The Parabolic/Near-Parabolic Renormalization and An invariant class of maps

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Siegel disks, parabolic implosion and area of Julia sets
In celebration of Adrien Douady's 70th birthday
Université de Cergy-Pontoise, May 9-11, 2006

$f_0(z)$ holomorphic near 0 $\quad f_0(0) = 0$

parabolic fixed point: $f_0'(0)$ a root of unity

1-parabolic and non-degenerate

$$f_0(z) = z + a_2 z^2 + \dots \qquad a_2 \neq 0$$

near-parabolic fixed point: $f'(0) = e^{2\pi i \alpha}$, α small

irrationally indifferent

$$f(z) = e^{2\pi i \alpha} z + O(z^2) \qquad \alpha \in \mathbb{R} \setminus \mathbb{Q}$$

$$\alpha = \cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{a_3 \pm \dots}}} \qquad \text{where } a_i \in \mathbb{N}$$

large continued fraction coefficients: $a_i \geq N$

The Goal of This Talk

Define

parabolic renormalization: $f_0 \rightsquigarrow \mathcal{R}_0 f_0$

$$\mathcal{R}_0 f_0(z) = z + O(z^2)$$

1-parabolic

near-parabolic renormalization: $f \rightsquigarrow \mathcal{R} f$

$$\mathcal{R} f(z) = e^{2\pi i \beta} z + O(z^2)$$

$\beta = -\frac{1}{\alpha}$

it is induced from the first return map
to a certain fundamental domain

Present the class $\mathcal{F}_1$ of 1-parabolic maps

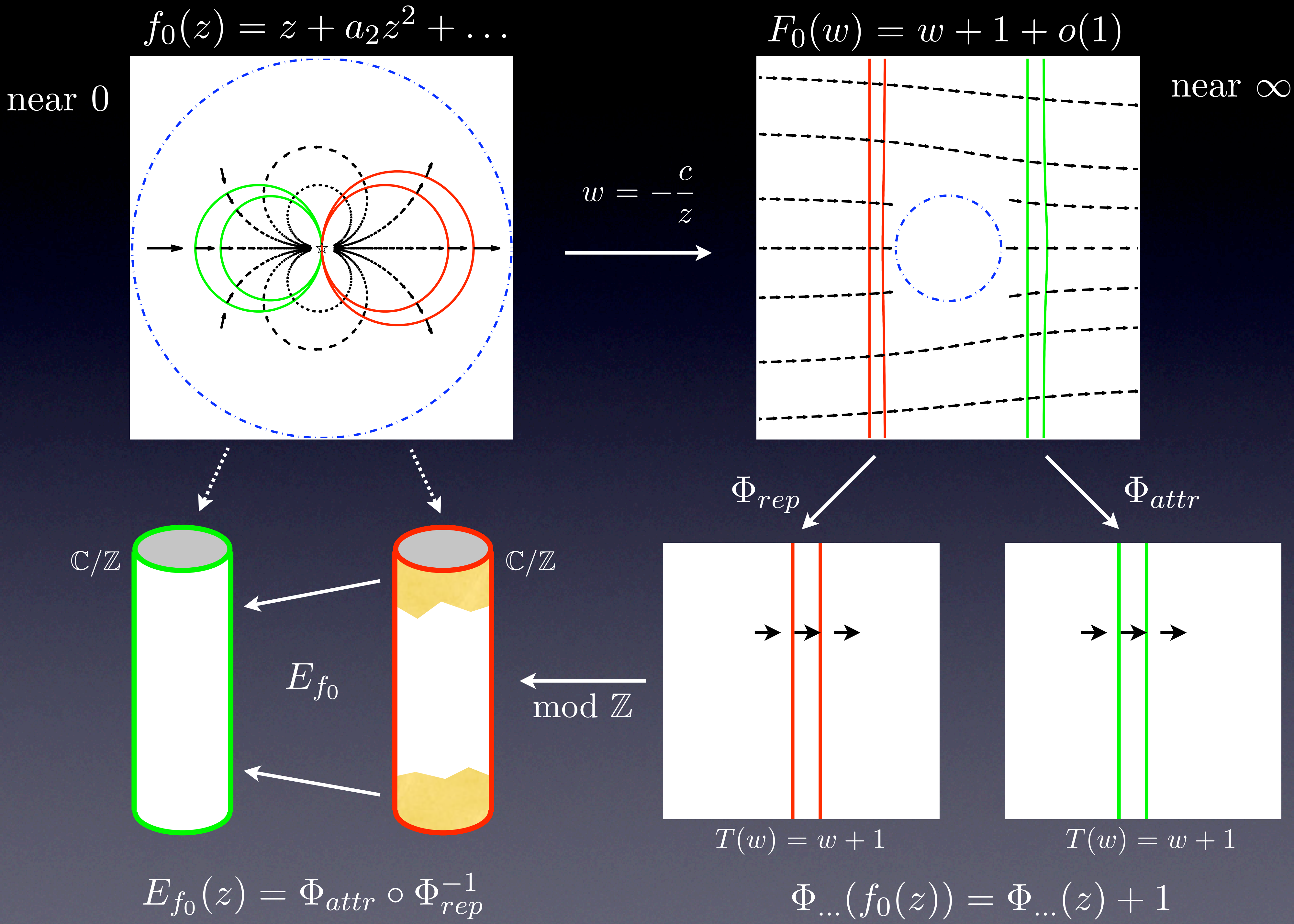
it is invariant under the renormalizations

the renormalizations are contracting/hyperbolic

Plan

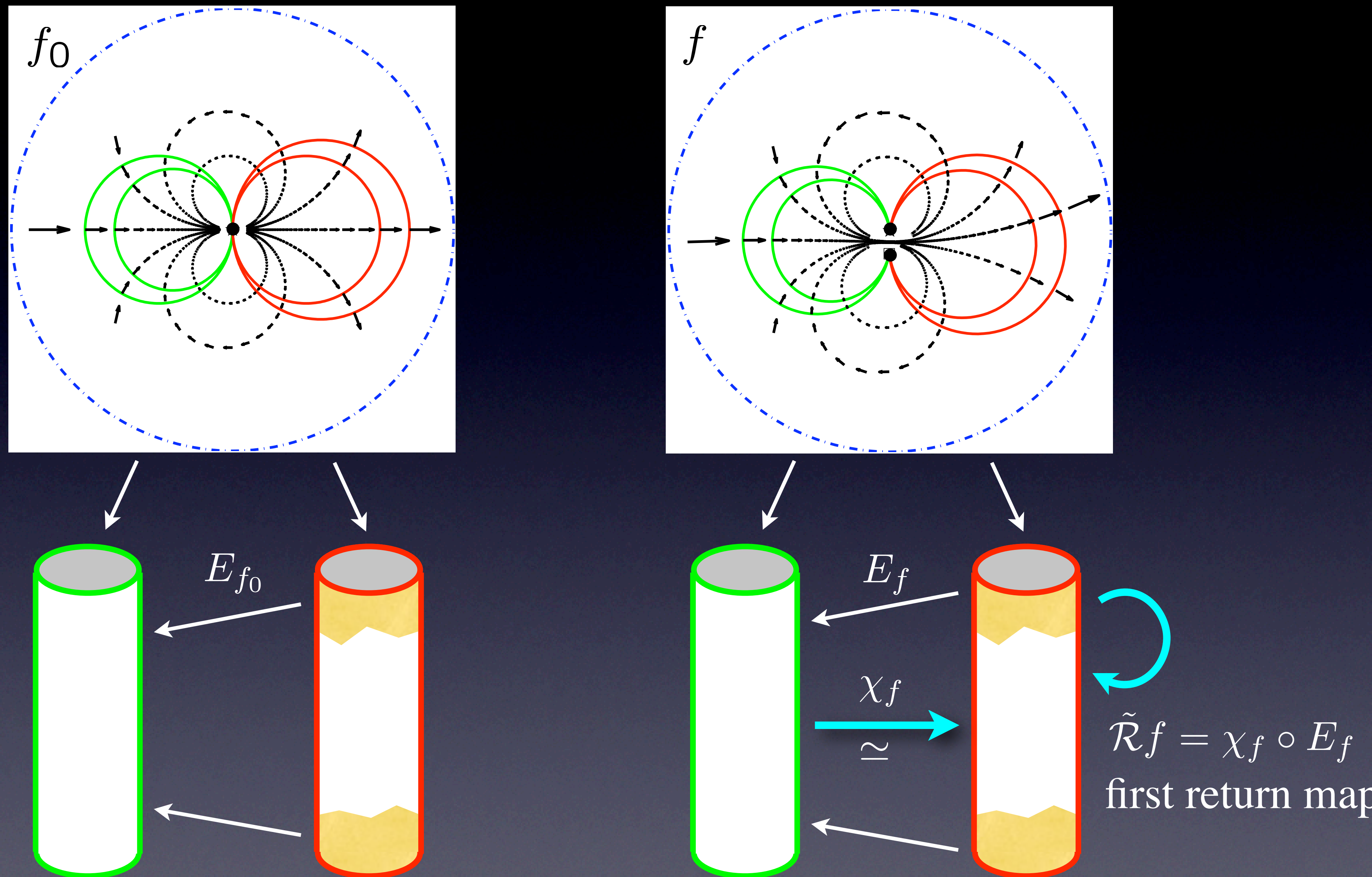
- Definition of Fatou coordinates and horn map
- Parabolic bifuracation/implosion and the return map (Douady-Hubbard-Lavaurs)
- Parabolic and near-parabolic Renormalizations
- Statements of Theorems
- Class $\mathcal{F}_1$ and its characterization
- About the proof of invariance

Fatou coordinates Φ_{attr}, Φ_{rep} and Horn map E_{f_0}



Perturbation

$f'(0) = e^{2\pi i\alpha}$, α small $|\arg\alpha| < \frac{\pi}{4}$



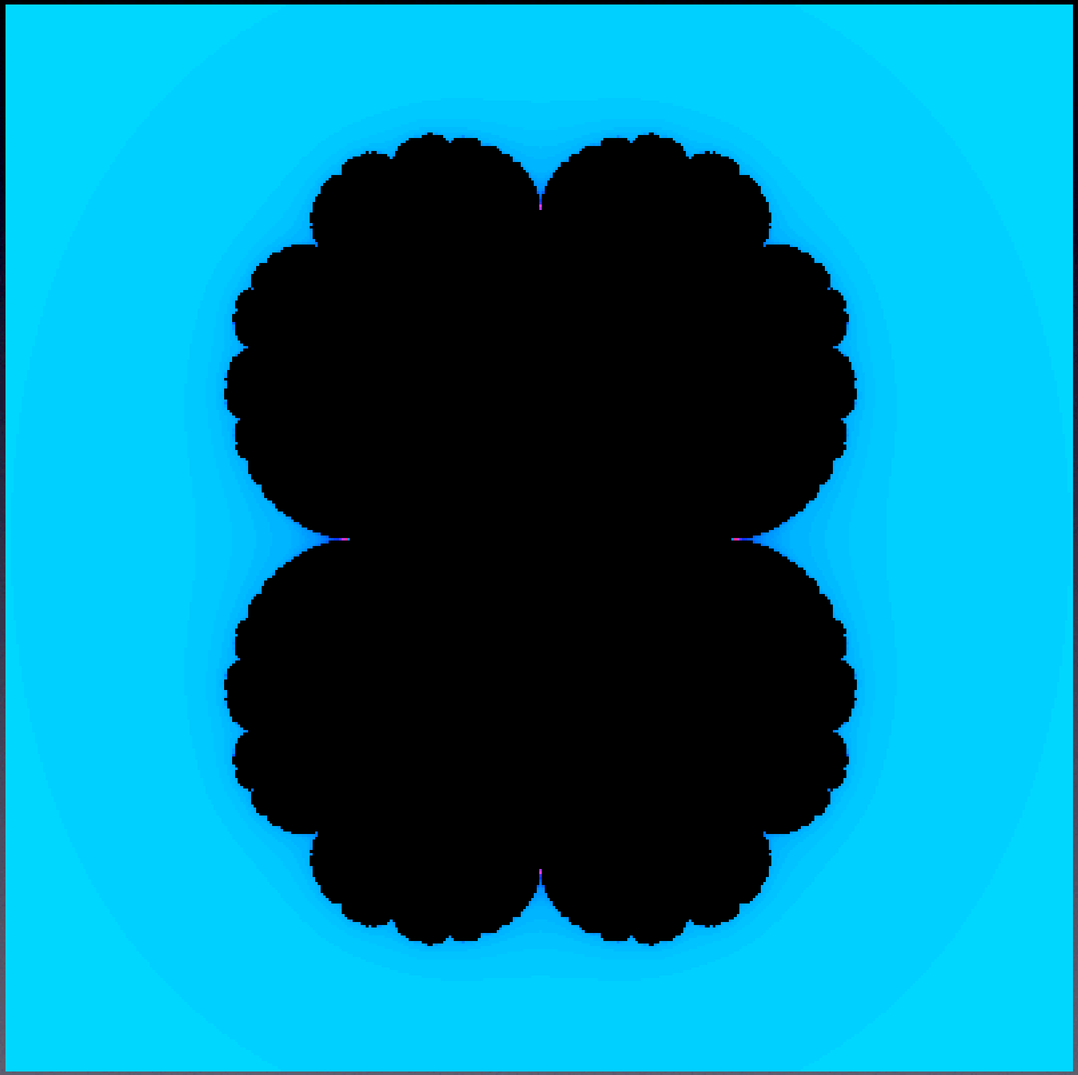
E_f depends continuously on f

(after a suitable normalization)

$$\chi_f(z) = z - \frac{1}{\alpha}$$

Parabolic Bifurcation/Implosion

When a parabolic point is perturnbed....

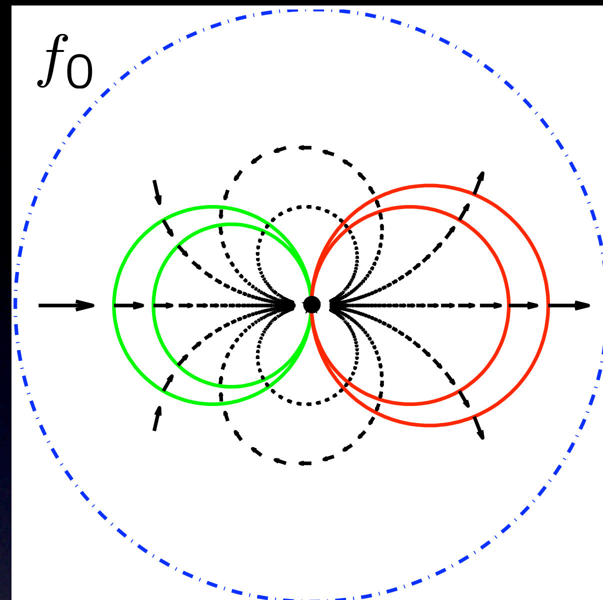


Discontinuous change of Julia sets

New complicated dynamics is created by the orbits going through the "gate"

New dynamics can be understood via the return map

Parabolic Renormalization



Normalize the Fatou coordinates so that

$$E_{f_0}(z) = z + o(1) \quad (\operatorname{Im} z \to +\infty)$$

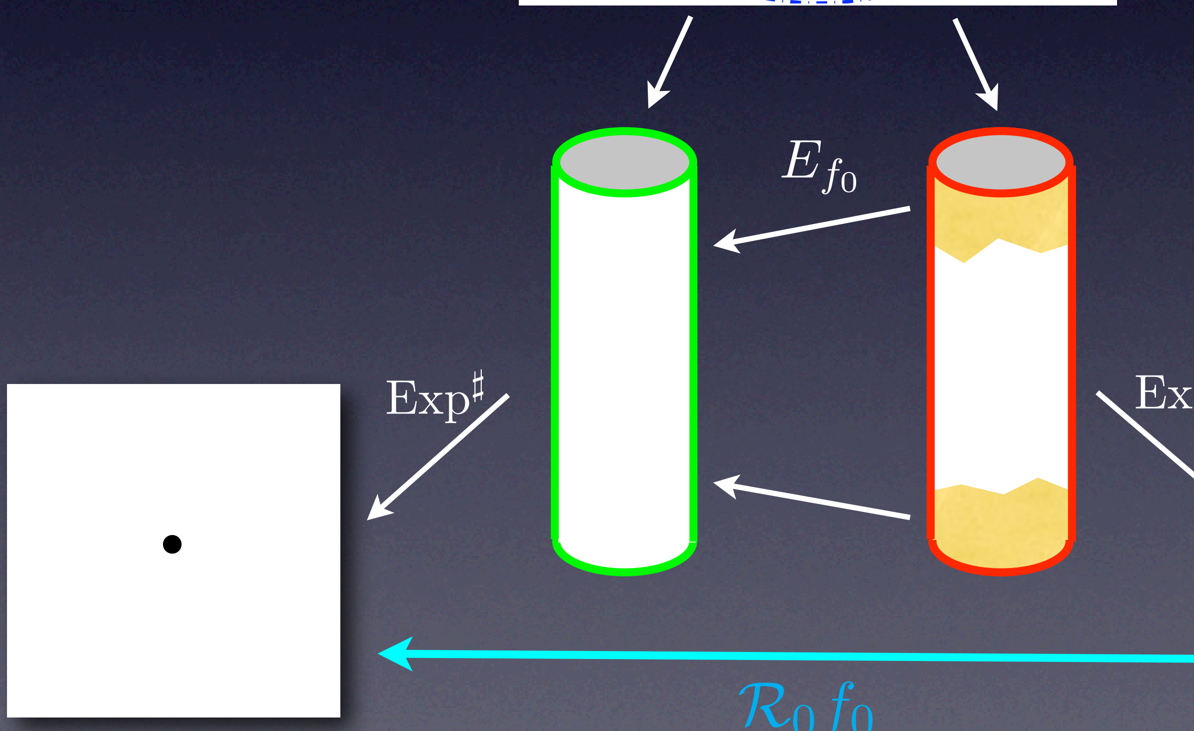
Let $\operatorname{Exp}^\sharp(z) = e^{2\pi i z}$ then

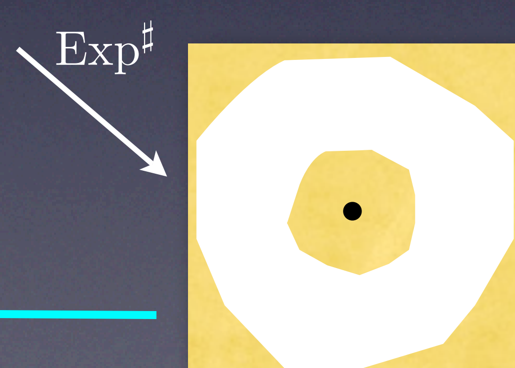
$$\operatorname{Exp}^\sharp : \mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*$$

Parabolic Renormalization

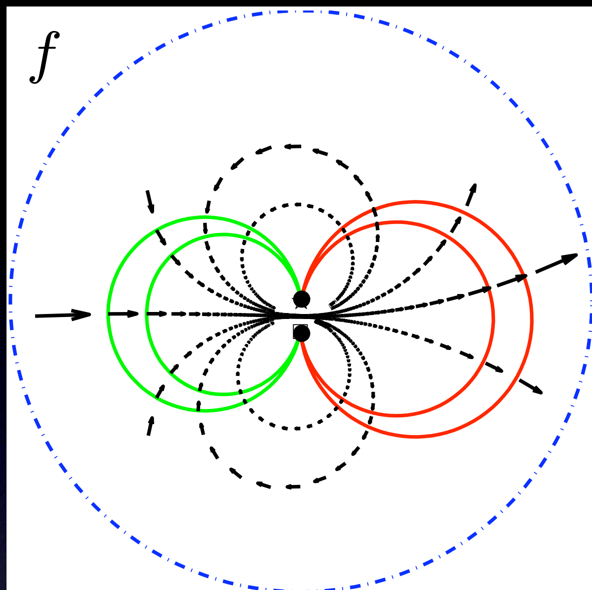
$$\mathcal{R}_0 f_0 = \operatorname{Exp}^\sharp \circ E_{f_0} \circ (\operatorname{Exp}^\sharp)^{-1}$$

0 is 1-parabolic fixed point





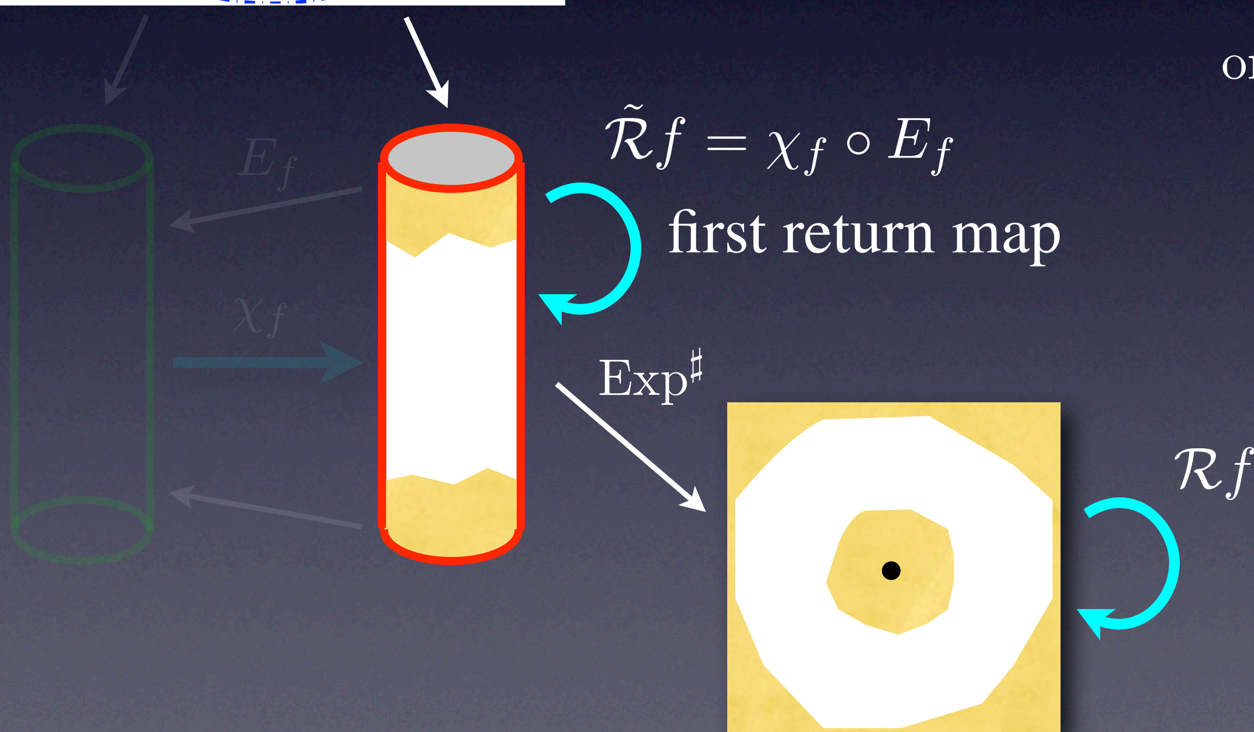
Near-parabolic Renormalization (cylinder renorm.)



$$\mathcal{R}f = \mathrm{Exp}^\sharp \circ \tilde{\mathcal{R}}f \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= \mathrm{Exp}^\sharp \circ \chi_f \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= e^{2\pi i\beta} z + O(z^2)$$

where $\beta = -\dfrac{1}{\alpha} \pmod{\mathbb{Z}}$

or $\alpha = \dfrac{1}{m-\beta}$ $(m \in \mathbb{N})$



Suppose the near-parabolic renormalization can be iterated:

$$f_0 \xrightarrow{\mathcal{R}} f_1 \xrightarrow{\mathcal{R}} f_2 \xrightarrow{\mathcal{R}} f_3 \xrightarrow{\mathcal{R}} \dots$$

high iterates of f_0 corresponds to low iterates of f_1

highly recurrent behavior of f_0 can be analyzed through f_i's

fine structure of orbits or invariant sets are magnified

$\mathcal{R}$ = a dynamical system in the space of certain type of dynamical systems

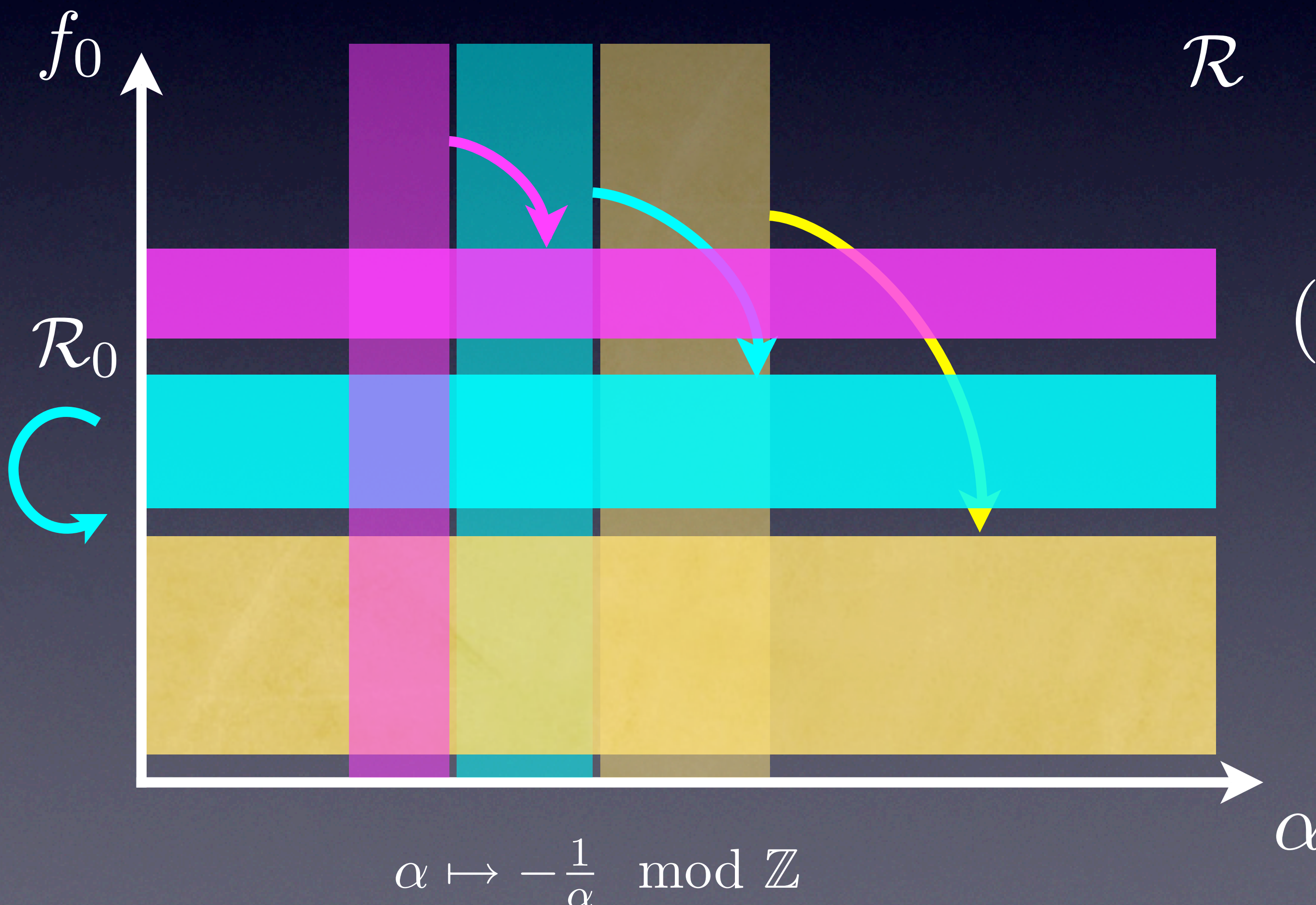
But, ... Can you really iterate infinitely many times?

Renormalization: The Picture

$f(z) = e^{2\pi i\alpha} z + O(z^2) = e^{2\pi i \alpha} f_0(z)$ where $f_0(z) = z + O(z^2)$ 1-parabolic

$f \ \leftrightarrow (\alpha, f_0)$

Write $\mathcal{R}f(z) = e^{-2\pi i \frac{1}{\alpha}} \mathcal{R}_\alpha f_0(z)$ then $\mathcal{R} : (\alpha, f_0) \mapsto (-\frac{1}{\alpha}, \mathcal{R}_\alpha f_0)$



$\mathcal{R}$ hyperbolic?

($\mathcal{R}_\alpha$ contracting?)

$\mathcal{R}_\alpha f_0 \to \mathcal{R}_0 f_0 \ (\alpha \to 0)$

$\mathcal{R}_0$ contracting?

YES for α small

Main Theorems

Theorem 1 *Let $P(z) = z(1+z)^2$. There exist bounded simply connected open sets V and V' with $0 \in V \subset \overline{V} \subset V' \subset \mathbb{C}$ such that the class*

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{c} \varphi : V \to \mathbb{C} \textit{ is univalent} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}$$

satisfies the following: univalent = holomorphic and injective

(0) *every $f \in \mathcal{F}_1$ is non-degenerate;*

(i) *$\mathcal{F}_0 \setminus \{$quadratic polynomial$\}$ can be naturally embedded into $\mathcal{F}_1$ (in particular, $\mathcal{R}_0^n(z+z^2) \in \mathcal{F}_1$ $n = 1, 2, \dots$);*

(ii) *The renormalization $\mathcal{R}_0$ is well defined on $\mathcal{F}_1$ so that $\mathcal{R}_0(\mathcal{F}_1) \subset \mathcal{F}_1$;*

(iii) *If we write $\mathcal{R}_0 f = P \circ \psi^{-1}$, then ψ can be extended univalently to V';*

(iv) *$f \mapsto \mathcal{R}_0 f$ is "holomorphic."*

Theorem 2 *The above statements hold for $\mathcal{R}_\alpha$ for α small. Hence there exists an N such that the above holds for*

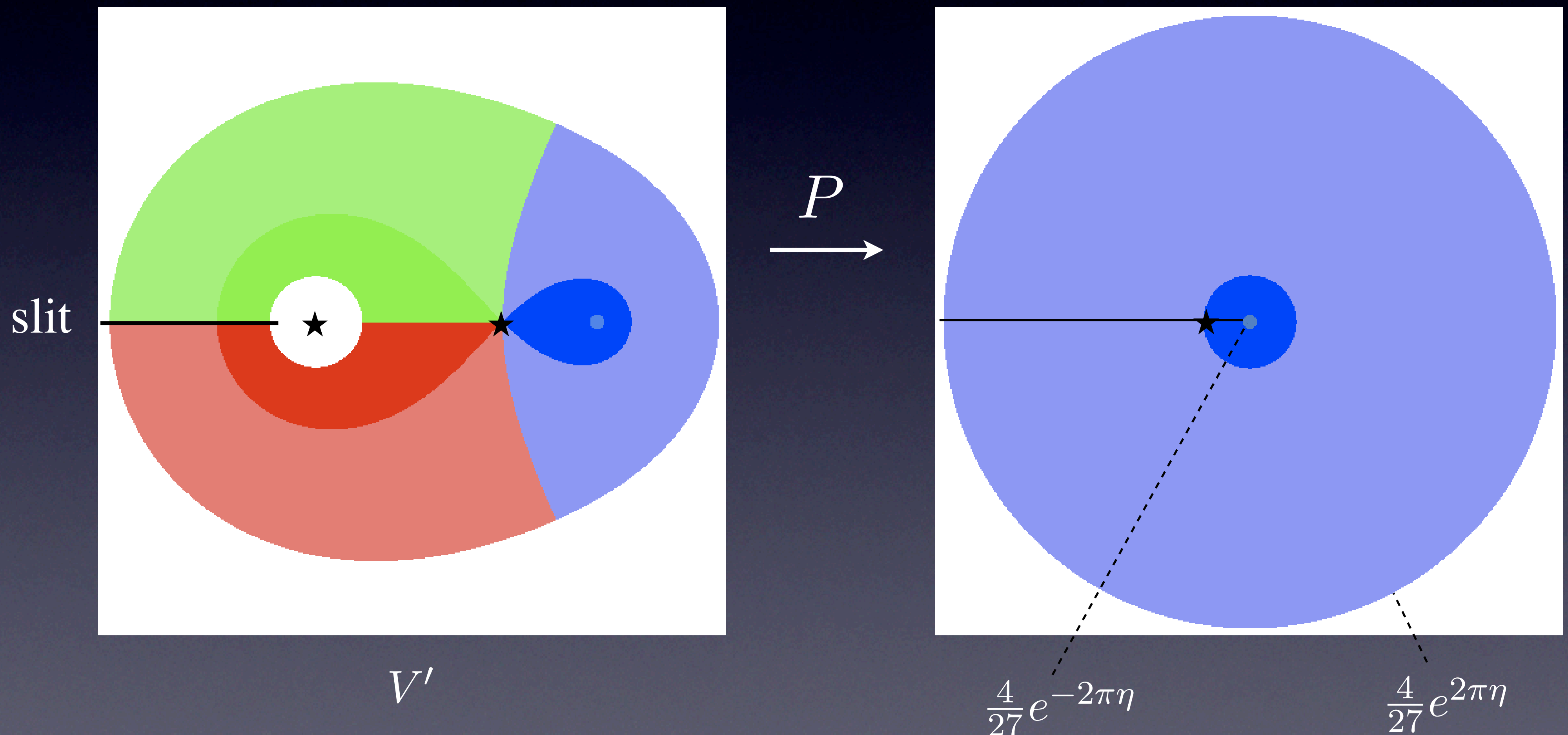
$$\alpha = \frac{1}{m+\beta} \quad \textit{with } m \in \mathbb{N},\ \beta \in \mathbb{C} \textit{ and } |\beta| \le 1.$$

$P(z) = z(1+z)^2$ and V, V'

$P(0) = 0, \quad P'(0) = 1$

critical points: $-\frac{1}{3}$ and -1 critical values: $P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$

$\eta = 2$



V slightly smaller domain than V'

$$\mathcal{F}_0 = \left\{ f : U_f \to \mathbb{C} \;\middle|\; \begin{array}{c} 0 \in U_f \text{ open and connected } \subset \mathbb{C}, \\ f \text{ is holomorphic in } U_f,\ f(0) = 0,\ f'(0) = 1, \\ f : U_f \setminus \{0\} \to \mathbb{C}^* \text{ is a branched covering map} \\ \text{with a unique critical value,} \\ \text{all critical points are of local degree } 2 \end{array} \right\}$$

$\mathcal{R}_0(\mathcal{F}_0) \subset \mathcal{F}_0$

$z + z^2, \mathcal{R}_0(z + z^2), \cdots \in \mathcal{F}_0$

This class was used in the proof of HD=2 for generic Julia sets on the boundary of the Mandelbrot set, and for the the boundary of the Mandelbrot set itself.

Also compare with the works on critical circle maps
(for example, Epstein-Yampolsky)

An Application

Corollary. *There exists an* N *such that if* $f(z) = e^{2\pi i\alpha}h(z)$ *with* $h \in \mathcal{F}_1$ *and* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *whose continued fraction has coefficients* $a_n \geq N$*, then the critical orbit stays in the domain of* f *and can be iterated infinitely many times. Moreover if* f *is (a part of) a rational map, then the critical orbit is not dense.*

Contraction and Hyperbolicity

Theorem 3 *Modifying the definition slightly (requiring that φ has a quasi-confomal extension to $\mathbb{C}$), $\mathcal{F}_1$ is in one to one correspondence with the Teichmüller space $Teich(W)$ of $W = \mathbb{C} \smallsetminus \overline{V} (\simeq \mathbb{D}^*)$. The induced map $\mathcal{R}_0^{Teich}$ is a uniform contraction with respect to the Teichmüller distance. (The Lipshitz constant $\leq \exp(-2\pi\, mod(V' \smallsetminus \overline{V}))$.)*

Theorem 4 *The above statements hold for the fiber map $\mathcal{R}_\alpha$ for α small. Hence the total renormalization $\mathcal{R}$ is hyperbolic in this region.*

$$\mathcal{F}_1 \ni f = P \circ \varphi^{-1} \rightsquigarrow [\tilde{\varphi}\,|_W] \in Teich(W) = \{\psi : W \to \mathbb{C} \text{ qc }\}/\sim$$

where $\tilde{\varphi}$ is a quasiconformal extension of φ to $\mathbb{C}$.

Royden-Gardiner Theorem (Teichmüller distance = Kobayashi distance)
cotangent space = {integrable holomorphic quadratic differentials}
modulus-area inequality for holom. quad. differentials
isoperimetric inequality for holom. quad. differentials
modified Carleman's inequality

Theorem 2 follows from Theorem 1 and the continuity of E_f with respect to f.

We outline the proof of Theorem 1.

one cannot compute $\mathcal{R}_0 f$!

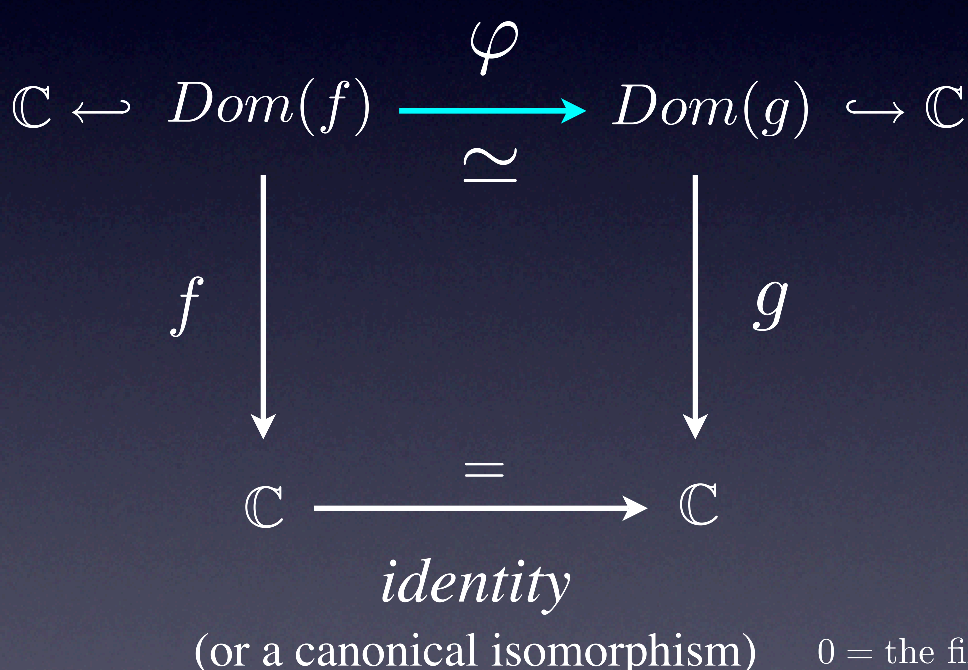
In order to define an invariant class of maps, we need a way to recognize that $\mathcal{R}_0 f$ belongs to this class.

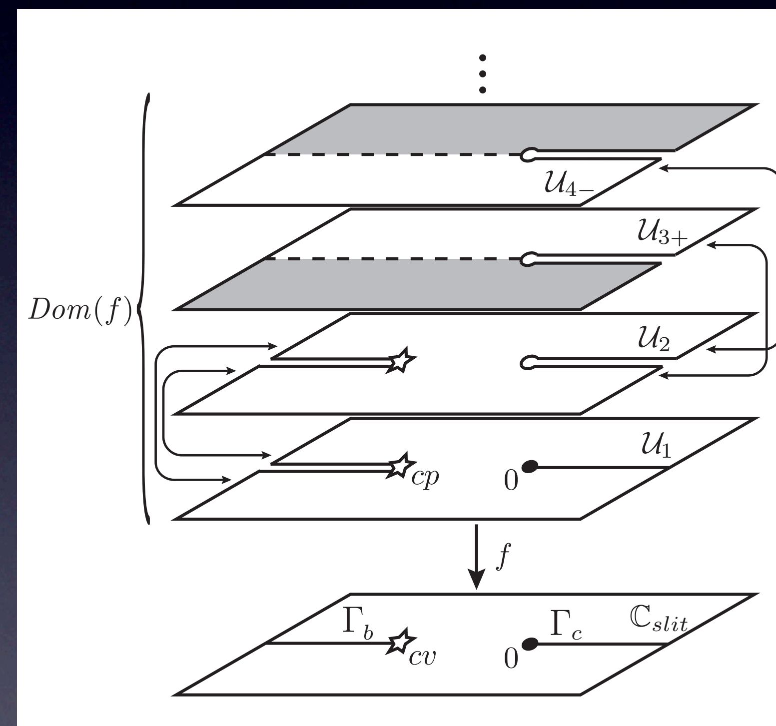
We characterize our class by covering property (as imcomplete/partial ramified covering over $\mathbb{C}$)

The Class $\mathcal{F}_1$ -- starting point and goal

We characterize our class by covering property
(as incomplete/partial ramified covering over $\mathbb{C}$)

"f and g have the same covering properties" or
$Dom(f)$ and $Dom(g)$ are the same when viewed as, in classical terms,
Riemann surfaces spread cover $\mathbb{C}$



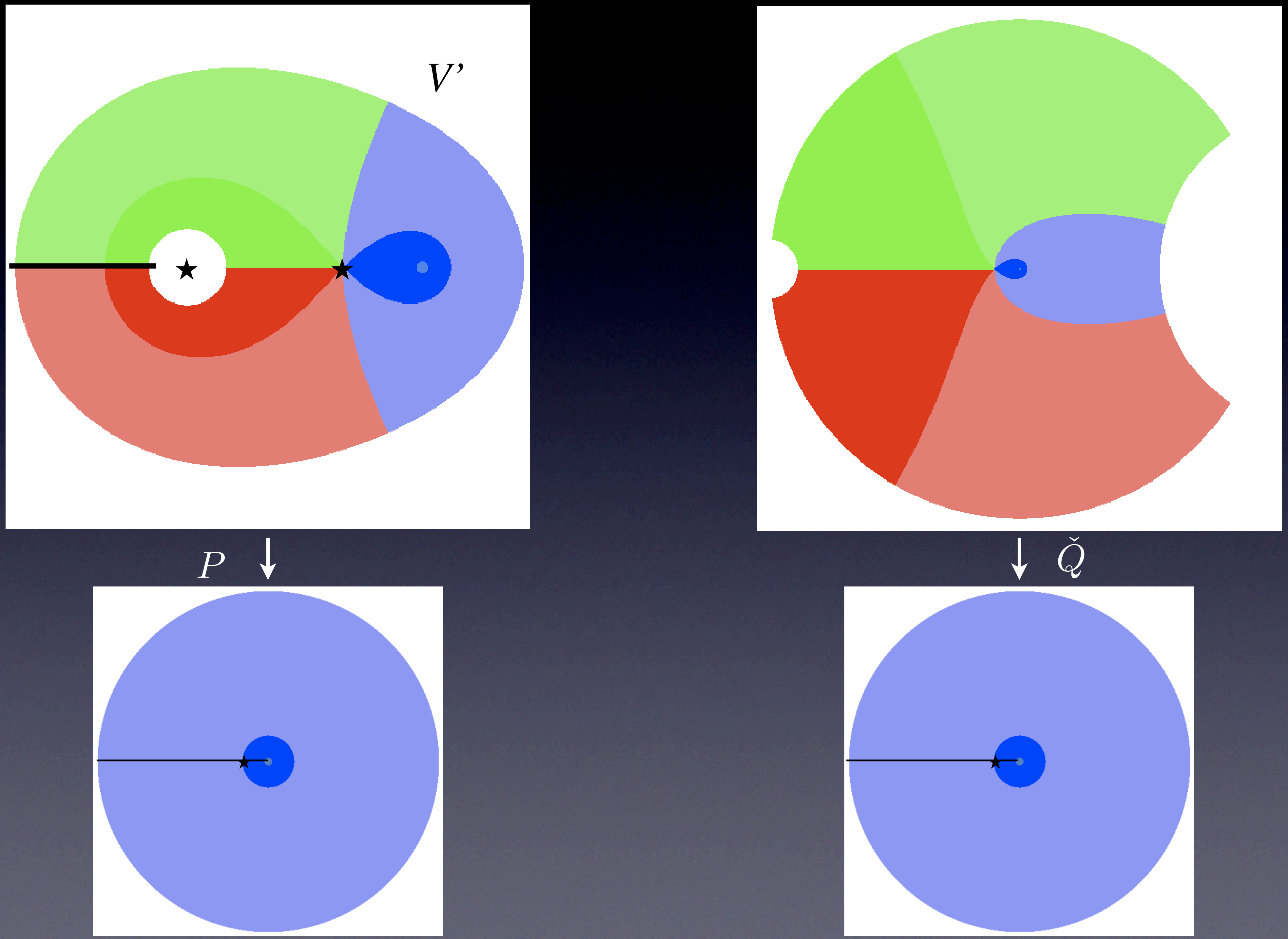


0 = the fixed point
∞ = designated omitted point
the critical value

$$g = f \circ \varphi^{-1}$$

We will characterize $f \in \mathcal{F}_1$ by color-tiling the domains (V replaced by V')

exercise: $\check{Q}(z) = \dfrac{z(1+z)^4}{(1-z)^6} \in \mathcal{F}_1'$ after rescaling (V is replaced by V' for $\mathcal{F}_1'$)



Before the proof...

send the parabolic fixed point to ∞

dynamics is already close to a translation

Fatou coordinates will be "close" to the identity or affine transformation

we must handle $f = P \circ \varphi^{-1}$ with arbitrary univalent function φ

it is easier to work with univalent functions in $\mathbb{C} \setminus \overline{\mathbb{D}}$ (Area theorem etc)

open the slit $(-\infty, -1]$ to the unit disk and obtain $Q(z) = z \dfrac{\left(1+\frac{1}{z}\right)^6}{\left(1-\frac{1}{z}\right)^4}$ with the same covering property

$Q = \psi_0^{-1} \circ P \circ \psi_1$ where $\psi_0(z) = -\dfrac{4}{z},\ \psi_1(z) = -\dfrac{4z}{(1+z)^2} = 4 f_{Koebe}\left(-\dfrac{1}{z}\right)$

We will work with $f \in \mathcal{F}_1^Q$ instead of $\mathcal{F}_1$

$$\mathcal{F}_1^Q = \left\{ f = Q \circ \varphi^{-1} : \varphi(V) \to \widehat{\mathbb{C}} \;\middle|\; \begin{array}{c} \varphi : \widehat{\mathbb{C}} \setminus E \to \widehat{\mathbb{C}} \text{ is univalent} \\ \varphi(\infty) = \infty,\ \lim_{z\to\infty} \frac{\varphi(z)}{z} = 1 \\ \text{and } 0 \notin Image(\varphi) \end{array} \right\}$$

$$E = \left\{ x + iy \in \mathbb{C} : \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \le 1 \right\} \qquad V = \psi_1(\widehat{\mathbb{C}} \setminus E)$$

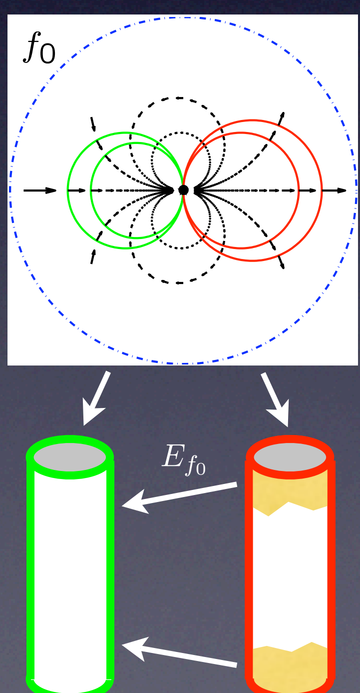
How to see that $\mathcal{R}_0 f$ is in $\mathcal{F}_1'$

$\mathcal{R}_0 f$ was defined via Horn map E_f and $\mathrm{Exp}^\sharp(z) = e^{2\pi i z}$

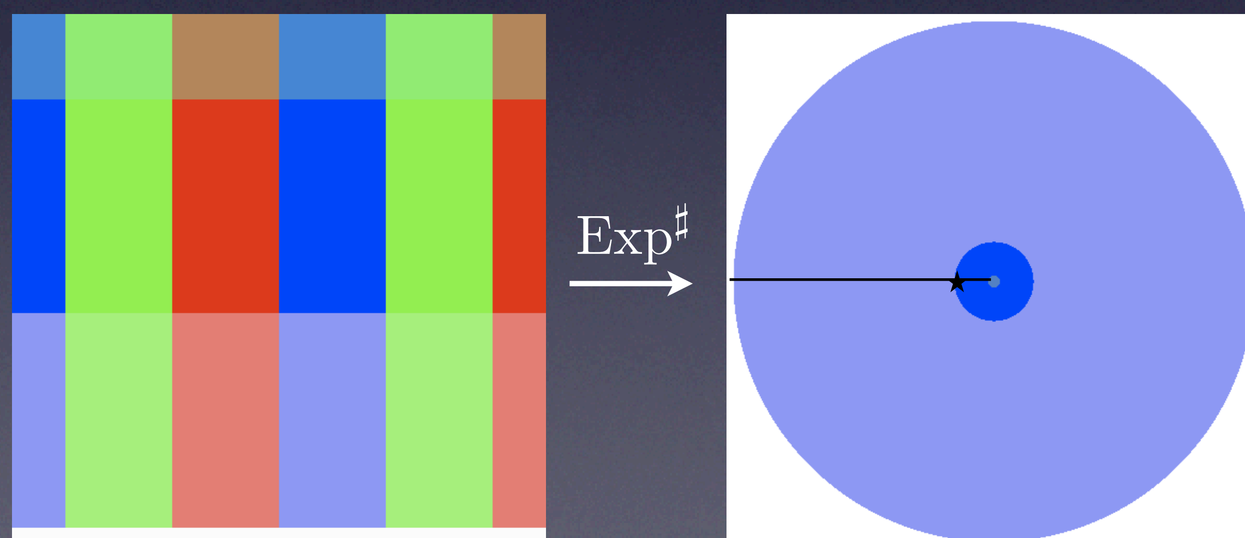
$$\mathcal{R}_0 f = \mathrm{Exp}^\sharp \circ E_f \circ (\mathrm{Exp}^\sharp)^{-1}$$

For E_f, domain = repelling Fatou coordinate
range = attracting Fatou coordinate

Make a color-tiling according to the range (= attracting Fatou coordinate) and compare with that of P

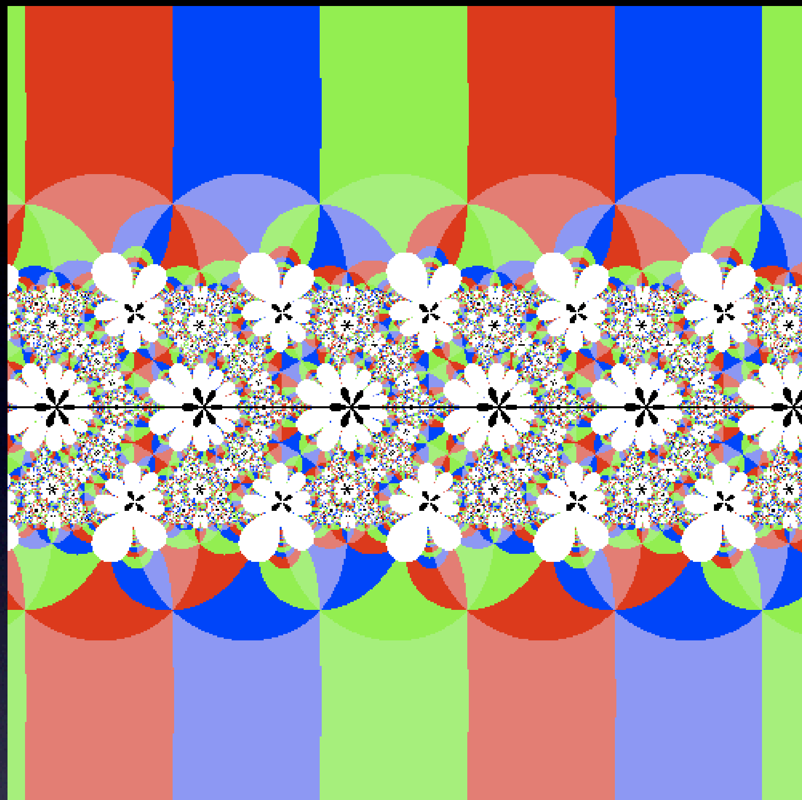


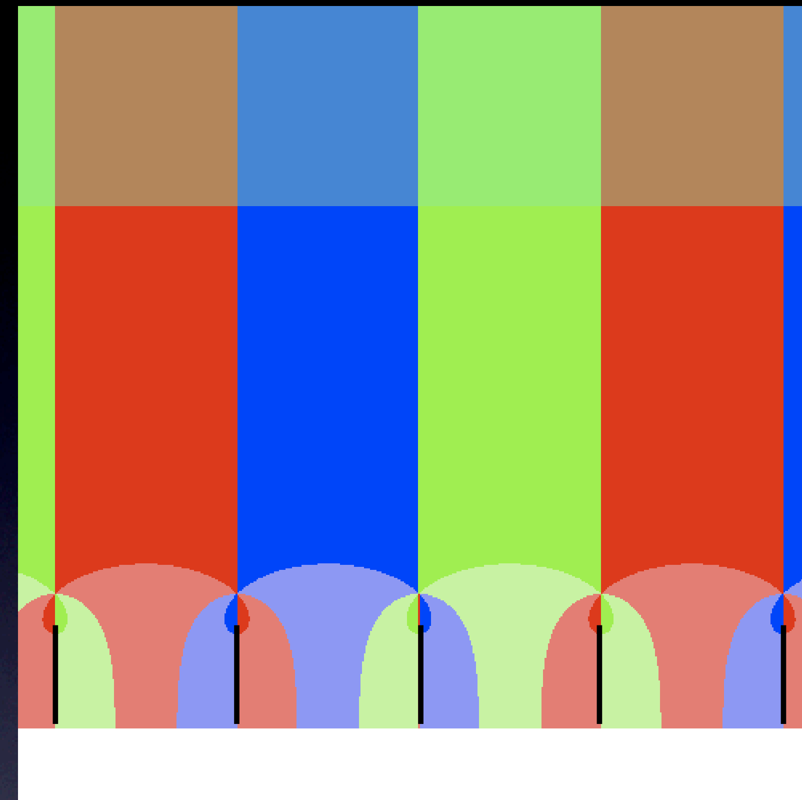
"checkerboard picture"



Compare $\mathcal{R}_0 f$ with P or its log lift

repelling side

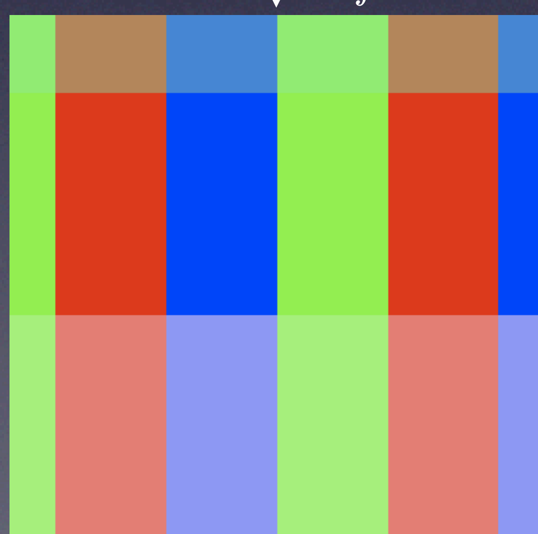


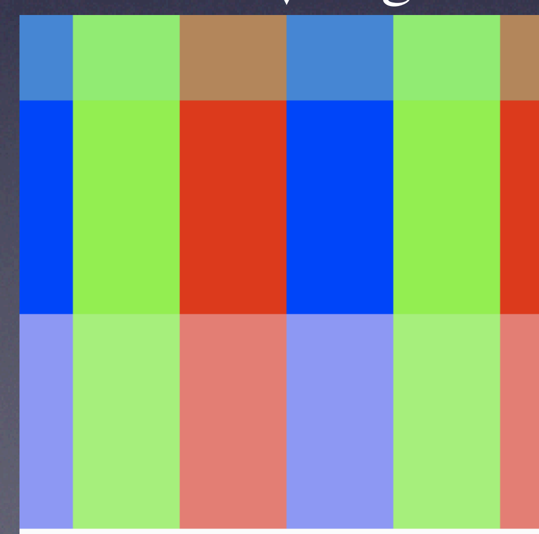


$P \downarrow E_f$

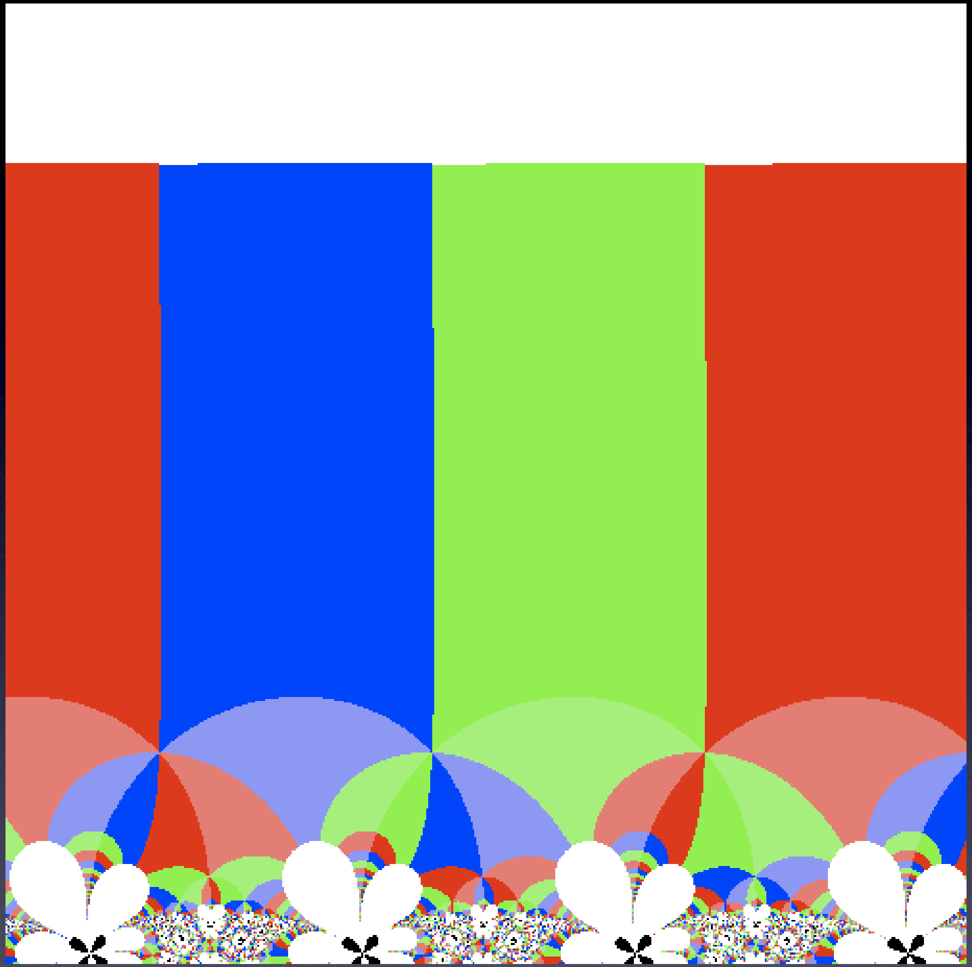
$\downarrow$ log lift of P

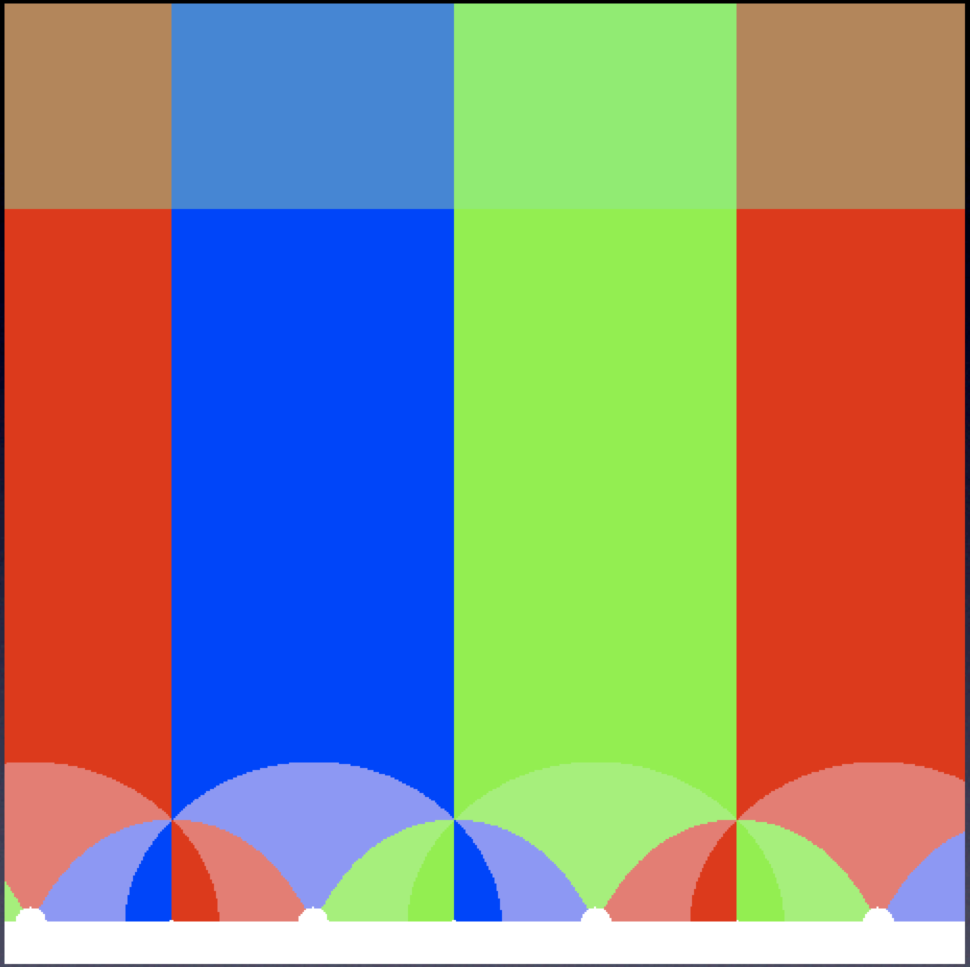
attracting side





Compare $\mathcal{R}_0 f$ with log lift of Q

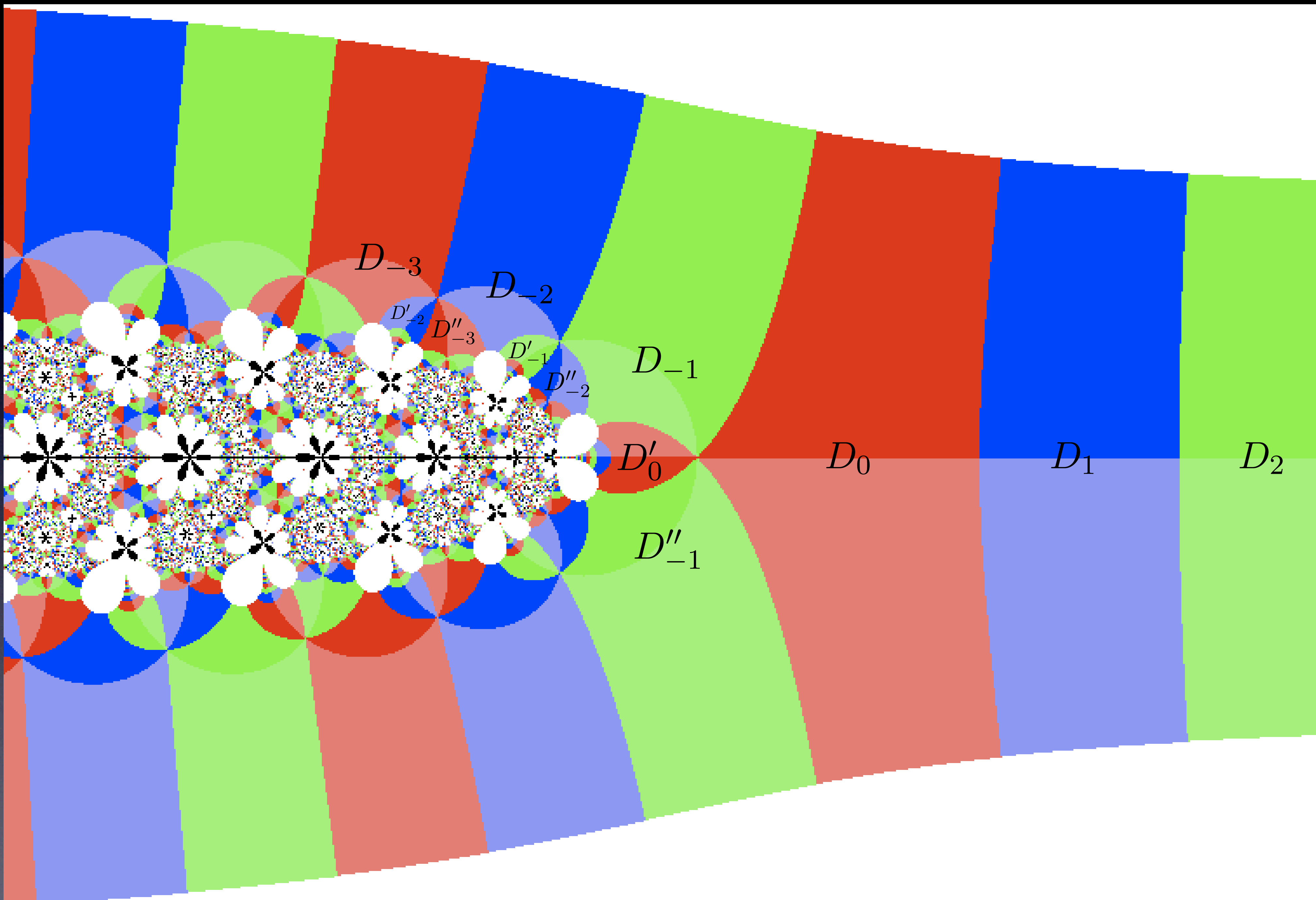




This is the starting point of the proof

Danger: inverse orbits may fall off from the domain of definition

More details: Checkerboard picture for Q



What we need to do

Left

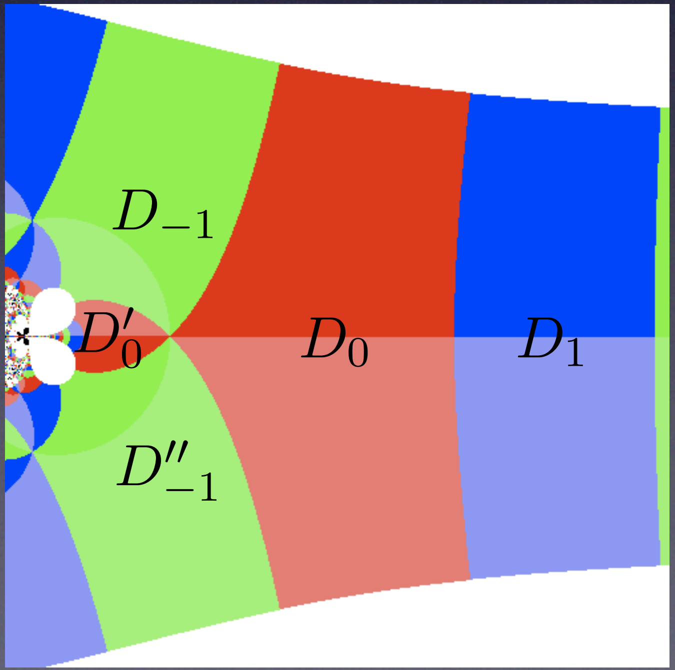
guarantee that certain inverse images arrive in the domain of repelling Fatou coordinate

construct a Riemann surface X on which an appropriate inverse branch of f can be lifted

Middle

take multiple inverse images of D_1 (D_0, D'_0, D_{-1}, D''_{-1}) and bound their location

they are glued together like the tiles for P



Right

distortion estimates for attracting Fatou coordinate

bound the location and shape of D_1

determine the domain where the attracting Fatou coord. is univalent and apply Golusin inequality

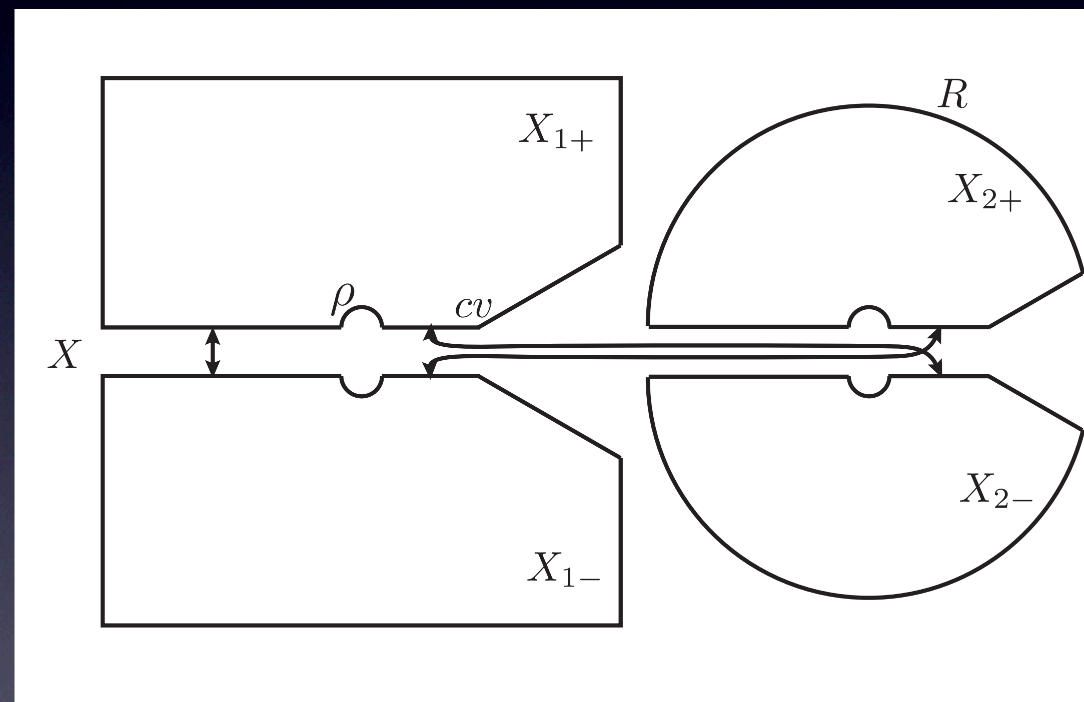
many inequalities (~30) needed to checked with help of computers

Left: repelling side

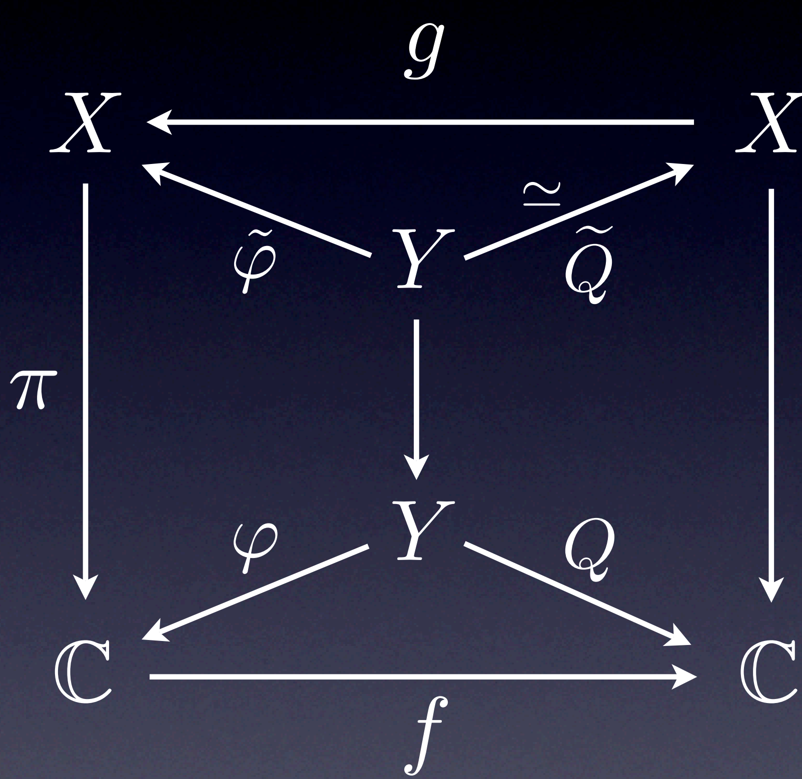
— Safety net —

Construct a Riemann surface X
where an appropriate inverse branch is defined

$$\begin{array}{ccc} X & \xleftarrow{\;g\;} & X \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{C} & \xrightarrow{\;f\;} & \mathbb{C} \end{array}$$



We can safely iterate a specific inverse branch of f


g
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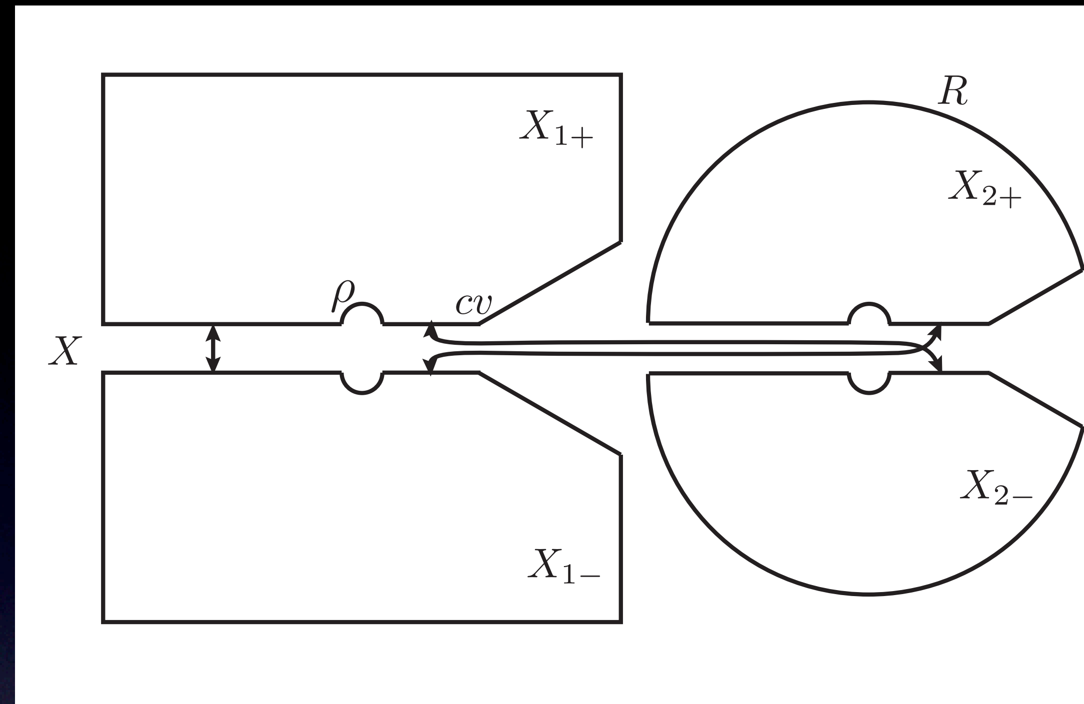
R
X_{1+}
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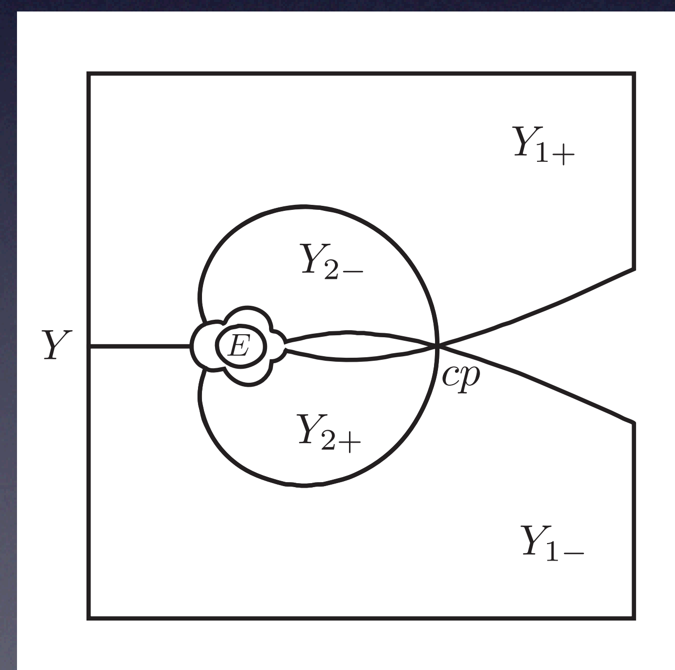
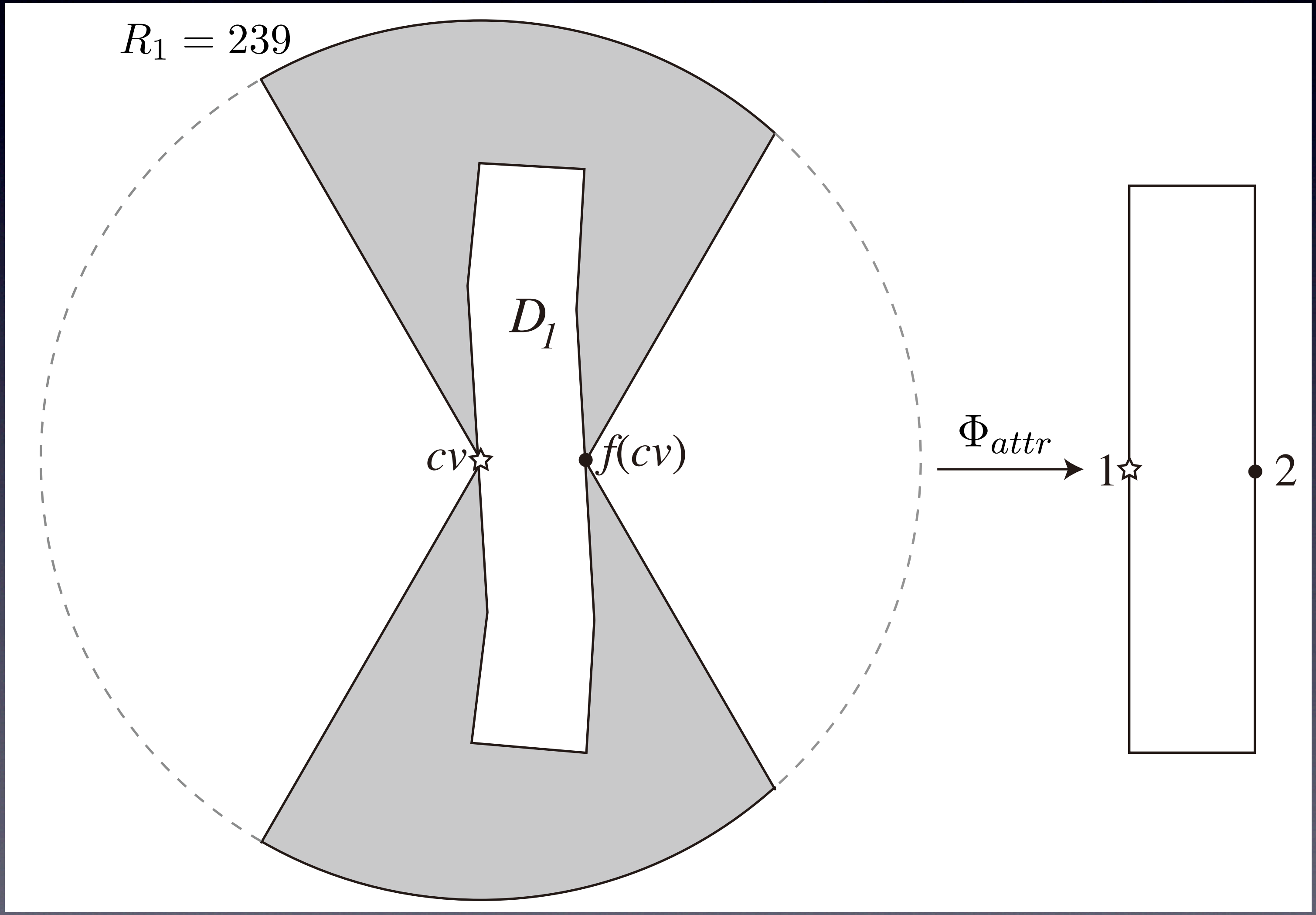

Y_{1+}
Y_{2-}
Y
E
cp
Y_{2+}
Y_{1-}

Right: attracting side

Normalization $\Phi_{attr}(cv) = 1$

Bound the shape and location of

$$D_1 = \Phi_{attr}^{-1}(\{z : 1 < \operatorname{Re} z < 2,\ |\operatorname{Im} z| < \eta = 1\})$$



For the estimate of D_1

Theorem (A consequence of Golusin inequalities). *Let Ω be a disk or a half plane in $\widehat{\mathbb{C}}$ (including the case of the complement of a closed disk). If $g : \Omega \to \widehat{\mathbb{C}}$ is a univalent holomorphic mapping, then for $z, \zeta \in \Omega$ with $z, \zeta, g(z), g(\zeta) \neq \infty$ and $z \neq \zeta$,*

$$\left| \log \frac{g'(z)g'(\zeta)(z-\zeta)^2}{(g(z)-g(\zeta))^2} \right| \leq 2 \log \cosh \frac{d_\Omega(z,\zeta)}{2}.$$

Theorem (A general estimate on Fatou coordinate). *Let Ω be a disk or a half plane and $f : \Omega \to \mathbb{C}$ a holomorphic function with $f(z) \neq z$. Suppose f has a univalent Fatou coordinate $\Phi : \Omega \to \mathbb{C}$, i.e., $\Phi(f(z)) = \Phi(z) + 1$ when $z, f(z) \in \Omega$. If $z \in \Omega$ and $f(z) \in \Omega$, then*

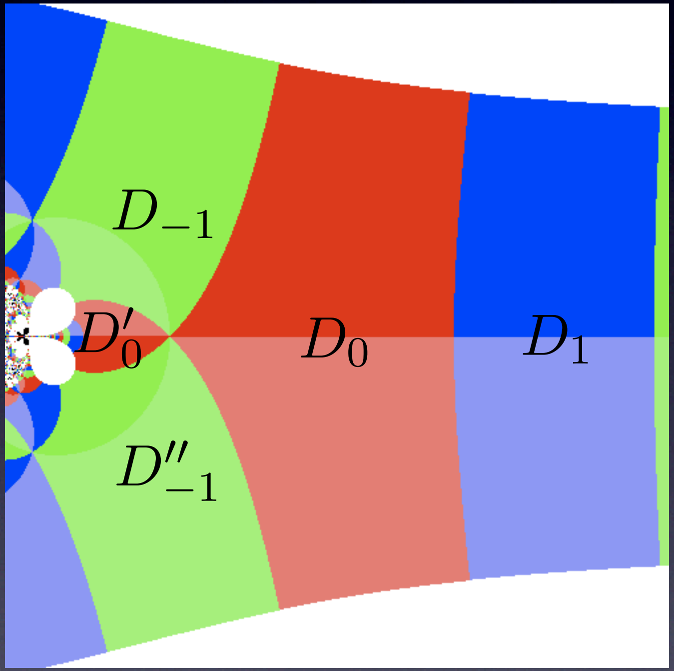
$$\left| \log \Phi'(z) + \log(f(z) - z) - \frac{1}{2} \log f'(z) \right| \leq \log \cosh \frac{d_\Omega(z, f(z))}{2} = \frac{1}{2} \log \frac{1}{1-r^2},$$

where r is a real number such that $0 \leq r < 1$ and $d_{\mathbb{D}}(0,r) = d_\Omega(z, f(z))$.

Middle

Just control so that

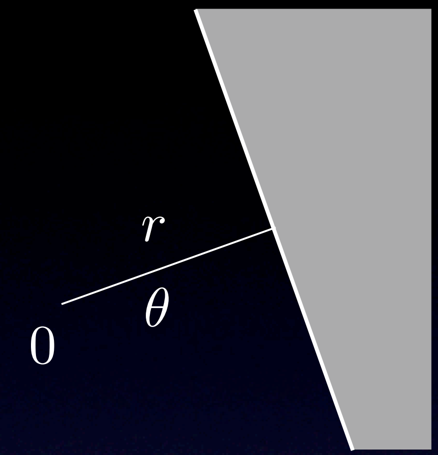
$D_0 \quad D'_0 \quad D_{-1} \quad D''_{-1}$ can be lifted to X



Then the horn map has the desired covering property

More about estimates

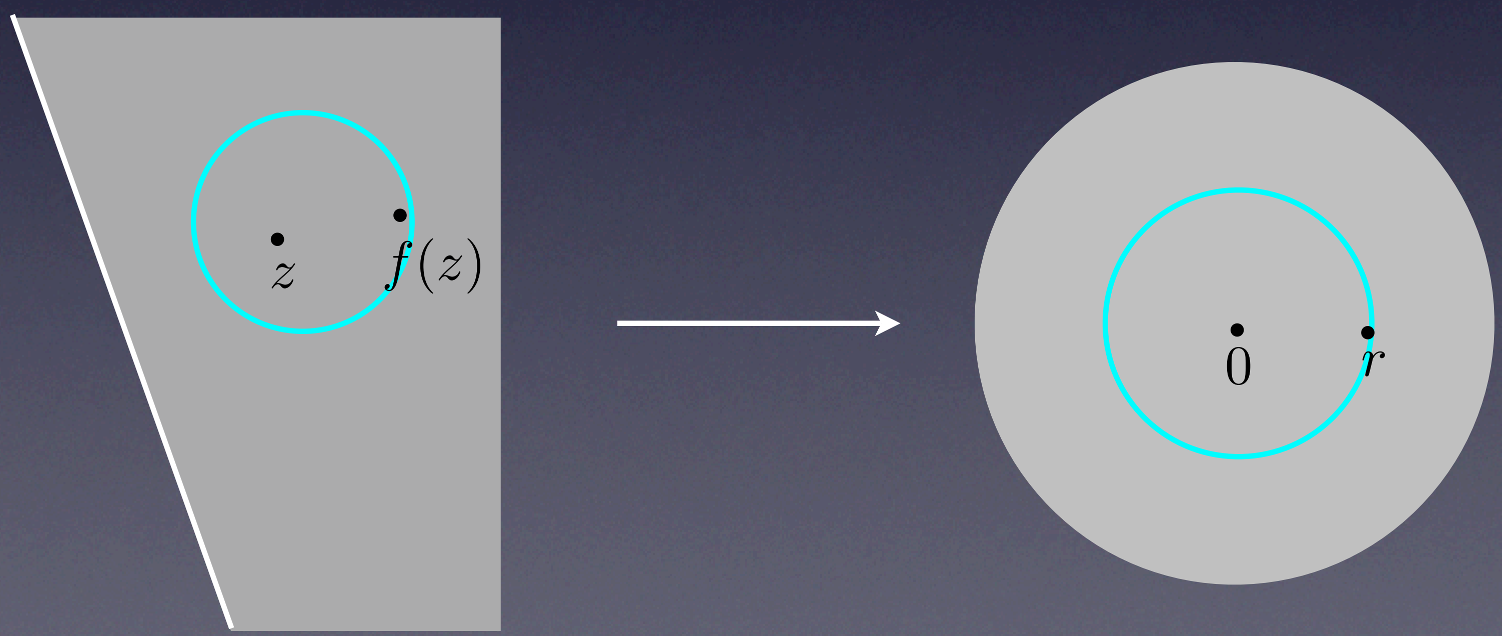
$$Q(\zeta) = \zeta + 10 + \frac{49}{\zeta} + \frac{160}{(\zeta-1)^2} + \frac{80\,\zeta + 32 - \frac{48}{\zeta}}{(\zeta-1)^4}$$



$$\mathrm{Re}(ze^{-i\theta}) > r$$

$$\Rightarrow \quad \left|\frac{1}{z}\right| \leq \frac{1}{r}$$

$$\frac{1}{z} \in \mathbb{D}\left(\frac{e^{-i\theta}}{2r}, \frac{1}{2r}\right)$$



Univalent functions

$g : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}} \setminus \{0\}$ **univalent,** $g(\infty) = \infty$ $\lim_{z\to\infty} \frac{g(z)}{z} = 1$

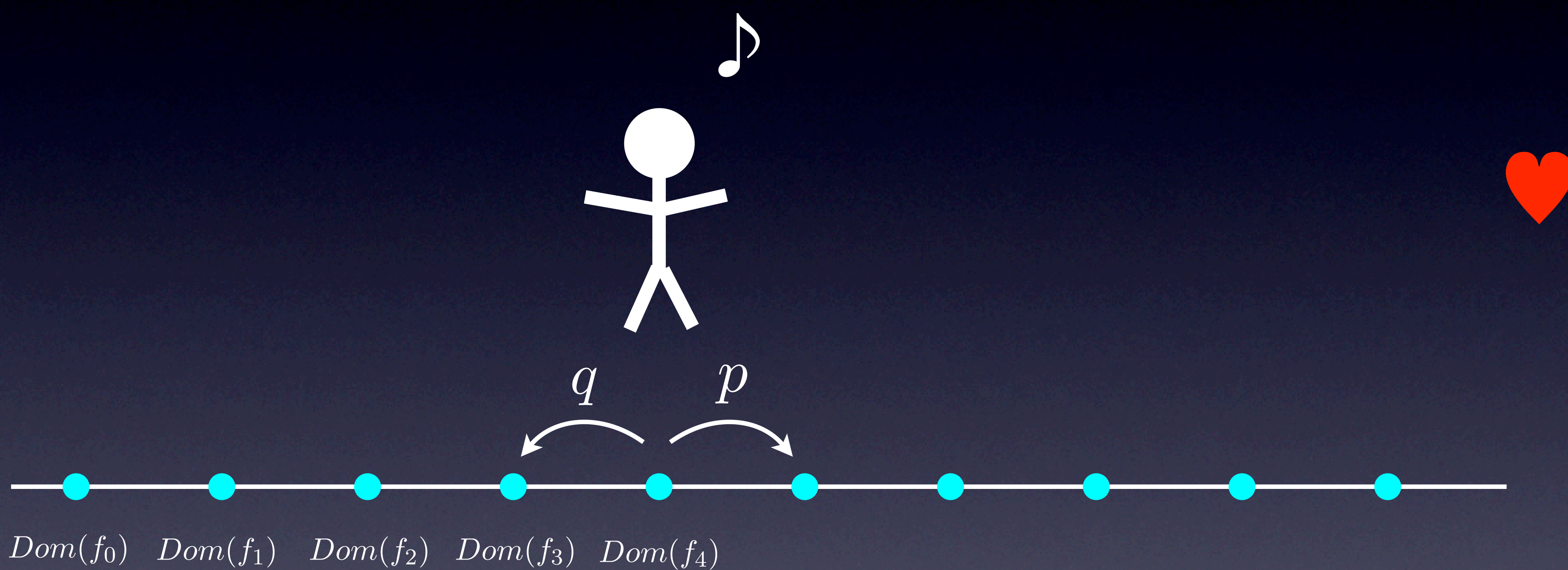
$g(z) = z + c_0 + g_1(z)$ $(\lim_{z\to\infty} g_1(z) = 0)$

then $|c_0| \le 2$

	$\lvert z\rvert \ge 1.25$	$\lvert z\rvert \ge 5$	$\lvert z\rvert \ge 10$
$\lvert g_1(z)\rvert \le$	$1.01\ldots$	$0.202\ldots$	$0.100\ldots$
$\lvert \log g'(z)\rvert \le$	$1.02\ldots$	$0.040\ldots$	$0.010\ldots$
$\lvert g(z)\rvert \ge$	0.05	3.2	8.1
$\arg \frac{g(z)}{z} \le$	$2.19\ldots$	$0.40\ldots$	$0.20\ldots$

PROBLEM: BY-PASS des disques de Siegel

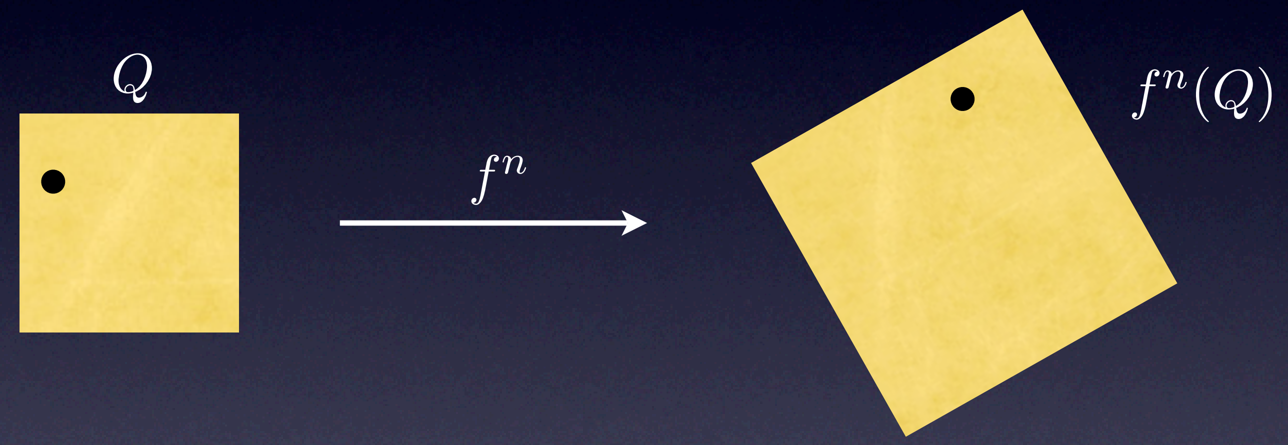
Program: Random Walk Model



$$\text{if } p > \tfrac{1}{2} > q \text{ then } \quad Prob(X_n > 0 \ (n \geq 1) \ \text{ and } \ \lim_{n \to \infty} X_n = +\infty) > 0$$

Lemma (Primitive Vitali Lemma). *$U \subset \mathbb{C}$ bounded open, $E \subset U$ measurable. If for any $x \in E$ there exists a dyadic square $Q \ni x$ contained in U such that $density_Q(E) < \lambda$, then*

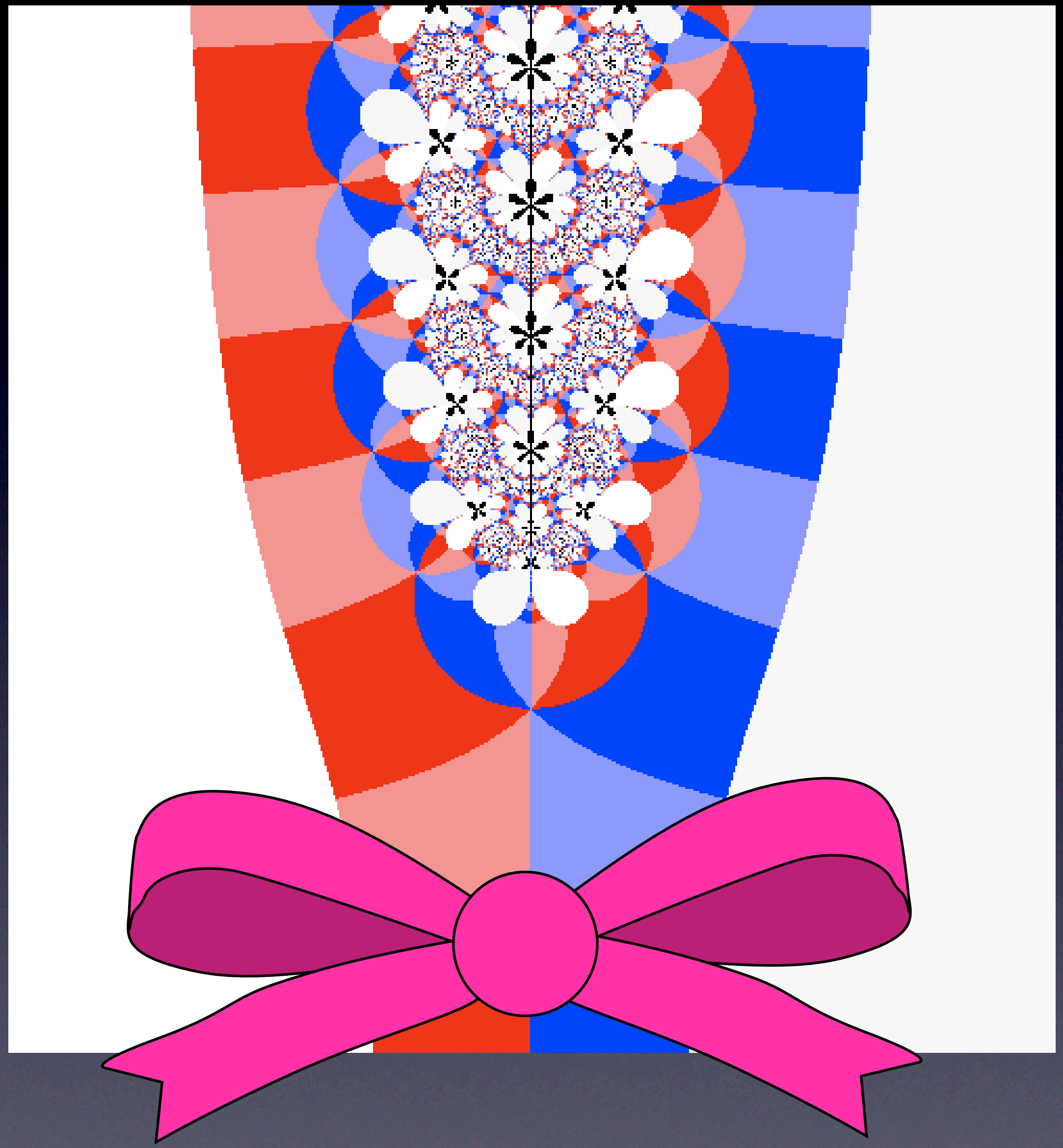
$$density_X(E) < \lambda.$$

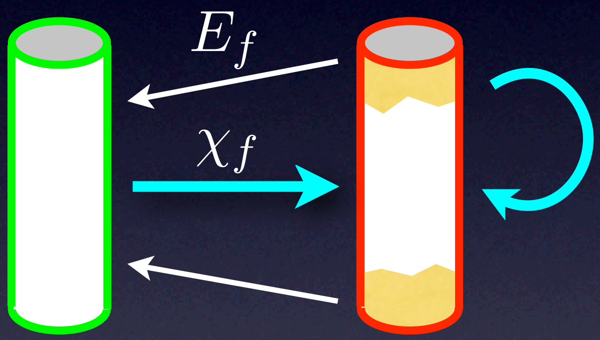


independence + $p > q$:

$$P(n\text{-th step} = +1 | \text{ itinerary up to } n-1) > \frac{1}{2}$$

Joyeux anniversaire!
Adrein.





Merci!

A draft is available at
http://www.math.kyoto-u.ac.jp/~mitsu/pararenorm/

Tomorrow at IHP:
Hyperbolicity of near-parabolic renormalization