k(\lambda + m\varpi_n)(M) \cong H^0(Z(k), \mathcal{M}) \cong H^0(G^0(k)/P^0(k), \mathcal{M}^\downarrow(M)),$$

for a $P(k)$-module M, where $\mathcal{M}$ is the $G(k)$-equivariant vector bundle on $G(k)/P(k)$ whose fiber at $P(k)/P(k)$ is $M^\vee$, and $\mathcal{M}^\downarrow(M)$ is the $G^0(k)$-equivariant vector bundle on $G^0(k)/P^0(k)$ whose fiber at $P^0(k)/P^0(k)$ is $\mathbb{C}[V(k)] \otimes M^\vee$.

Thus, we can replace

$$\varinjlim_m \mathbb{C}_{-m\varpi_n} \otimes \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\bullet) \tag{5.14}$$

with $H^0(G^0(k)/P^0(k), \mathcal{M}^\downarrow(\bullet))$ in order to calculate the desired limit.

Originally, $\mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\bullet)$ is defined for $\widetilde{\mathbf{P}}(k)$-modules. Since the base space is shrinked to $G^0(k)/P^0(k) \subset G(k)/P(k)$, our reinterpretation of (5.14) extends to an endofunctor on the category of $P^0(k)\widetilde{\mathbf{B}}$-modules.

In this setting, the construction of $H^0(G^0(k)/P^0(k), \mathcal{M}^\downarrow(\bullet))$ involves forming a vector bundle, that replace the effect of the application of

$$\varinjlim_m \mathbb{C}_{-m\Lambda_n} \otimes \mathcal{D}_0(\mathbb{C}_{m\Lambda_n} \otimes \bullet)$$

followed by $\mathcal{D}_j$ for $0 \le j < \mathtt{h}_k(\Psi)$, and the application of the Demazure functors of shape $\mathcal{D}_j$ for $k \le j < n$. The former operation (forming vector bundles) commutes with arbitrary character twists, while the latter operations (Demazure functors) commute with the character twist by $\mathbb{C}_{\Lambda_i}$ for $i < k$.

This shows that in the inductive limit, the character twist by $\mathbb{C}_{m_i\Lambda_i}$ commutes with the operations defining $\mathcal{C}_{\mathtt{h}_k(\Psi),k}$. Hence, we obtain the desired functorial isomorphism:

$$\mathbb{C}_{\Lambda_i} \otimes \varinjlim_m \mathbb{C}_{-m\varpi_n} \otimes \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\bullet) \cong \varinjlim_m \mathbb{C}_{-m\varpi_n} \otimes \mathcal{C}_{\mathtt{h}_k(\Psi),k}(\lambda + m\varpi_n)(\mathbb{C}_{\Lambda_i} \otimes \bullet).$$

This completes the proof. $\square$

Corollary 5.18. *We have a surjective graded $\mathfrak{sl}(n, \mathbb{C}[z])$-module map*

$$U(\mathfrak{sl}(n, \mathbb{C}[z])) \otimes_{U(\mathfrak{sl}(n,\mathbb{C}))} V^*_\lambda \twoheadrightarrow \varprojlim_m H^0(\mathcal{X}_\Psi(w_0), \mathcal{O}_{\mathcal{X}_\Psi(w_0)}(\lambda + m\varpi_n)) \otimes \mathbb{C}_{m\varpi_n}.$$

Proof. By repeatedly applying Proposition 5.17, we move the character twists to the initial term $\mathbb{C}$ in (2.2) to identify

$$\varinjlim_m \mathcal{D}_{w_0}(\mathcal{C}_{\mathtt{h}_{e_1(\Psi)},e_1(\Psi)}(\lambda+m\varpi_n)\circ\cdots\circ\mathcal{C}_{\mathtt{h}_n(\Psi),n}(\lambda+m\varpi_n))(\mathbb{C}))\otimes\mathbb{C}_{-m\varpi_n},$$

that calculates the (restricted) dual of

$$\varprojlim_m H^0(\mathfrak{X}_\Psi(w_0),\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda+m\varpi_n))\otimes\mathbb{C}_{m\varpi_n},$$

with $\varinjlim_m \mathcal{D}_{\mathbf{i}}(\mathbb{C}_{\lambda+m\varpi_m})$, where $\mathbf{i}$ is a sequence in $\mathtt{I}_{\mathrm{af}}$ that records a reduced expression of w_0 followed by the sequence read out from (2.2).

By Theorem 1.15(2), we have

$$\varinjlim_m \mathcal{D}_{\mathbf{i}}(\mathbb{C}_{\lambda+m\varpi_m})\cong\varinjlim_m L_w(\lambda+m\varpi_n)\otimes\mathbb{C}_{-m\varpi_n}\subset\varinjlim_m L(\lambda+m\varpi_n)\otimes\mathbb{C}_{-m\varpi_n}$$

for some $w\in\widetilde{\mathfrak{S}}_n$ determined uniquely by $\mathbf{i}$. In particular, we conclude that

$$\varprojlim_m L(\lambda+m\varpi_n)^\vee\otimes\mathbb{C}_{m\varpi_n}\twoheadrightarrow\varprojlim_m H^0(\mathfrak{X}_\Psi(w_0),\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda+m\varpi_n))\otimes\mathbb{C}_{m\varpi_n}$$

by taking (restricted) dual. Now Lemma 5.15 yields the result. □

5.3.3 The final step

To conclude the proof of Theorem 5.8, we combine the surjective map from Corollary 5.18 with the surjectivity of $\eta(\lambda)$ established in Corollary 5.13. This yields a sequence of surjective maps:

$$\begin{aligned}U(\mathfrak{sl}(n,\mathbb{C}[z]))\otimes_{U(\mathfrak{sl}(n,\mathbb{C}))}V_\lambda^*&\twoheadrightarrow\varprojlim_m H^0(\mathfrak{X}_\Psi(w_0),\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda+m\varpi_n))\otimes\mathbb{C}_{m\varpi_n}\\&\twoheadrightarrow H^0(\mathfrak{X}_\Psi(w_0),\mathcal{O}_{\mathfrak{X}_\Psi(w_0)}(\lambda)),\end{aligned}$$

all regarded as graded $\mathfrak{gl}(n,\mathbb{C}[z])$-modules. Since the leftmost term has simple head V_λ^* (by Lemma 5.15), the same holds for the final term. This completes the proof of the theorem.

5.4 Monotonicity of multiplicities

Proposition 5.19. *Let $\Psi'\subset\Psi\subset\Delta^+$ be root ideals, and let $w',w\in\mathfrak{S}_n$ be Ψ-tame permutations such that $X(w')\subset X(w)$. For each $\lambda\in\mathtt{Par}$, we have an inclusion*

$$N_{w'}^{\Psi'}(\lambda)\subset N_w^\Psi(\lambda)\quad\text{as}\quad\widetilde{\mathbf{B}}\text{-modules.}$$

Proof. Note that a Ψ-tame element is automatically Ψ'-tame by $d_1(\Psi')\geq d_1(\Psi)$, and hence w' is Ψ'-tame. Thanks to Proposition 2.9, we can replace $N_w^\Psi(\lambda)$ and $N_{w'}^{\Psi'}(\lambda)$ with $M_w^\Psi(\lambda)$ and $M_{w'}^{\Psi'}(\lambda)$.

By interpreting (2.3) as successive applications of Demazure functors together with the character twists, we have a sequence $\mathbf{i}$ of elements of $\mathtt{I}_{\mathrm{af}}$ such that

$$(M_w^\Psi(\lambda))^*=H^0(X(\mathbf{i}),\mathcal{L}_\lambda),$$

where $X(\mathbf{i})$ is defined in (1.6), and $\mathcal{L}_\lambda$ is defined as: $\{\mathcal{L}_{\varpi_i}\}_{i \in \mathtt{I}_{\mathrm{af}}}$ is a collection of $\widetilde{\mathbf{B}}$-equivariant line bundles on $X(\mathbf{i})$ obtained as the pullback of $\mathcal{O}_{\mathbf{i}[k]}(\Lambda_k)$ on $X(\mathbf{i}[k])$ (using the maps offered in Lemma 1.10), where $\mathbf{i}[k]$ denotes the truncation of the sequence $\mathbf{i}$ up to the place where $\mathbb{C}_{m_k \Lambda_k}$ appears in (2.3). For general $\lambda \in \mathtt{Comp}$, we extend this definition by tensor products to obtain $\mathcal{L}_\lambda$.

Examining the sequence offered in (2.3), we have its subsequence $\mathbf{i}'$ that realizes $M_{w'}^{\Psi'}(\lambda)$. In particular, we have $X(\mathbf{i}') \subset X(\mathbf{i})$. Thus, we obtain a restriction map

$$M_w^{\Psi}(\lambda)^* = H^0(X(\mathbf{i}), \mathcal{L}_\lambda) \longrightarrow H^0(X(\mathbf{i}'), \mathcal{L}_\lambda) = M_{w'}^{\Psi'}(\lambda)^*. \tag{5.15}$$

The $\widetilde{T}$-weights of simple heads of both sides of (5.15), provided by Corollary 5.9, coincide, as both are realized as the (dual of the) fiber of $\mathcal{L}_\lambda$ at the $\widetilde{T}$-fixed point $X(\emptyset) \in X(\mathbf{i}') \subset X(\mathbf{i})$. Hence, we deduce that (5.15) is surjective. Taking duals then yields the desired inclusion of $\widetilde{\mathbf{B}}$-modules. $\square$

Proposition 5.20. *Let $\Psi' \subset \Psi \subset \Delta^+$ be root ideals, and let $w', w \in \mathfrak{S}_n$ be Ψ-tame elements such that $X(w') \subset X(w)$. Then we have an inclusion $\mathfrak{X}_{\Psi'}(w') \subset \mathfrak{X}_\Psi(w)$ that induces a surjection*

$$H^0(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) \twoheadrightarrow H^0(\mathfrak{X}_{\Psi'}(w'), \mathcal{O}_{\mathfrak{X}_\Psi(w')}(\lambda)) \qquad \lambda \in \mathtt{Par}.$$

Proof. Recall that the homogeneous coordinate ring of $\mathfrak{X}_\Psi(w)$ is $\bigoplus_{\lambda \in \mathtt{Par}} (N_w^\Psi(\lambda))^*$. By Proposition 5.19, the natural map

$$N_w^\Psi(\lambda)^* \longrightarrow N_{w'}^{\Psi'}(\lambda)^*$$

is surjective for each $\lambda \in \mathtt{Par}$. This implies that the homogeneous coordinate ring of $X_{\Psi'}(w')$ is a quotient of that of $X_\Psi(w)$. In view of Theorem 4.1, we conclude the desired surjection. $\square$

Corollary 5.21. *Let $\Psi' \subset \Psi \subset \Delta^+$ be root ideals, and let $w', w \in \mathfrak{S}_n$ be Ψ-tame elements such that $X(w') \subset X(w)$. The natural restriction map*

$$H^0(T^*_\Psi X(w), \mathcal{O}_{X_\Psi(w)}(\lambda)) \longrightarrow H^0(T^*_{\Psi'} X(w'), \mathcal{O}_{X_{\Psi'}(w')}(\lambda)) \qquad \lambda \in \mathtt{Par}$$

is surjective. Furthermore, there is a scheme-theoretic identification

$$\mathfrak{X}_{\Psi'}(w') = \overline{T_{\Psi'} X(w')} \subset \mathfrak{X}_\Psi(w).$$

Proof. Note that w' is Ψ'-tame. By Proposition 5.20, we have the following commutative diagram

$$\begin{tikzcd}
H^0(T^*_\Psi X(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda)) \arrow[r] & H^0(T^*_{\Psi'} X(w'), \mathcal{O}_{\mathfrak{X}_\Psi(w')}(\lambda)) \\
H^0(\mathfrak{X}_\Psi(w), \mathcal{O}_{\mathfrak{X}_\Psi(w)}(\lambda + m\varpi_n)) \otimes \mathbb{C}_{m\varpi_n} \arrow[r, two heads] \arrow[u, hook] & H^0(\mathfrak{X}_{\Psi'}(w'), \mathcal{O}_{\mathfrak{X}_{\Psi'}(w')}(\lambda + m\varpi_n)) \otimes \mathbb{C}_{m\varpi_n} \arrow[u, hook]
\end{tikzcd}$$

for each $\lambda \in \mathtt{Par}$ and $m \geq 0$. Thus, Theorem 5.1 yields the first assertion. The second assertion follows directly from a comparison of the homogeneous coordinate rings via the above commutative diagram. $\square$

Definition 5.22. For a root ideal $\Psi \subset \Delta^+$ and weights $\lambda, \mu \in \mathsf{P}^+$, we define the graded multiplicity series

$$K^{\Psi}_{\lambda,\mu}(q) := \sum_{m \in \mathbb{Z}} q^m \dim \mathrm{Hom}_{G \times \mathbb{G}_m^{\mathrm{rot}}}(V(\lambda) \boxtimes \mathbb{C}_{-m\delta}, H^0(T^*_\Psi X, \mathcal{O}_{X_\Psi(w_0)}(\mu))^\vee) \in \mathbb{Z}[\![q]\!].$$

The following assertion generalizes and proves [35, Conjecture 12]:

Corollary 5.23. *Let $\Psi' \subset \Psi \subset \Delta^+$ be root ideals. We have*

$$K^{\Psi'}_{\lambda,\mu}(q) \le K^{\Psi}_{\lambda,\mu}(q) \qquad \lambda, \mu \in \mathsf{P}^+.$$

Proof. Since rational representations of $(G \times \mathbb{G}_m^{\mathrm{rot}})$ are completely reducible, $K^{\Psi}_{\lambda,\mu}(q)$ counts the graded multiplicities of $V(\lambda)$ in

$$H^0(T^*_\Psi X(w_0), \mathcal{O}_{X_\Psi(w_0)}(\mu))^\vee.$$

Therefore, the $w = w' = w_0$ case of Corollary 5.21 yields the assertion. □

Remark 5.24. By Corollary 5.21, we deduce that the composition map

$$\mathrm{Spec}\, H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}) \to \mathrm{Spec}\, H^0(T^* X, \mathcal{O}_{T^* X}) \subset \mathfrak{sl}(n)$$

defines an irreducible and reduced closed subscheme[1]. Thus, it must be the closure of a nilpotent orbit, denoted by $\mathbb{O}_\Psi$.

Taking into account the fact that ϖ_n is the determinant character of G, we find that

$$\mathcal{O}_{X_\Psi(w_0)}(\varpi_n)|_{T^*_\Psi X} \cong \mathcal{O}_{T^*_\Psi X} \otimes \mathbb{C}_{-\varpi_n}$$

for each $\Psi \subset \Delta^+$.

From these, we have

$$K^{\Psi'}_{\lambda,k\varpi_n}(q) \le K^{\Psi}_{\lambda,k\varpi_n}(q) \qquad k \in \mathbb{Z}, \lambda \in \mathsf{P}^+$$

if $\Psi, \Psi' \subset \Delta^+$ satisfies $\mathbb{O}_{\Psi'} \subset \overline{\mathbb{O}_\Psi}$ (and the equality holds when $\mathbb{O}_{\Psi'} = \mathbb{O}_\Psi$). This recovers (or generalizes) [35, Conjecture 13] for $\gamma = (k^n)$, that is implicit in Fenn-Sommers [11, §5.1]. In a similar manner, Corollary 5.21 provides a wide extension of the speculations presented in [35, §2.10].

The following assertion generalizes and proves [35, Conjecture 13]:

Corollary 5.25. *Let $1 \le a < b \le n$ and let $\mu \in \mathtt{Par}$ be a partition such that*

$$\mu_a = \mu_{a+1} = \cdots = \mu_b.$$

Let $\Psi', \Psi \subset \Delta^+$ be two root ideals with the following properties:

1. *$E_{a-1,j}, E_{i,b+1} \in \mathfrak{n}(\Psi') \cap \mathfrak{n}(\Psi)$ for $j \ge a, i \le b$;*
2. *When $i < a$ or $j > b$, then we have $E_{i,j} \in \mathfrak{n}(\Psi')$ if and only if $E_{i,j} \in \mathfrak{n}(\Psi)$.*

[1] This feature no longer holds if we replace G with other type even when we employ an equivariant vector subbundle of $T^*(G/B)$ corresponding to the pullback of $T^*(G/P)$ for a parabolic subgroup $P \subset G$ (see e.g. [10, 29]).

Consider the subgroup $G_{a,b} := SL(b-a+1) \subset G$ whose set of T-weights are $\{\epsilon_i - \epsilon_j\}_{a \le i,j \le b}$. If we have

$$G_{a,b}\mathfrak{n}(\Psi') \subset \overline{G_{a,b}\mathfrak{n}(\Psi)},$$

then we have

$$K^{\Psi'}_{\lambda,\mu}(q) \le K^{\Psi}_{\lambda,\mu}(q) \qquad \lambda \in \mathsf{P}^+. \tag{5.16}$$

Proof. We consider the parabolic subgroup $P \subset G$ defined by $P = G_{a,b}B$. By assumption, we have

$$\mathfrak{g}_{a,b} + \mathfrak{n}(\Psi) = \mathfrak{g}_{a,b} + \mathfrak{n}(\Psi'),$$

and they are stable under the action of P. Hence, we have a map

$$f_\Psi : T^*_\Psi X = G \times^B \mathfrak{n}(\Psi) \longrightarrow G \times^P (\mathfrak{g}_{a,b} + \mathfrak{n}(\Psi)),$$

and similarly a map $f_{\Psi'}$ for Ψ'. By the discussion in Remark 5.24, we have a surjection of $(P \times \mathbb{C}^\times)$-equivariant sheaves

$$(f_\Psi)_* \mathcal{O}_{T^*_\Psi X}(\mu) \longrightarrow (f_{\Psi'})_* \mathcal{O}_{T^*_{\Psi'} X}(\mu),$$

defined as the restriction. Taking their global sections, we obtain a map

$$\imath : H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\mu)) \longrightarrow H^0(T^*_{\Psi'} X, \mathcal{O}_{T^*_{\Psi'} X}(\mu)). \tag{5.17}$$

This is enhanced into a commutative diagram induced by the restrictions of sheaves on $G \times^P (\mathfrak{g}_{a,b} + \mathfrak{n}(\Psi))$:

$$\begin{tikzcd}
H^0(T^* X, \mathcal{O}_{T^* X}(\mu)) \arrow[d, two heads] \arrow[dr, two heads] & \\
H^0(T^*_\Psi X, \mathcal{O}_{T^*_\Psi X}(\mu)) \arrow[r, "\imath"] & H^0(T^*_{\Psi'} X, \mathcal{O}_{T^*_{\Psi'} X}(\mu)),
\end{tikzcd}$$

where the vertical maps are surjective by Corollary 5.21. This implies that $\imath$ is surjective. Taking the graded characters of (5.17), we conclude (5.16). This completes the proof. ☐

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