Homotopy shadowing

Yutaka Ishii
Kyushu University

Complex Dynamics and Related Topics

Research Institute for Mathematical Sciences, Kyoto University

September 3–6, 2007

Joint with John Smillie

<u>Def.</u>

$f : X \to X$ is <u>semi-conjugate</u> to $g : Y \to Y$

$\underset{def}{\Longleftrightarrow}$ $\exists \varphi : X \to Y$ s.t. $\varphi f = g \varphi$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "g"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

<u>Remark</u> not assume surjectivity of φ.

<u>The goal</u> is to prove :

A "semi-conjugacy up to a homotopy"
to a "generalized dynamical system"
with "some dynamical structure"
induces a semi-conjugacy.

1. Generalized Dynamical System

<u>Def</u> X^0, X^1 : spaces , $\iota, f : X^1 \to X^0$ a pair of maps

The quadruplet $(X^0, X^1 ; \iota\, f)$ is called a <u>multivalued system</u>.

Orbit spaces :

$$X^{+\infty} \equiv \{ (x_i) \in (X^1)^{\mathbb{N}} : f(x_i) = \iota(x_{i+1}) \}$$

$$X^{\pm\infty} \equiv \{ \quad " \quad (X^1)^{\mathbb{Z}} : \quad " \quad \}$$

<u>notation</u>

$+\infty$ "one-sided"

$\pm\infty$ "two-sided"

∞ "~~both~~ either"

Shif map :

$$\hat{f} : X^{\infty} \to X^{\infty}$$

Examples

- (classical) dynamical systems:

  $X^0 = X^1 = X, \quad f = f, \quad \iota = id_X.$

- SFT's:

  $G = (V, E)$ finite directed graph

  $X^0 = V, \ X^1 = E, \ f = \text{head}, \ \iota = \text{tail}$

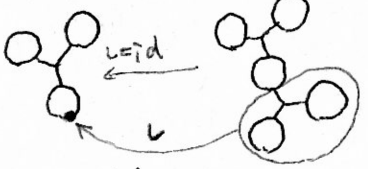
  $\Longrightarrow X^\infty$: edge shift.

- polynomial-like maps:

  $U \Subset V \subset \mathbb{C}$ $\quad p : U \to V$ proper, $\quad \iota : U \to V$ inclusion

- Hubbard trees: $\qquad \mathcal{J}^0 \qquad \mathcal{J}^1$

  rabbit 

  $\tau : \mathcal{J}^1 \to \mathcal{J}^0$ covering

  $\iota : \mathcal{J}^1 \to \mathcal{J}^0$ squashing unnecessary part

- complex Hénon:

  $A \subset \mathbb{C}^2$ open

  $\iota, f : A \cap f^{-1}A \to A$

  f : Hénon, $\quad \iota$: inclusion $\quad$ 15

2. Semi-Conjugacy up to Homotopy

$\mathcal{X} = (X^0, X^1; \iota, f), \quad \mathcal{Y} = (Y^0, Y^1; \iota, g)$

Def. $\mathcal{X}$ is <u>homotopy semi-conjugate</u> to $\mathcal{Y}$

$\underset{\text{def}}{\Longleftrightarrow} \exists h^0 : X^0 \to Y^0, \ \exists h^1 : X^1 \to Y^1$ s.t.

$$\begin{cases} \cdot\ h^0 f \sim g h^1 & \text{by } G = G_t \ (G_0 = h^0 f, \ G_1 = g h^1) \\ \cdot\ h^0 \iota \sim \iota h^1 & \text{by } H = H_t \ (H_0 = h^0 \iota, \ H_1 = \iota h^1) \end{cases}$$

$h = (h^0, h^1; G, H)$: <u>homotopy semi-conjugacy</u>

Examples

- semi-conjugacy
- identity semi-conjugacy $id_{\mathcal{X}}$ from $\mathcal{X}$ to itself.

$$\begin{array}{ccc} X^1 & \xrightarrow{h^1} & X^1 \\ \iota \downarrow\downarrow f & & \iota\downarrow\downarrow f \\ X^0 & \xrightarrow[h^0]{} & X^0 \end{array}$$

$h^1 := id_{X^1}$, $h^0 := id_{X^0}$

$G_t := f$, $H_t := \iota$

(constant homotopies)

- composition

$$\mathcal{X} \longrightarrow \mathcal{Y} \longrightarrow \mathcal{Z}$$

$h = (h^0, h^1; G, H)$ $\quad k = (k^0, k^1; G', H')$

concatenation

$$kh : \mathcal{X} \to \mathcal{Z}$$

$$kh \equiv (k^0 h^0, k^1 h^1; k^0 G \cdot G' h^1, k^0 H \cdot H' h^1)$$

Def $\quad h, k : \mathcal{X} \to \mathcal{Y}$

h is homotopic to k

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists S = S_t : X^1 \to Y^1 \quad S_0 = h^1, \; S_1 = k^1$

$\exists T = T_s : X^0 \to Y^0 \quad T_0 = h^0, \; T_1 = k^0$

s.t. (i) $gS \cdot (G')^{-1} \sim G^{-1} \cdot Tf$

(ii) $H \cdot \iota S \sim T\iota \cdot H'$.

Def. $\mathcal{X}$ and $\mathcal{Y}$ are homotopy equivalent

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists h : \mathcal{X} \to \mathcal{Y}$, $\exists k : \mathcal{Y} \to \mathcal{X}$ homotopy semi-conjugacies

s.t. $kh \sim id_{\mathcal{X}}$, $hk \sim id_{\mathcal{Y}}$.

("conjugacy up to homotopy")

3. Some Dynamical Structure

(X^0, d^0), (X^1, d^1) : length spaces

Def.

$\mathcal{X}$ is an expanding system

$\underset{def}{\iff}$ (i) $f : X^1 \to X^0$ is a covering.

(ii) $\exists \lambda > 1$, $\exists \delta > 0$ s.t.

$$d^1(x, y) < \delta \implies d^0(f(x), f(y)) \geq \lambda \cdot d^0(\iota(x), \iota(y)).$$

Examples
- expanding maps on cpt mfds, $\partial = \phi$ (Shub).
- poly. like maps without branching.
- Hubbard trees.

M_x, M_y : cpt connected orientable mfds

M_y : simply connected

$X^0 \equiv M_x \times M_y$, projections $\pi_{x/y} : X^0 \to M_{x/y}$

$X^1 \subset X^0$: open subset

$f : X^1 \to X^0$ smooth diffeo onto its image

$\iota : X^1 \to X^0$ inclusion

Def. $\iota, f : X^1 \to X^0$ is a crossed mapping

$\underset{def}{\iff}$ $P_f \equiv (\pi_x \circ f, \pi_y \circ \iota) : X^1 \to X^0$ is proper.

For $p \in X^0$

$$C_h(p) \equiv \{ v = (v_x, v_y) \in T_p X^0 : |v_x|_{M_x} > |v_y|_{M_y} \}$$

$\|v\|_h \equiv |v_x|_{M_x}$ horizontal cone field

$C_v(p) \equiv \cdots$, $\|v\|_v \equiv \cdots$ vertical cone field

Def. $\iota, f : X^1 \to X^0$ is a hyperbolic system

$\underset{def}{\iff}$ (i) crossed mapping

(ii) $f \cdot \iota^{-1}$ uniformly expands the horizontal cone field

" contracts the vertical "

Example
- complex Hénon maps with (CMC) & (NTC).
  (compactness is not satisfied, but o.k.)

4. Main Results

Thm A. A homotopy equivalence between exp./hyp. systems induces a conjugacy between their orbit spaces.

Applications

- Shub's theorem
- structural stability of expanding rational maps of $\mathbb{CP}^1$
- Hubbard trees for expanding poly. of $\mathbb{C}$.

Take $y_0 \in M_y$ and let $\tau_{y_0} : M_x \to X^0$, $\tau_{y_0}(x) = (x, y_0)$.

$$\sigma_{y_0} \equiv \pi_x \circ f \circ \tau_{y_0} : \underbrace{M_x \cap \sigma_{y_0}^{-1} M_x}_{\overset{\parallel}{Y^1}} \longrightarrow \underbrace{M_x}_{\overset{\parallel}{Y^0}}$$

Def. $(Y^0; Y^1; \iota, \sigma_{y_0})$ is called an associated expanding system at $y_0 \in M_y$.

Remark By Thm A, the homotopy equiv. class of ass. exp. syst. is independent of $y_0 \in M_y$. Thus, we drop y_0.

Thm B. $\mathcal{X}$: hyperbolic system with M_y : contractible

$\mathcal{Y}$: associated exp. system

$\Rightarrow$ $\mathcal{X}$ and $\mathcal{Y}$ are homotopy equivalent.

In particular, $\hat{f} : X^{\pm\infty} \circlearrowleft$ and $\hat{\sigma} : Y^{\pm\infty} \circlearrowleft$ are conjugate.

Applications

- quantitative version of [Ho]

★ Hubbard trees for some complex Hénon. (my Nagoya talk)

5. Idea of Proof

"homotopy shadowing"

Def. $\mathcal{X} = (X^0, X^1; \iota, f)$

(x, α) : homotopy pseudo-orbit of $\mathcal{X}$

$$\overset{\text{def}}{\iff} \begin{cases} x = (x_i) & x_i \in X^1 \\ \alpha = (\alpha_i) & \alpha_i : [0,1] \to X^0 \text{ s.t.} \\ & \alpha_i(0) = f(x_{i-1}),\ \alpha_i(1) = \iota(x_i) \\ & \text{length}(\alpha_i) < C. \end{cases}$$

(orbit $\Rightarrow$ homotopy pseudo-orbit)

(**Lemma** A (homotopy pseudo-) orbit is sent to a homotopy p-orbit by a homotopy semi-conjugacy.)

$\mathcal{X} \underset{k}{\overset{h}{\rightrightarrows}} \mathcal{Y}$

$h \sim k$

$\Rightarrow h(x,\alpha) \sim k(x,\alpha)$

Lemma (virelangue, Tongue twister)

Two homotopy semi-conjugacies which are homotopic send a homotopy p-orbit to two homotopy p-orbits which are homotopic.

Homotopy Shadowing Theorem

$\forall$ Homotopy p-orbit of an exp./hyp. system

$\exists!$ orbit which is homotopic to the homotopy p-orbit.

unique!

Assume $\mathcal{X} \underset{k}{\overset{h}{\rightleftarrows}} \mathcal{Y}$ homotopy equivalent.

$X^\infty \ni x = (x_i) \rightsquigarrow h(x)$: homotopy p-orbit

$\rightsquigarrow \exists y \in Y^\infty$: shadowing orbit

This defines a semi-conjugacy $h^\infty : X^\infty \to Y^\infty$

similarly $k^\infty : Y^\infty \to X^\infty$

functor

$$\left[\begin{array}{ccc} \infty : \mathrm{HSC} & \to & \mathrm{SC} \\ \Psi & & \Psi \\ h & \longmapsto & h^\infty \end{array} \right]$$

Functorial properties of ∞

- $\mathcal{X} \underset{h_2}{\overset{h_1}{\rightrightarrows}} \mathcal{Y}$ $\quad h_1 \sim h_2 \Rightarrow h_1^\infty = h_2^\infty$ (by uniqueness).

  hyp. or exp.

Thus,

- $\mathcal{X} \xrightarrow{h} \mathcal{X}$ $\quad h \sim id_{\mathcal{X}} \Rightarrow h^\infty = id_{\mathcal{X}^\infty}$.

  hyp. or exp.

- $\mathcal{X} \xrightarrow{h_1} \mathcal{Y} \xrightarrow{h_2} \mathcal{Z} \quad \Rightarrow (h_2 h_1)^\infty = h_2^\infty h_1^\infty$.

  hyp. or exp.

From these we have

$$k^\infty h^\infty = (kh)^\infty = (id_{\mathcal{X}})^\infty = id_{\mathcal{X}^\infty}$$

$$h^\infty k^\infty = \qquad\qquad = id_{\mathcal{Y}^\infty}$$

// Proof of Thm A.