Lyapounov exponents and meromorphic maps

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Lyapunov exponents and meromorphic maps

$(X,\omega)$ compact Kähler manifold of dimension $k$.

$f : X \to X$ dominating meromorphic map

$I_f = \text{the indeterminacy set of } f.$

$C_f = \text{the critical set of } f.$

I) Dynamical quantities:

10) Dynamical degrees:

(Russakovskii-Schiffman)

for $x \in \mathbb{C} P^k$

$\omega^e = \underbrace{\omega \wedge \ldots \wedge \omega}_{\text{times}}$

$0 \leq e \leq k = \dim X$

$f^* (\omega^e)$ form with $L^2_{\text{loc}}$ coefficients

$\text{Sel}(f) = \int f^* (\omega^e) \wedge \omega^{k-e}$

$\text{deg}_e = \lim_{n \to \infty} \left( \text{Sel}(f^n) \right)^{1/n} \in \text{this limit exists}$

(Dinh-Sibony)

$e^{\text{th}}$ dynamical degree
\( q \to \log dq \) is concave.

It implies that the dynamical degrees look like:

\[
\begin{align*}
  d_0 < d_1 < \ldots < d_s \geq d_{s+1} \geq \ldots \geq d_k
\end{align*}
\]

Examples:

- \( f \) holomorphic endomorphism of \( \mathbb{C}P^k \) of degree \( d \geq 2 \)
  \[
  d_0 = 1 < d_1 < d_2 = d^2 < \ldots < d_k = d^k
  
  \]

- \( f \) birational map of \( \mathbb{C}P^2 \) of degree \( d \geq 2 \)
  \( f \) algebraically stable
  \[
  d_0 = 1 < d_1 = d > d_2 = 1
  \]
2°) Entropy:

a) Topological entropy:

\[ d_n(x,y) := \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y)) \]

F is a \((n, s)\)-separated set if

\[ \forall x, y \in F, x \neq y \Rightarrow d_n(x,y) \geq s. \]

\[ \begin{array}{cccc}
  & f^{-1}(x) & f^0(x) & f^{n-1}(x) \\
  x & \times & \times & \times \\
  y & \times & \times & \times \\
  f^i(y) & \times & \times & \times \\
  f^{i-1}(y) & \times & \times & \times \\
  & f^i(y) & f^i(y) & f^i(y) \\
\end{array} \]

\[ h_{\text{top}}(f) := \sup_{s > 0} \lim_{n \to \infty} \frac{1}{n} \log \max \{ \text{Card } F, F \text{ \((n,s)\)-separated set} \} \]

\[ h_{\text{top}}(f) \leq \max_{0 \leq p \leq s} \log d_p \text{ dynamical degrees} \]

Th (Gromov: holomorphic case
Dinh-Sibony: meromorphic case)

h_{\text{top}}(f) \leq \max_{0 \leq p \leq s} \log d_p \text{ dynamical degrees}
b) Metric entropy:

\[ \mu \text{ measure } \quad \mu(\mathcal{I}_f) = 0 \]
\[ \mathcal{I}_f \mu = \mu \quad (\mu \text{ is invariant}) \]

\[ B_n(x, s) = \text{ball with center } x \text{ and radius } s \text{ for the metric } d_n \]

\[ h_\mu(f) : = \sup_{s > 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu B_n(x, s) \]

metric entropy

\[ x \text{ generic for } \mu \quad (\text{Brin-Katok}) \]

Fact: \( h_\mu(f) \leq \text{top } h(f) \).
II) Meromorphic maps:

$f: X \to X$ dominating meromorphic map

$(X,\omega)$ compact Kähler manifold dimension $k$

$d_0 \leq d_1 \leq \cdots \leq d_s < d_s > d_{s+1} \geq \cdots \geq d_k$

Example: holomorphic endomorphisms of $\mathbb{CP}^k$

of degree $d \geq 2$

$s = k$ in this case.

We consider a measure $\mu$ invariant, ergodic

with $\log d\mu \perp \text{If} \in L^1(\mu)$

Also

$\mathbb{P} \uparrow$

necessary condition

for the existence

of the Lyapounov exponents

$t > x_1 \geq \cdots \geq x_k > -\infty$

$R_k: h_{\rho}(f) \leq h_{\text{top}}(f) \leq \log ds$

Oin. Sibony
Th (5.1)
If \( h_\mu (f) \geq \max \{ \log d_{s-1}, \log d_{s+1} \} \) (or \( h_\mu (f) \geq \max \log d_{k-1} \) if \( s = k \))

Then:
\[
X_1 \geq \ldots \geq X_s \geq \frac{1}{2} (h_\mu(f) - \log d_{s-1}) > 0
\]
\[
0 > \frac{1}{2} (\log d_{s+1} - h_\mu(f)) \geq X_{s+1} \geq \ldots \geq X_k
\]

In particular \( \mu \) is hyperbolic.

\( X_1 \geq \ldots \geq X_s > 0 \) \( \sim \) \( s \) directions with expansion
\( 0 \geq X_{s+1} \geq \ldots \geq X_k \) \( \sim \) \( k-s \) directions with contraction.

Cor:
If \( h_\mu (f) = \log d_s \)

\[
X_1 \geq \ldots \geq X_s \geq \frac{1}{2} \log \frac{d_s}{ds-1} > 0
\]

\[
0 > \frac{1}{2} \log \frac{d_{s+1}}{d_s} \geq X_{s+1} \geq \ldots \geq X_k
\]

\( \text{Th.} \) These bounds are sharp.
Examples of dynamical systems and measure $\mu$ which satisfy the hypothesis of the corollary:

- Holomorphic endomorphisms of $\mathbb{C}^k$ of degree $d \geq 2$
  
  $d_0 = 1 < d_1 = d < \cdots < d_k = d^k$

$\mu$ is Green measure (Fornaess - Sibony)

$\mu$ is invariant and ergodic, $\log d(x, \xi^g) \in L^1(\mu)$

$\alpha$ is a generic point

$$\frac{1}{d^{kn}} \sum \delta_{a_i} \to \mu$$

$\frac{d^k}{2}$ = topological degree of $f$

Dinh - Sibony

Bridge - Duval

The Gromov / Miocorencz - Przytycki

$h_\mu (f) = k \log d = \log d^k = \log d_k$.

We can apply the corollary

$X_1 \geq \cdots \geq X_k \geq \frac{1}{2} \log \frac{d^k}{d^{k-1}} = \frac{1}{2} \log d$

As we found the Bridge - Duval's inequality.
X projective manifold with dimension k

\[ f : x \rightarrow x \] dominating meromorphic map with \( dk > d_{k-1} \geq \ldots \geq d_0 = 1 \)

Guedj constructed an invariant ergodic measure \( \mu \) with \( \log d(x, x) \in L^2(\mu) \)
and \( h_\mu(f) = \log dk \).

Corollary \( k = 1 \Rightarrow x_1 \geq \ldots \geq x_k \geq \frac{1}{2} \log \frac{dk}{d_{k-1}} > 0 \)

\( \Rightarrow \) this is the Guedj's inequality.

\( f \) regular birational map of \( \mathbb{C}^{1 \times k} \)
(Dinh - Sibony)

They constructed an invariant ergodic measure \( \mu \) with \( \log d(x, x) \in L^1(\mu) \)
and with maximal entropy

\( \Rightarrow \) \( \mu \) is hyperbolic.

Cor.

Other examples: holomorphic automorphisms in Kähler manifolds (Dinh - Sibony) \ldots
III) A general inequality:

\( d_1 \leq d_2 \leq \ldots \leq d_s \leq \ldots \leq d_k \)

Th (D.)

Let \( \mu \) be an invariant ergodic measure

\[ \log d(x, \Omega) \in L^2(\mu). \]

\( x_1 \geq \ldots \geq x_k \) the Lyapunov exponents

Fix \( s \) \( 1 \leq s \leq k \) and we define

\[ e = e(s) \text{ with:} \]

\[ e' = e'(s) \]

\( x_1 \geq \ldots \geq x_{s-1} > x_s = \ldots = x_s = \ldots = x_{s+e'} > x_{s+e'+1} \geq \ldots \geq x_k \)

( with \( s - e' = 1 \) if \( x_1 = \ldots = x_s 

\( s + e' = k \) if \( x_s = \ldots = x_k \) )

Then

\[ h_\mu(f) \leq \max_{0 \leq q \leq s-1} \log d_q + 2X_{s-p}^+ + \ldots + 2X_k^+ \]

\[ X_c^+ = \max (x_i, 0) \]

\[ h_\mu(f) \leq \max_{s \leq q \leq k} \log d_q - 2X_1^- - \ldots - 2X_{s-e'}^- \]

\[ X_i^- = \min (x_i, 0). \]
Corollary 1:
Let \( \mu \) be an invariant ergodic measure
\[ \log d(x, \mathcal{F}) \leq \lambda^{1/\mu} \]
\[ h_\mu (f) \leq 2x_i^+ + \ldots + 2x_k^+ \]
(Ruelle's inequality)

Proof: take \( s = 1 \) in the first inequality
\[ d_0 = 1. \]

Corollary 2:
Same hypothesis
\[ h_\mu (f) \leq \log d_k^{1/\mu} - 2x_i^- - \ldots - 2x_k^- \]
Topological degree

"Inverse Ruelle's inequality"

Proof: set \( s = k \) in the second inequality.
Proof of the previous theorem

\[ d_0 = 1 \leq d_1 \leq \cdots \leq d_{s-1} < d_s > d_{s+1} \geq \cdots \geq d_k \]

measure \( \mu \) invariant ergodic

with \( \log d(x, \mathcal{E}) \in L^1(\mu) \)

\( h_{\mu}(f) > \max(\log d_{s-1}, \log d_{s+1}) \)

\[ \Rightarrow \mu \text{ is hyperbolic} \quad x_I \geq \cdots \geq x_S > 0 \]

\[ 0 > x_{S+1} \geq \cdots \geq x_k \]

**Proof:**

Suppose \( x_S \leq 0 \)

1st formula:

\[ x_{s-1}^+ = \cdots = x_S^+ = 0 = \cdots = x_k^+ \]

because \( 0 \geq x_{s-1} \geq \cdots \geq x_S \geq \cdots \geq x_k \)

\[ \log d_{s-1} < h_{\mu}(f) \leq \log d_{s-1} \leq \log d_{s-1} \leq \log d_{s-1} \]

\( \sim \) contradiction.

\[ \Rightarrow x_S > 0 \]

By using the second formula with \( s = s+1 \)

\[ \Rightarrow x_{s+1} < 0. \]
IV. Ideas for the proof of the inequalities:

An easier case:

\( X = CIP^2 \)

\( X_1 \geq X_2 \) we suppose \( X_1 > X_2 \) \( S = 2 \)

the first inequality becomes \( l = 0 \)

\[ h_\mu(f) \leq \log d_1 + 2x_2^+ \]

\[ h_\mu(f) = \lim_{s \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu B_n(x,s) \]

\[ \mu B_n(x,s) \geq e^{-h_\mu \ln x} \text{ generic for } \mu \]

we can find \( x_1, \ldots, x_n \in X \)

with \( N \geq e^{h_\mu \ln n} d_n(x_i, x_j) \geq S \) if \( i \neq j \)

and \( x_i = \) good points for Pesin's theory.

\( B_n(x_1, s) \)

\( B_n(x_2, s) \)

\( \ldots \)

\( B_n(x_n, s) \)
\( x_1 > x_2 \) so through each \( x_i \) we can construct an "approximate stable manifold"

\[ \text{diameter } f^n(\text{Vol}(W)) \leq \frac{s}{4} \]

\[ f_{-1}(x_1), f(x_1), \ldots, f^n(x_1) \]

To realize that, we use the graph transform.

\( x = x_i \)

\( x \quad \text{graph} \quad f(x) \quad \text{graph over the } \) "\( x_2 \)" direction

\( f^n(x_1) \quad f(x_1) \)

\( x_1 > x_2 \)

If \( x_2 \leq 0 \) → we do a cut-off.
We keep the part in a box of size \( \frac{s}{4} \).

If \( x_2 > 0 \) → we keep all.

\[ \text{side } \frac{s}{4} e^{-x_2} \]

\[ \text{diameter } 4 \]
We start again this process

\[ x \in \text{Is}(x_1) \]

\[
\text{size diameter} \geq \frac{3}{4} \text{ if } x_2 \leq 0 \\
\frac{8}{\pi} e^{-x_2} \text{ if } x_2 > 0
\]

area \geq e^{-2x_2^+ n} c(\delta^1 x_2^+ = \max(x_0)}

So we have the approximate stable manifold which have area \geq e^{-2x_2^+ n} c(\delta^1}

\[ \Pi \]

\[ \text{projection on N} \]

\[ \text{chart in} \ C \mathbb{P}^2 \]

The area of \( \Pi(\bigcap_{i=1}^{n} \text{Is}(x_i)) \geq e^{-\text{hyp}(1-2x_2^+ n)} \).

if \( a_n \leq d_1 \)

\[ \Rightarrow \log d_1 \geq \text{hyp}(1-2x_2^+) \rightarrow \text{OK.} \]
We can find a line $L$ with
\[ \# \{ \ell \in \mathcal{L} \mid \text{w} \leq x^{2+} \} \geq \frac{n}{2} \]

**Fundamental remark:** Buzzi / Newhouse

\[ d_n(y_i, y_j) \geq S/2 \]

The diameter of $f_e(\text{w}^s(x, 1)) \leq S/4$

\[ e = 0, \ldots, n-1 \]

and $d_n(x_i, x_j) \geq S + \exists e \in \{0, \ldots, n-1\}$

\[ d(f_e(x_i, 1), f_e(x_j)) \geq S \]
\[ f(\xi_1) \quad f(\xi_2) \quad f(\xi_3) \quad f(\xi_4) \]

\[ \leq \frac{c}{2} \]

\[ d\left( f(\xi_1), f(\xi_2) \right) \leq \frac{c}{2} \]

\[ \text{we proved} \quad c \leq 1 - x \to n - 2x^2 + n \]

\[ \leq \max_{0 \leq q \leq k} \log dq \]

\[ X \to \mathbb{L} \]