Near-parabolic Renormalization and Rigidity

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Irrationally indifferent fixed points

We consider holomorphic functions of one variable with fixed point \( z=0 \).

\[
f(z) = \lambda z + a_2 z^2 + \ldots
\]

If \( |\lambda| = 1 \), \( z=0 \) is called \textit{indifferent fixed point}.

If \( \lambda \) is a root of unity, \textit{parabolic}; otherwise \textit{irrationally indifferent}.

\[
\lambda = e^{2\pi i \alpha} \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}
\]

If conjugate to a rotation (linearizable), then it has a Siegel disk.

Otherwise, very complicated invariant sets (hedgehogs).

Earlier works: Siegel, Bruno, Herman, Yoccoz, Perez Marco, Petersen, McMullen, Buff, Chéritat, ...

Consider

\[
f_0(z) = z + a_2 z^2 + O(z^3) \quad a_2 \neq 0
\]

and its perturbation

\[
f(z) = e^{2\pi i \alpha} z + a_2 z^2 + \ldots
\]
Linearizability of irrationally indifferent fixed points

Siegel (under Diophantine cond.), Bruno (under Bruno condition),
Yoccoz (a new proof using renormalization and converse);
Cremer (nonlinearizable ex.)

Boundary of Siegel disks (Jordan curve in known cases)

Herman (quadratic polynomial, bounded type rotation number => J. curve)
Petersen (quad. poly., bounded type => locally connected J, measure 0)
Herman-Yoccoz, Petersen-Zackeri (weaker cond. for J. curve w. crit. pt.)
Herman (quadratic polynomial, no critical point on bdry)
Buff-Cheritat (various smoothness)

Universality/Rigidity at the boundary

Manton-Nauenberg (experiments, heuristic argument)
McMullen (quadratic-like map => rigidity and differentiability)
This talk (a new class, high type rotation number
  => rigidity and differentiability)
Physicists’ motivation

KAM torus \[\rightarrow\] Chaos

last KAM torus destroyed?

Physicists expect a “universal phenomenon” at critical parameter

Simpler model (no parameter, only in the phase space)

Irrationally indifferent fixed point

\[f(z) = e^{2\pi i \alpha} z + a_2 z^2 + \ldots\] holomorphic near 0

Boundary of Siegel Disk is the closure of critical orbit (for polynomials)

Physicists expect a “universal phenomenon” at the boundary of SD

Manton-Nauenberg (physicists), McMullen (for bounded type)

\[f(z) = e^{2\pi i \alpha} z + a_2 z^2 + \ldots\]
**Theorem** (McMullen). Let \( f \) and \( \hat{f} \) be quadratic-like maps with Siegel disks of period one with the same rotation number \( \alpha \) of bounded type. Then \( f \) and \( \hat{f} \) are conjugate by a quasiconformal mapping \( \varphi \) which is \( C^{1+\gamma} \)-conformal on the boundary of the Siegel disk, i.e.

\[
\varphi(z) = \varphi(z_0) + A(z - z_0) + O(|z - z_0|^{1+\gamma}) \text{ as } z \to z_0
\]

where \( z_0 \) is on the boundary of the Siegel disk and \( A \) is a non-zero constant.

**Theorem.** Let \( f = e^{2\pi i \alpha} h \) and \( \hat{f} = e^{2\pi i \alpha} \hat{h} \) where \( h \) and \( \hat{h} \) are in the class \( \mathcal{F}_1 \) which will be defined later, and the rotation number \( \alpha \) is of high type \( (N) \) with sufficiently large \( N \) (also defined later). Then \( f \) and \( \hat{f} \) are asymptotically conformally conjugate on the closure of critical orbit. Moreover the conjugacy is \( C^{1+\gamma} \)-conformal on the critical orbit. Furthermore there exists \( 0 < \lambda < 1 \) such that if the continued fraction coefficients of \( \alpha \) satisfies \( a_n \leq C \lambda^n \) with some \( C > 0 \) then the conjugacy is \( C^{1+\gamma'} \)-conformal on the closure of the critical orbit.

**Remark.** The closure of critical orbit contains boundary of Siegel disk. The above theorem follows from Rigidity result (Theorem 5) via a differentiability result on quasiconformal mappings.
Differentiable functions

Graph looks like a line

In small scale...

- Homeomorphism: can do anything
- Quasi-symmetric, quasi-conformal: bounded ratio
- Asymptotically conformal: ratio -> 1

\[ C^{1+\alpha} : \text{ratio} \to 1 \text{ "fast"} \]

For conjugacies between dynamical systems...

- Compare orbits
- To see details, need to iterate many times
Return map

\[ \mathcal{R} f = (\text{first return map of } f) \text{ after rescaling} \]
\[ = g \circ f^k \circ g^{-1} \quad (\text{if return time } \equiv k) \]

**Renormalization**

high iterates of \( f \) \quad \leftrightarrow \quad \text{fewer iterates of } \mathcal{R} f

fine orbit structure for \( f \) \quad \leftrightarrow \quad \text{large scale orbit structure for } \mathcal{R} f

Successive construction of \( \mathcal{R} f, \mathcal{R}^2 f, \ldots \), helps to understand the dynamics of \( f \) (orbits, invariant sets, rigidity, bifurcation, \ldots)

If \( \mathcal{R} f = f \) (fixed point of renormalization),
then \( f = g \circ f^k \circ g^{-1} \) (fixed point equation)
Renormalization and Rigidity (an oversimplified view)

Suppose $f$ and $\tilde{f}$ have “the same combinatorial type” and admit successive construction of renormalizations.

\[ f_0 = f \]
\[ f_1 = R f_0 \]
\[ f_2 = R f_1 \]
\[ f_3 = R f_2 \]
\[ \tilde{f}_0 = \tilde{f} \]
\[ \tilde{f}_1 = R \tilde{f}_0 \]
\[ \tilde{f}_2 = R \tilde{f}_1 \]
\[ \tilde{f}_3 = R \tilde{f}_2 \]

\{ h_n \} “bounded” $\quad$ $f$ and $\tilde{f}$ quasi-conformally conjugate

\[ d(f_n, \tilde{f}_n) \to 0 \quad \Rightarrow \quad h_n \to \text{linear} \quad \Rightarrow \quad \text{conjugacy is asymptotically conformal or smooth, etc.} \]
Yoccoz renormalization for Siegel-Bruno Theorem

\[ f(z) = e^{2\pi i \alpha} z + \ldots, \quad \sum_n \frac{\log q_{n+1}}{q_n} < \infty \quad \text{where} \quad \frac{p_n}{q_n} \to \alpha \quad \text{(convergents)} \]

\[ \implies \quad f \text{ is conjugate to } z \mapsto e^{2\pi i \alpha} z \]

Yoccoz’s proof: construct the sequence of renormalizations \( f_n \)

\[ f_0 = f, \quad \alpha_0 = \alpha, \quad \alpha_{n+1} = \text{dist}\left(\frac{1}{\alpha_n}, \mathbb{Z}\right) \]

\[ f_n(z) = e^{2\pi i \alpha_n} z + \ldots \quad \sim \quad f_{n+1}(z) = e^{2\pi i \alpha_{n+1}} z + \ldots \]
Yoccoz renormalization for Siegel-Bruno Theorem

\[ f_n(z) = e^{2\pi i\alpha_n}z + \ldots \sim f_{n+1}(z) = e^{2\pi i\alpha_{n+1}}z + \ldots \]

Cylinder/Near-parabolic renormalization

\[ \mathcal{R}f \] can be defined when \( f(z) = e^{2\pi i\alpha}z + \ldots \) is a small perturbation of \( z + a_2z^2 + \ldots \) (\( a_2 \neq 0 \)) and \( |\arg \alpha| < \pi/4 \).
Renormalization: The Picture

Write \( f(z) = e^{2\pi i\alpha}z + O(z^2) = e^{2\pi i\alpha}h(z) \) where \( h(z) = z + O(z^2) \).

\( f \leftrightarrow (\alpha, h) \)

Then \( \mathcal{R} f(z) = e^{-2\pi i \frac{1}{\alpha}} \mathcal{R}_\alpha h(z) \) where \( \mathcal{R}_\alpha h = \text{Exp}^\# \circ E(e^{2\pi i \alpha} h) \circ (\text{Exp}^\#)^{-1} \).

Hence \( \mathcal{R} : (\alpha, h) \mapsto (-\frac{1}{\alpha}, \mathcal{R}_\alpha h) \) (skew product)

\( \mathcal{R} \) hyperbolic?

(\( \mathcal{R}_\alpha \) contracting?)

\( \mathcal{R}_\alpha h \to \mathcal{R}_0 h \) \((\alpha \to 0)\)

\( \mathcal{R}_0 \) contracting?

YES for \( \alpha \) small
Horn map and Parabolic Renormalization

\[ f_0(z) = z + a_2 z^2 + \ldots \]
\[ a_2 \neq 0 \]

**Horn map**
\[ E_{f_0} = \Phi_{\text{attr}} \circ \Phi_{\text{rep}}^{-1} \]

**Parabolic Renormalization**
\[ R_0 f_0 = \text{Exp}^\# \circ E_{f_0} \circ (\text{Exp}^\#)^{-1} \]
\[ \text{Exp}^\#(z) = e^{2\pi i z} : \mathbb{C}/\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^* \]

\[ R_0 f_0(z) = z + \ldots \]
by normalization
\[ E_{f_0}(z) = z + o(1) \quad (\text{Im } z \rightarrow +\infty) \]
Perturbation (Douady-Hubbard-Lavaurs) $f'(0) = e^{2\pi i \alpha}$, $\alpha$ small $\mid \arg \alpha \mid < \frac{\pi}{4}$

$E_f$ depends continuously on $f$

(after a suitable normalization)

$\chi_f(z) = z - \frac{1}{\alpha}$

$\tilde{R}_f = \chi_f \circ E_f$

first return map
Main Theorems 1-4 (with H. Inou)

We define a class of functions $\mathcal{F}_1$, (and $\mathcal{F}'_1 \subset \mathcal{F}_1$) such that if $f \in \mathcal{F}_1$, then $f$ is holomorphic, $f(0) = 0$, $f'(0) = 1$, $f$ has a unique critical point $c_f$ in its domain of definition and the critical value $f(c_f) = -\frac{4}{27}$ (fixed). Moreover $f''(0) \neq 0$.

Theorem 1. $\mathcal{F}_1 \xrightarrow{\mathcal{R}_0} \mathcal{F}'_1 \subset \mathcal{F}_1$.

Moreover $\mathcal{R}_0$ is “holomorphic” and $\mathcal{R}_0(z + z^2) \in \mathcal{F}'_1$.

Theorem 2. For small $\alpha \in \mathbb{R}$, $\mathcal{F}_1 \xrightarrow{\mathcal{R}_\alpha} \mathcal{F}'_1 \subset \mathcal{F}_1$.

Hence there exists a large $N$ such that if $f = e^{2\pi i \alpha} h$ with $\alpha$ of high type ($N$) and $h \in \mathcal{F}_1$, then the sequence of renormalizations

$$f = f_0 \xrightarrow{\mathcal{R}} f_1 \xrightarrow{\mathcal{R}} f_2 \xrightarrow{\mathcal{R}} f_3 \xrightarrow{\mathcal{R}} \ldots$$

is defined so that $f_n = e^{2\pi i \alpha_n} h_n(z)$, $h_n \in \mathcal{F}_1$. (Here $\alpha_{n+1} = ||\frac{1}{\alpha_n}||$ and $h_{n+1} = \mathcal{R}_{\alpha_n} h_n$, possibly after complex conjugation.)
“Irrational numbers of high type” (N)

\[ \alpha = \frac{1}{a_1 \pm \frac{1}{a_2 \pm \frac{1}{a_3 \pm \ldots}}} \text{ where } a_i \geq N \]
Definition of $\mathcal{F}_1$ and $\mathcal{F}_1'$

Let $P(z) = z(1 + z)^2$. We take specific simply connected open sets $V$ and $V'$ with $0 \in V \subset \overline{V} \subset V' \subset \mathbb{C}$.

$$\mathcal{F}_1 = \left\{ f = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \middle| \varphi : V \to \mathbb{C} \text{ is univalent} \right\}$$

$$\varphi(0) = 0, \quad \varphi'(0) = 1$$

Define $\mathcal{F}_1'$ with $V$ replaced by $V'$.

$P(0) = 0, \quad P'(0) = 1$ critical points: $\frac{-1}{3}$ and $-1$ critical values: $P(-\frac{1}{3}) = -\frac{4}{27}$ and $P(-1) = 0$

$V' \quad V \text{ slightly smaller domain than } V'$

$\eta = 2$
Theorem 3. After modifying the definition slightly, $\mathcal{F}_1$ is in one to one correspondence with the Teichmüller space of a punctured disk. With respect to the Teichmüller distance (which is complete), $\mathcal{R}_0$ is a uniform contraction.

Theorem 4. The same statement for small $\alpha (\in \mathbb{R})$. Hence when restricted to the subset where $|\alpha|$ is small, the renormalization $\mathcal{R}$ is hyperbolic.

Teichmüller space is like the unit disk with Poincaré metric. holomorphic self map does not expand the distance.

(Royden-Gardiner Theorem: Teichmüller distance = Kobayashi distance)

\[
\mathcal{F}_1 \xrightarrow{\mathcal{R}_0} \mathcal{F}'_1 \subset \mathcal{F}_1
\]

Estimate of contraction of $\mathcal{F}'_1 \hookrightarrow \mathcal{F}_1$ via cotangent space which is the space of integrable holomorphic quadratic differentials. + modulus-area inequality
Applications

Theorem. Under the assumption of Theorem 2, the critical orbit stays in the domain of $f$ and can be iterated infinitely many times. Moreover if $f$ is (a part of) a rational map, then the critical orbit is not dense.

Theorem (Buff-Chéritat). There exists an irrational number $\alpha$ such that the Julia set of the quadratic polynomial $P_{\alpha}(z) = e^{2\pi i \alpha}z + z^2$ has positive Lebesgue measure.

Theorem. Suppose $f$ and $f'$ satisfy the assumption of Theorem 2, with the same rotation number $\alpha$. Then they have small periodic cycles $\zeta_n$ and $\zeta'_n$ around 0 with period $q_n$. Let $\lambda(\zeta_n)$, $\lambda(\zeta'_n)$ be their multipliers. The differences

$$|\lambda(\zeta_n) - \lambda(\zeta'_n)|$$

and

$$\left|\frac{1}{1 - \lambda(\zeta_n)} - \frac{1}{1 - \lambda(\zeta'_n)}\right|$$

tends to 0 exponentially fast as $n \to \infty$ with a uniform rate.
Application 2: Rigidity

Theorem 5 (Rigidity). If $h, \tilde{h} \in \mathcal{F}_1$ and $\alpha$ satisfies the hypothesis of Theorem 2, then there exists a quasiconformal homeomorphism $\varphi$ which conjugates $f = e^{2\pi i \alpha}h$ and $\tilde{f} = e^{2\pi i \alpha}\tilde{h}$ along their critical orbits, and asymptotically conformal on the closure of critical orbits.

Within this class of maps, the same rotation number implies a better conjugacy.

$g_n$’s, $\tilde{g}_n$’s are “exponential-like” (very expanding).
Various Renormalizations

Feigenbaum

- Proper subintervals
- Cantor set

Circle map

- Partition of interval

Near-parabolic

- Covering by sector or croissant-like domains
- Gluing/identification needed to define the renormalization
Return to Theorem 5

**Theorem 5 (Rigidity).** If \( h, \tilde{h} \in \mathcal{F}_1 \) and \( \alpha \) satisfies the hypothesis of Theorem 2, then there exists a quasiconformal homeomorphism \( \varphi \) which conjugates \( f = e^{2\pi i \alpha} h \) and \( \tilde{f} = e^{2\pi i \alpha} \tilde{h} \) along their critical orbits, and asymptotically conformal on the closure of critical orbits.

\[ g_n \text{'s, } \tilde{g}_n \text{'s are "exponential-like" (very expanding).} \]

Need to *reconstruct* the dynamics of \( f \) in subdomains (with control on geometry) from \( f_n = \mathcal{R}^n f \). Because the relation between \( f \) and \( f_n = \mathcal{R}^n f \) is less obvious.
**Difficulty** in proving rigidity for irrationally indifferent fixed pts.

Knowing $\mathcal{R}f$, what can be said about $f$?

How to transfer information (e.g. geometry) on $\mathcal{R}^n f$ to previous generations of renormalizations $\mathcal{R}^{n-1} f, \mathcal{R}^{n-2} f, \ldots, f$?

Fundamental domains (and their boundary curves) are not unique.

Need to cover previous fundamental regions with next generation fundamental regions WITH OVERLAP. (not partition)

Need to reconstruct the dynamics of $f$ from that of $\mathcal{R}f$ so that one can understand $f$ better.

this is like ...
Thank you!