Dynamics on character varieties

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Complex Dynamics and Related Topics
Research Institute for Mathematical Sciences, Kyoto University
September 3–6, 2007
GOAL

. Study an action of the group

\[ \mathbb{Z}_2^* = \{ M \in \text{PGL}(2, \mathbb{Z}) \mid M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod(2) \} \]

on the family of surfaces

\[(S_{A,B,C,D}) \quad x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D \]

by polynomial diffeomorphisms.

. Painlevé Equations \#VI, Monodromy of P\(\text{VI}\).

Iwasaki and Uehara, Inaba, Iwasaki, Saito, ...

. Quasi-Fuchsian Groups, Character Varieties

Goldman, Benedetto, Brown, Neumann, Stantchev, Pickrell, Previte, Xia, Souto, Storm, Tan, Wong, Zhang, Yamashita, ...

. Holomorphic Dynamics

Bedford, Diller, Dinh, Dujardin, Forcadel, Ljubich, Sibony, Smillie, ...

. Certain kind of "discrete Schrödinger Operators"

Bellissard, Roberto, Casdagli, Mackay, ...

Thanks to Frank Loray (partly a joint work) with him
The Torus and The Sphere.

\[ \pi_1(T_1) : \text{the once punctured torus.} \]
\[ \pi_1(T_1) = \langle \alpha, \beta \mid \phi \rangle \cong \mathbb{F}_2 \]  
(free group of rank 2)
\[ [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1} \] makes one turn around the puncture.

\[ S_4 : \text{the four punctured sphere} \]
\[ \pi_2(S_4) = \langle \alpha, \beta, \gamma, \delta \mid \alpha \beta \gamma \delta = 1 \rangle \cong \mathbb{F}_3 \]  
(free group of rank 3)

If \( X = T_1 \) or \( S_4 \), then \( \text{euler}(X) = -1 \) or \( -2 < 0 \).

\[ \exists \ \rho : \pi_1(X) \rightarrow \text{PSL}(2, \mathbb{R}) = \text{Isom}^+ (\mathbb{D}) \]

such that \( \rho(\pi_1(X)) \) is a discrete subgroup of \( \text{PSL}(2, \mathbb{R}) \) and \( \mathbb{D}/\rho(\pi_1(X)) \cong X \).

Moreover, the Teichmüller space of \( X \) has real dimension 2.

Since \( \pi_1(X) \) is free, representations \( \rho : \pi_1(X) \rightarrow \text{PSL}(2, \mathbb{R}) \) can be lifted to \( \text{SL}(2, \mathbb{R}) \).

The Mapping Class Group of \( X \) coincides with \( \text{Aut}(\pi_1(X)) \) / \( \text{Inn} \), where \( \text{Inn} \) = inner automorphisms (= conjugations).

It acts on the space of representations \( \{ \rho : \pi_1(X) \rightarrow \text{SL}(2, \mathbb{C}) \} \)

\text{modulo} \( \text{SL}(2, \mathbb{C}) \)-conjugations.

\text{GOAL:} \text{ STUDY THIS ACTION!}
Character Varieties.

\[ \text{Rep} \left( \pi_1(X), SL(2, \mathbb{C}) \right) = \left\{ \rho: \pi_1(X) \to SL(2, \mathbb{C}) ; \rho \text{ morphism} \right\} \]
\[ = \begin{cases} \{ (\rho(\alpha), \rho(\beta)) \in SL(2, \mathbb{C})^2 \} = SL(2, \mathbb{C})^2 & \text{or} \text{ SL}(2, \mathbb{C})^3 \text{ if } X \cong S_4 \end{cases} \]

\[ X(X) = \text{Rep} \left( \pi_1(X), SL(2, \mathbb{C}) \right) \]

\[ \text{Quotient in the sense of Geometric Invariant Theory } \]

\[ \text{SL}(2, \mathbb{C}) \text{ acts by conjugation: } (\rho, A) \mapsto A \rho A^{-1}. \]

The Torus $\mathbb{T}_4$:

- $\tau_1(\rho(\alpha)), \tau_1(\rho(\beta)), \tau_1(\rho(\alpha \beta))$ are invariant functions.
- They generate the algebra of invariant functions.
- There are no relations between these functions.

\[ \Rightarrow \int X(\mathbb{T}_4) = \mathbb{C}^3, \ (x,y,z) = (\tau_1(\alpha), \ldots) \]

Remark: $\tau_1(\rho(\alpha \beta \gamma)) = x^2 + y^2 + z^2 - xyz - z$

The Sphere $S_4$:

- $a = \tau_1(\alpha), \ b = \tau_1(\beta), \ c = \tau_1(\gamma), \ d = \tau_1(\delta)$
- $x = \tau_1(\alpha \beta), \ y = \tau_1(\beta \gamma), \ z = \tau_1(\gamma \delta)$

They generate the algebra of invariant functions.

They satisfy the equation

\[ \begin{align*} x^2 + y^2 + z^2 + xyz &= Ax + By + Cz + D \\
\text{with } A &= ab + cd \quad \quad B = bc + ad \\
C &= ac + bd \quad \text{and } D = 4 - a^2 - b^2 - c^2 - d^2 - abc \end{align*} \]

\[ \Rightarrow X(S^2_4) \text{ is a 6-dimensional complex quartic hyper surface in } \mathbb{C}^7. \]
The group $\text{Aut}(\pi_1(X))$ acts on $\text{Rep}(\pi_1(X), \text{SL}(2, \mathbb{C}))$ by composition:

$\varphi \in \text{Rep}(\pi_1(X), \text{SL}(2, \mathbb{C}))$, $\alpha \in \text{Aut}(\pi_1(X)) \mapsto \varphi \circ \alpha$.

$\text{Inn}(\pi_1(X)) = \text{Inner automorphisms} = \{ \gamma \mapsto \alpha \gamma \alpha^{-1}, \alpha \in \pi_1(X) \}$

The group $\text{Inn}(\pi_1(X))$ does not act on $X(X)$.

$\Rightarrow \text{Out}(\pi_1(X)) := \text{Aut}(\pi_1(X)) / \text{Inn}(\pi_1(X))$ acts on $X(X)$.

The group $\text{Out}(\pi_1(X))$ coincides with the mapping class group of $X$.

**Example:** The 4-punctured sphere $S_4$.

$T = \mathbb{R}^2 / \mathbb{Z}^2$

$\pi \downarrow$

$S = T / \sigma$

where $\sigma(x, y) = (-x, -y)$

$\text{GL}(2, \mathbb{Z})$ acts on $T$ and commutes with $\sigma$.

$\downarrow$

$\text{PGL}(2, \mathbb{Z})$ acts on the sphere.

$H = \{(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\} = \text{2-torsion of } T$

also acts $\Rightarrow [\text{PGL}(2, \mathbb{Z}) \ltimes H \text{ acts on } S_4]$

**Fact:** This is $\text{MLCG}^*(S_4)$.

**Remark:** $\Gamma_2^* = \{ M \in \text{PGL}(2, \mathbb{Z}) \mid M \equiv (0, 0) \mod(2) \}$

This group acts on $S_4$ and preserves the punctures.

$\Rightarrow \text{Act on } X(S_4)$ and preserves $a, b, c, d$, i.e.

$A, B, C, \text{and } D$. 
Summary:
The group $\Gamma_2^*$ acts on the family of cubic surfaces
$(S_{A,B,C,D})\ x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D$
where $A, B, C,$ and $D$ are parameters (complex or real).
One wants to describe this dynamical system.
Tools from holomorphic dynamics are useful for that!!

Automorphisms ($= \text{polynomial diffeomorphisms}$)

- $S_x : (x,y,z) \in S_{A,B,C,D} \longrightarrow (-x - yz + A, y, z)$
- $S_y : (x,y,z) \in S_{A,B,C,D} \longrightarrow (x, -y - 2x + B, z)$
- $S_z : (x,y,z) \in S_{A,B,C,D} \longrightarrow (x, y, -2 - 2yz + C)$

THM (El' - Hurwitz 1974)
- There are no relations between $S_x, S_y, S_z$:
  \[ \langle S_x, S_y, S_z \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \subset \text{Aut}(S_{A,B,C,D}) \]
- The index of $\langle S_x, S_y, S_z \rangle$ in $\text{Aut}(X)$ is $\leq 24$
- For generic $A, B, C, D$, $\text{Aut}(X) = \langle S_x, S_y, S_z \rangle$.

Fact (easy computation): The group $\Gamma_2^*$ acts on $S_{A,B,C,D}$.
Its image in $\text{Aut}(X)$ coincides with $\langle S_x, S_y, S_z \rangle$.

- $S_x$ corresponds to $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$
- $S_y$ corresponds to $\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$
- $S_z$ corresponds to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Those 3 matrices generate $\Gamma_2^*$.

Example: $S_x \circ S_y \circ S_z$ corresponds to $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ and is given by

$(x, y, z) \mapsto \begin{pmatrix} -x - (y + xz + x^2 y - Cz + A) + B \\ -y + xz + x^2 y - Cz + B \\ -z - xy + C \end{pmatrix}$
The Cayley Cubic.

1. Choose $A, B, C, D = 0, 0, 0, 4$, then $S$ is given by $z^2 + y^2 + z^2 + xy z = 4$

2. Consider $\eta : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*, \eta (u, v) = \left( \frac{1}{u}, \frac{1}{v} \right)$

Then the map $\mathbb{C}^* \times \mathbb{C}^* \to S_{0,0,0,4}$

$(u, v) \mapsto \left(-\frac{u}{u}, -\frac{1}{v}, -\frac{1}{u}, -\frac{1}{uv}\right)$

provides an isomorphism between $S_{0,0,0,4}$ and $\mathbb{C}^* \times \mathbb{C}^*/\eta$

3. $S_{0,0,0,4}$ has 4 singularities corresponding to the 4 fixed points of $\eta : (1, -1) \in \mathbb{C}^* \times \mathbb{C}^* \mapsto (-2, 2, 2) \in \text{Sing}(S)$.

THM (Cayley, n 1880)

$S_{0,0,0,4}$ is the unique surface in the family $S_{A,B,C,D}$ with 4 singularities

We shall call it the Cayley cubic and denote it $S_c$

4. The group $\text{GL}(2, \mathbb{Z})$ acts on $\mathbb{C}^* \times \mathbb{C}^*$ by monomial transformations:

$\eta = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), (u,v) \in \mathbb{C}^* \times \mathbb{C}^* \mapsto (u^a v^b, u^c v^d)$

$\Rightarrow \text{PGL}(2, \mathbb{Z})$ acts on $S_c$ by polynomial diffeomorphisms

$\Rightarrow \Gamma_2^* \text{ acts on } S_c$: this is the same action!

Consequence: When $A, B, C, D = 0, 0, 0, 4$,

the dynamics of $\Gamma_2^*$ is "uniformized" by its usual linear action on $\mathbb{C} \times \mathbb{C}$:

$\mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*, s, t \mapsto \exp s, \exp t \mapsto \left( \frac{1}{s}, \frac{1}{t}, \frac{1}{st}, -\frac{1}{uv}\right)$
Action of $\Gamma_2^*$ at infinity (I)

. Description of $\Gamma_2^*$.

$\Gamma_2^* \subset \text{PGL}(2, \mathbb{R}) = \text{Isom}_0(D)$ is the group of symmetries of the tessellation of $D$ by ideal triangles.

. Compactification of $S$; consider $\bar{S} \subset \text{P}^3(\mathbb{C})$.

$$\bar{S} : \quad (x^2 + y^2 + z^2)w + xyz = (Ax + By + Cz)w^2 + Dw^3$$

At infinity: $xyz = 0, w = 0$.

The group $\Gamma_2^*$ acts on $\bar{S}$ by birational transformations.

. Action of $S_x$ at infinity:

$$D_x : \{ x = 0 \}$$

$$\{ y = z = 0 \} = v_x$$

Ind $S_x = \{ v_x \}$

$D_x$ is blown down on $v_x$.

$D_y$ and $D_z$ are invariant.

. Action of $S_x \circ S_y = g_x$

$$\text{Ind} (g_x) = \{ v_y \}$$

$D_y$ and $D_z \sim v_2$.

$D_2$ is invariant.
Action of $\Gamma_2^*$ at infinity (II)

Let $\gamma \in \Gamma_2^*$: $\gamma$ corresponds to an isometry of $\mathbb{D}$
$\gamma$ corresponds to a $2 \times 2$ real matrix.

$\lambda(\gamma) := \text{largest \ eigenvalue of } \gamma$.

$\gamma$ is said to be hyperbolic if $\lambda(\gamma) > 1$
$\gamma$ is said to be parabolic if $\lambda(\gamma) = 1$ and $\gamma \approx (1 \ast \ast 0)$
$\gamma$ is said to be elliptic otherwise.

Fact: elliptic $\iff$ conjugated to $s_x$, $s_y$ or $s_z$
parabolic $\iff$ an iterate of $s_z \circ s_y$ or $s_y \circ s_x$ or $s_x \circ s_z$.

If $\gamma$ is hyperbolic then $\gamma$ has two fixed points on $\partial \mathbb{D}$
and the dynamics is:

hyperbolic isometry

up to conjugacy $\gamma(\mathbb{D})$ and $\omega(\gamma)$ are in 2 different segments.
**Topological Entropy**

**Summary:** Let \( f \) be an automorphism of \( SA_{A,B,C,D} \). Assume that \( f \) is determined by an hyperbolic element of \( \Gamma_2^* \). Then, after conjugacy in \( \text{Aut}(SA_{A,B,C,D}) \) we have:

\[
\text{Ind}(f^{-1}) \text{ is another vertex.}
\]

\[
\text{Ind}(f) \text{ is one vertex.}
\]

\[
\text{The whole triangle (minus Ind}(f)) \text{ is sent to Ind}(f^{-1}).
\]

**Consequence:** Up to conjugacy in \( \text{Aut}(SA_{A,B,C,D}) \), \( f \) is algebraically stable.

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**THM (a new version of Iwasaki & Uehara)**

For any set of parameters \( A, B, C, D \in \mathbb{C} \)
For any \( \text{hyperbolic} \) element \( f \) in \( \text{Aut}(SA_{A,B,C,D}) \),
The topological entropy of \( f : SA_{A,B,C,D}(\mathbb{C}) \to SA_{A,B,C,D}(\mathbb{C}) \)
is given by \( h_{\text{top}}(f) = \log (\lambda(f)) \)

**Remark:** \( \lambda(f) := \lambda(f^k)^{\frac{1}{k}} \) for any \( k \geq 1 \)
such that \( f^k \) is induced by \( \gamma \in \Gamma_2^* \).
proof 1: (Smillie, Bedford & Diller, Dujardin; Dinh & Sibony)

- \( f : S \to S \) a birational transformation of a complex projective surface.
- \( \text{Ind}(f^{-1}) \cap \text{Ind}(f) = \emptyset \), \( f^{-1}(\text{Ind} f) = \text{Ind}(f) \)
- \( f^* : H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \)
- \( \lambda(f^*) = \limsup_{m \to +\infty} \| (f^m)^* \|^{1/m} \)

Then \( h_{\text{top}}(f) = \log(\lambda(f^*)) \).

- Moreover: \( H \subset S \) a hyperplane section, then \( h_{\text{top}}(f) = \log(\limsup_{m \to +\infty} \| (f^m)^* [H] \|^{1/m}) \)

proof 2: Assume that \( f \) is induced by \( g \in \Gamma^* \).

- The triangle at infinity is a hyperplane section of \( \overline{S}\).
- The action of \( f^* \) on the triangle at infinity does not depend on \( A, B, C, D : f^* : \text{Vect}([D_1], [D_2], [D_3]) \).
- We compute \( \lambda(f^*) \) in a specific case: The Cayley cubic case \( S_c \).
- In this case, the dynamics is linear:

\[
\begin{align*}
\mathbb{C}^* & \to \mathbb{R}^2 / \mathbb{Z}^2 \\
(1, 1) & \to (0, 0)
\end{align*}
\]

\( h_{\text{top}}(f) = \log(\lambda(f)) \)
Normal forms at infinity (I)

- Germs of contracting holomorphic transformations (Dloussky, Favre).

\[ f : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C} \] is a germ of holomorphic map near the origin.

Assume that \( f \) contracts both axes on \((0,0)\):

\[ f(\{x=0\}) = f(\{y=0\}) = (0,0). \]

Let \( \tilde{f}_* : \mathbb{P}_1(\mathbb{C}^* \times \mathbb{C}^*) \to \mathbb{P}_1(\mathbb{C}^* \times \mathbb{C}^*) \)

be the linear map induced by \( f \):

\[ \tilde{f}_*(p) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \quad \det(c,d) = \pm 1 \]

\[ \text{THM (Dloussky, Favre)}: \exists \Psi \text{ a germ of holomorphic diffeomorphism } \Psi : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C} \text{ such that } \]

\[ \Psi \left( (x,y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = f(\Psi(x,y)) \]

i.e. \( \Psi \) conjugates \( f \) to \( (x,y) \mapsto (x^ay^b, x^cy^d) \)

- Consequence (for \( f \in \text{Aut}(S_{A,B,C,D}) \))

\[ \exists N_f \in \text{GL}(2, \mathbb{Z}) \]

\[ (m_0, n_0) \mapsto (m_0, n_0)^{N_f} \]

\( f \) hyperbolic (after a good conjugacy in \( \text{Aut}(S) \))
Proposition. Let $A, B, C, D \in \mathbb{C}$.
Let $M$ be an element of $\Gamma_2^*$. Let $f : SA, B, C, D \to SA, B, C, D$ be the automorphism corresponding to $fM$.
Assume that $M$ is hyperbolic and $\text{Ind} f \neq \text{Ind} f^{-1}$.
Then

(i) $\exists N f$ a $2 \times 2$ integer matrix with $\geq 0$ entries which is conjugate to $\pm M$.

(ii) $\exists \Psi : (C^2, 0) \to (\overline{S}_{A, B, C, D}, \text{Ind} f^{-1})$ a germ of holomorphic diffeomorphism such that

$$f(\Psi(u, v)) = \Psi((u, v)^N f)$$

Remark: $\forall M \in \text{PSL}(2, \mathbb{Z}) \exists N$ with $\geq 0$ entries such that $M$ is conjugate to $N$ in $\text{PSL}(2, \mathbb{Z})$.

Unbounded orbits:
Let $(x, y, z) \in SA, B, C, D(C)$. Assume that the forward orbit of $(x, y, z)$ is not bounded, then

$$f^m(x, y, z) \to \text{Ind}(f^{-1})$$

and the following limit is well defined:

Green

$$G_f^+(x, y, z) = \lim_{m \to +\infty} \frac{1}{m} \log \| f^m(x, y, z) \|$$

(Here $\| (x, y, z) \| = 1x^2 + 1y^2 + 1z^2$.}
Basin of attraction of $\text{Ind}(f^{-1})$

. Basin of attraction of $\text{Ind}(f^{-1})$:

$$\Omega^* (\text{Ind}(f^{-1})) = \left\{ m \in S_{A, B, C, D}(G) ; \lim_{n \to \infty} f^n(m) \to \text{Ind}(f^{-1}) \right\}$$

$$\Omega (\text{Ind}(f^{-1})) = \left\{ m \in \overline{S_{A, B, C, D}(G)} ; \lim_{n \to \infty} f^n(m) \to \text{Ind}(f^{-1}) \right\}$$

. Monomial Model:

$$\Omega^* (N_f) = \left\{ (\mu, v) \in \mathbb{C}^* \times \mathbb{C}^* ; |v| < |\mu|^s(\ell) \right\}$$

where $N_f(s(\ell)) = \lambda(\ell) (s(\ell))$

(i.e. $\lambda(\ell)$ is the slope of the eigenline of $N_f$ corresponding to the eigenvalue $\lambda(\ell)$)

Proposition:

The conjugacy $\Psi$ extends to a holomorphic diffeomorphism between $\Omega^* (N_f)$ and $\Omega^* (\text{Ind}(f^{-1}))$. 
Julia Sets and Currents.

- If the orbit of a point \( m \in S_{A,B,C,D} (f) \) is unbounded, then
  
  either \( f^m (m) \to \text{Ind}(f^{-1}) \) and \( m \in \Omega_{+}^* (\text{Ind} f^{-1}) \)
  or \( f^m (m) \to \text{Ind}(f) \) and \( m \in \Omega_{-}^* (\text{Ind} f) \)

- **Notations.**

  - \( K^+(f) = \{ m \mid \text{the forward orbit of} \ m \text{ is bounded}\} \)
    
    \( K^-(f) = \{ m \mid \text{the backward orbit is bounded}\} \)

  - \( K(f) = K^+(f) \cap K^-(f) \)

  - \( J^+(f) = \partial K^+(f) \)
    
    \( J^-(f) = \partial K^-(f) \)

  - \( J(f) = J^+(f) \cap J^-(f) \subseteq \partial K(f) \)

  - \( J^*(f) = \text{closure of the set a saddle periodic points of} \ f. \)

- **Eigen currents**

  - \( T^+_f = dd^c G^+_f \) where \( G^+_f (m) = \lim_{m \to +\infty} \frac{1}{\alpha(f)^m} \log \| f^m (m) \| \)

  - \( T^-_f = dd^c G^-_f \) where \( G^-_f (m) = \lim_{m \to -\infty} \frac{1}{\alpha(f)^m} \log \| f^m (m) \| \)

  - \( \mu_f = T^+_f \land T^-_f \)

If \( T^+_f \) and \( T^-_f \) are normalized correctly, then \( \mu_f \) is an \( f \)-invariant probability measure.
Results from holomorphic dynamics
(Bedford, Diller, Dinah, Dujardin, Fornaess, Lyubich, Sibony, Smillie,...)

- $G^f$ and $\overline{G}^f$ are Hölder continuous.
  $\Rightarrow \mu_f$ is well defined.

- $\mu_f$ is the unique $f$-invariant probability measure with maximal entropy:
  $$h_{\mu_f}(f) = h_{\text{top}}(f) = \log 2\lambda_f$$

- The number of periodic points of $f$ of period $N$ is finite (Iwasaki-Uehara: explicit formula) $\approx 2\lambda_f^N$.
  Most of them are hyperbolic saddle points.

  $$\frac{1}{2\lambda_f^N} \sum_{m \in \text{Per}(f, N)} \text{Sinc}_m \xrightarrow{N \to \infty} \mu_f$$

  Where $\# \text{Per}(f, N) = \{ \text{saddle periodic points} \}$.

- $J^*(f)$ coincides with the support of $\mu_f$.
  Any periodic saddle point is in the support of $\mu_f$.
  If $p, q$ are periodic saddle points then
  $$W^s(p) \cap W^u(q) = J^*(f)$$

  stable manifold of $p$  unstable manifold of $q$
• If $p$ is a saddle periodic point of $f$, then $W^u(p)$ is parametrized by $\xi$:

\[ \exists \xi : C \rightarrow S_{A,B,C,D}(C) \]

with $\xi$ injective, $\xi(0) = p$ and $\xi(C) = W^u(p)$

Let $D \subset C$ be the unit disk, let $X$ be a smooth non-negative function on $\xi(D)$ with $X(m) > 0$ and $X \equiv 0$ along $\partial D$.

Let $[\xi(D)]$ be the current of integration on $\xi(D)$:

\[ \left\langle [\xi(D)] \right\rangle = \int_D \xi^* X. \]

Then

\[ \frac{1}{A(f)^m} \xi^m (X, [\xi(D)]) \quad m \rightarrow \infty \]

• Since $f$ is area preserving, we have

\[
\text{Interior } (K(f)) = \text{Interior } (K^+(f)) = \text{Interior } (K^-(f)) = \text{bounded open subset of } S_{A,B,C,D}(C).
\]
The Quasi-Fuchsian Space.

Quasi-Fuchsian Space. (for the once punctured torus).

We consider $X(T_4) = \text{Rep}(\pi_1(T_4), \text{SL}(2, \mathbb{C}))/\text{SL}(2, \mathbb{C})$ and we add the condition $t(\epsilon(x, \beta)) = -2$.

The real surface $S(R)$:

$x^2 + y^2 + z^2 = 2xy$

$x = t(\epsilon(x, \beta))$

$y = t(\epsilon(\beta))$

$z = t'(\epsilon(x, \beta))$

Each connected component $\mathfrak{c} \neq \{(0, 0, 0)\}$ is homeomorphic to $\Delta$.

The action of $\text{PGL}(2, \mathbb{Z}) \times \mathbb{R}^*_+$ on $S(R) \cap (\mathbb{R}^*_+)^3$ is conjugate to the action of $\text{MCG}^+(T_4)$ on $\text{Teich}(T_4)$, i.e. to the action of $\text{PGL}(2, \mathbb{Z})$ on $\Delta$: In particular, it is totally discontinuous.

Quasi-Fuchsian deformation.
Small deformations of fuchsian representations
→ quasi fuchsian representations:
\[ \rho : \mathbb{F}_2 = \langle \alpha, \beta \rangle \rightarrow \text{SL}(2, \mathbb{C}) \]
\[ \rho \text{ is faithful} \]
\[ \rho(F_2) \text{ is discrete} \]
\[ \rho(F_2) \text{ preserves a Jordan Curve } \Lambda \text{ and } \mathbb{P}^1(\mathbb{C}) \setminus \Lambda \]
\[ \text{is the union of } \mathbb{Z} \text{-invariant disks } \Delta^+ \text{ and } \Delta^- \]

\[ \text{QF is an open subset of } S(\mathbb{C}) \]
\[ \overline{\text{QF}} = \text{DF} := \{ [\rho] : \mathbb{F}_2 \rightarrow \text{SL}(2, \mathbb{C}) \text{ discrete faithful} \} \]

Bers Parametrization.

\[ T_1' = \text{the once punctured torus, with the opposite orientation.} \]
\[ \text{Teich}(\mathbb{T}_1') \cong H^+ , \quad \text{Teich}(\mathbb{T}_1') \cong H^- . \]
\[ \text{GL}(2, \mathbb{Z}) \text{ acts on } H^+ \text{ and } H^- \text{ simultaneously.} \]

\[ \text{Thm (Bers)} \quad \exists \text{ Bers } : H^+ \times H^- \rightarrow \text{QF a holomorphic} \]
\[ \text{diffeomorphism such that} \]
\[ \text{Bers}(f(X), f(Y)) = f(\text{Bers}(X, Y)) \]
\[ \forall (X, Y) \in H^+ \times H^- = \text{Teich}(\mathbb{T}_1') \times \text{Teich}(\mathbb{T}_1') \]
\[ \forall f \in \text{GL}(2, \mathbb{Z}) = \text{MCG}(\mathbb{T}_1) \]

This action also conjugates the action of \( \text{MCG}(\mathbb{T}_1) \) on
\[ \{ (z_1, z_2) \in H^+ \times H^- / z_1 = \overline{z}_2 \} \]
\[ \text{Teich}(\mathbb{T}_1) \]
to the action of \( \text{PGL}(2, \mathbb{Z}) \) on \( S(\mathbb{C}) \cap (\mathbb{R}_+ \setminus \mathbb{R}_+^3) \).
Dynamics on $\mathcal{QF}$

**Theorem (Minsky)**

The Bers map extends up to

$$\mathcal{E}^*(\mathcal{H}^+ \times \mathcal{H}^-) = \mathcal{E}(\overline{\mathcal{H}^+ \times \mathcal{H}^-}) \setminus \{ (x,x); x \in \mathcal{P}^1(\mathbb{R}) \}$$

and provides a continuous bijection between

$$\overline{\mathcal{H}^+ \times \mathcal{H}^-} \setminus \{ (x,x) \in \mathcal{P}^1(\mathbb{R}) \}$$

and $\mathcal{D}$. 

**Consequence:** Take $\gamma \in \text{PGL}(2,\mathbb{Z})$, hyperbolic.

For example, $\gamma = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right)$. 

\[ \gamma \text{ acts on } \mathcal{H}^+ \times \mathcal{H}^- \]

\[ \text{Bers}(\omega, \omega) \]

\[ \text{Bers}(\omega, \omega) \]

\[ \text{Bers}(\omega, \omega) \]

\[ \text{Bers}(\omega, \omega) \]

\[ \text{dynamics of } f \text{ (corresponding to } \gamma) \]

**Fact:** 

- $\text{Bers}(\omega, \omega)$ and $\text{Bers}(\omega, \omega)$ are two hyperbolic fixed points of $f$.
- $\text{Bers}(\omega, \omega) \subset W^u(\text{Bers}(\omega, \omega))$
- $\text{Bers}(\omega, \omega) \subset W^s(\text{Bers}(\omega, \omega))$
Nice Orbits.

The origin \((0,0,0)\)

The point \((0,0,0) \in S\) is a singular point
\[(S) \quad x^2 + y^2 + z^2 = xyz.\]

It corresponds to the finite representation \(p_0 : F_2 \rightarrow SL(2,\mathbb{C})\) defined by:
\[p_0(\alpha) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad p_0(\beta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\]

**THM:** Let \(\gamma \in PGL(2,\mathbb{Z})\) be any hyperbolic element.

Let \(f\) be the automorphism of \(S\) determined by \(\gamma\).

Let \(q\) be one of the 2 fixed points of \(f\) on \(E\).

There exists \([p] \in S(C)\) such that the closure of the orbit \(MCG(T_4) \cdot [p]\) contains both \(q\) and the origin \((0,0,0) = [p_0]\).

**Proof:**

**Step 1 (Bowditch):** \(\exists\) a neighborhood \(U\) of the origin \((0,0,0) \in U \subseteq S(C)\) such that
\[\forall m \in U \quad MCG(T_4) \cdot m \ni (0,0,0).\]
Step 2:

1. $(0,0,0) \in K(f)$ because this is a fixed point.
2. If $(0,0,0) \in \text{Int}(K^-(f)) = \text{Int}(K(f))$, then $f$ is linearizable at the origin.
3. But $Df(0,0,0)$ has finite order and $f$ is not periodic, so $(0,0,0) \notin \text{Int}(K^-(f))$.

$\Rightarrow (0,0,0) \in J K^-(f)$.

Conclusion:

Since $W^s(q)$ is dense in $J K^-(f)$, $W^s(q)$ intersects the open set $U$.

Choose $[p] \in U \cap W^s(q)$ to conclude
Another Example (Orbifold Structure on T_2)

- Impose the condition \( t_i (p[x, \beta]) = 0 \).
  i.e. \( p[x, \beta]^4 = \text{Id} \)

- The surface is now \( x^2 + y^2 + z^2 - xyz = 2 \).

- We can use Teichmüller theory + quasi-fuchsian deformations in the orbifold category.

- New feature: The topology of \( x^2 + y^2 + z^2 - xyz = 2 \).

**THM:**

\[\forall \gamma \in \text{PGL}(2, \mathbb{Z}) \text{ hyperbolic} \]
\[\forall q \text{ one of the } 2 \text{ fixed points of } f \text{ on } \mathcal{QF} \]

If \( f: \bigcirc \rightarrow \bigcirc \) has a periodic saddle point then
\[\exists \ m \in \{ x^2 + y^2 + z^2 - xyz = 2 \} \text{ such that} \]
\[f^m (m) \underset{m \to \infty}{\longrightarrow} \bigcirc \]
\[f^m (m) \underset{m \to \infty}{\longrightarrow} q \]

Moreover, if \( \gamma = (2 1) \), this works and \( \text{MCG}(T_2) \cdot m \)
contains the whole bounded component \( \bigcirc \).
**REAL versus COMPLEX Dynamics.**

- Now we focus on the one parameter family
  \[ x^2 + y^2 + z^2 = xyz + D \]  
  (S_D)

- **Topology of S_D(R), D \in \mathbb{R}** (Benedetto, Goldman)

  \[ \begin{array}{c|c|c|c|c} 
  D<0 & D=0 & 0<D<4 & D=4 & D>4 \\
  \hline
  \text{4 connected} & \text{A sphere} & \text{Cayley} & \text{Only one} & \text{All periodic} \\
  \text{components,} & \text{appears} & & \text{connected} & \text{points are real.} \\
  \text{all unbounded} & & & \text{connected} & \\
  \end{array} \]

- Description of the real dynamics. (for f \in \text{Aut}(S_D), \text{hyper})

<table>
<thead>
<tr>
<th>D &lt; 0</th>
<th>D = 0</th>
<th>0 &lt; D &lt; 4</th>
<th>D &gt; 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>All periodic points of f are complex: ( \text{Per}(f) \subseteq S_D(C) \setminus S_D(R) )</td>
<td>The origin is the unique real periodic point</td>
<td>There are always complex (= non real) periodic points.</td>
<td>All periodic points are real.</td>
</tr>
<tr>
<td>( \text{Supp}(\mu_f) \cap S_D(R) = \emptyset )</td>
<td>( \text{Supp}(\mu_f) ) may intersect ( S_D(C) ) but is not contained in ( S_D(R) )</td>
<td>( \text{Supp}(\mu_f) ) is contained in ( S_D(R) )</td>
<td></td>
</tr>
<tr>
<td>( h_{top}(f</td>
<td>_R) = 0 )</td>
<td>( h_{top}(f</td>
<td>_R) = 0 )</td>
</tr>
<tr>
<td>Totally discontinuous</td>
<td></td>
<td>Totally discontinuous on the 4 disks</td>
<td>Uniformly hyperbolic on the Julia Set.</td>
</tr>
</tbody>
</table>
Corollary:
Assume that $A, B, C, D$ are real parameters.
Let $\gamma \in \Gamma_2^*$ be hyperbolic.
Let $f$ be the automorphism of $S_{A,B,C,D}$ induced by $\gamma$.
If $S_{A,B,C,D}(R)$ is connected then the measure
$\mu_f$ is singular with respect to the Lebesgue measure
of $S_{A,B,C,D}(R)$; $\text{Haus-Dim}(\text{Supp}\, \mu_f) < 2$.

Sketch of the proof. (When $A, B, C, D = 0, 0, 0, 0, D$)
Since the surface is connected, $D \geq 4$ and by the
previous theorem the dynamics is uniformly hyperbolic.
If the Hausdorff dimension of $\text{Supp}(\mu_f) = 2$,
then a result of Bowen and Ruelle implies that
$K(f) \cap S_D(R)$ is an attractor for $f : S_D(R) \to D$.
This contradicts the fact that $K(f)$ is compact and
that $f$ is area preserving.

Consequence (Answer to a question by Iwasaki).
There are parameters $k_1, k_2, k_3, k_4$ of the sixth
Painlevé equation such that the monodromy along
any loop with $\lambda(\gamma) > 1$ has a singular
measure of maximal entropy.
Sketch of the proof of the theorem I.

- **Goal:** Prove that the dynamics is uniformly hyperbolic if \( D > 4 \), and that \( \text{htop}(f|_R) = \log(A(f)) \) (if \( D > 4 \))

- The Cayley Cubic
  - Blow Up Singularities.

\[ \text{Cayley}(IR) \]

- Cut along the green unstable manifold:
  - Heteroclinic connection

- **Wandering dynamics**
- **Julia set**

- **Sphere**

- **Unstable**
- **Stable foliation**
- **Unstable**
Sketch of the proof of the Theorem II

Entropy:

- To compute the entropy we know

$$h_{\text{top}}(f_R) \leq h_{\text{top}}(f_C) = \log(A(f))$$

New Version of Iwasaki-Iwahara.

- The estimate from below comes from Bowen's inequality:

In the Cayley case, we remark that if you take a generic loop \( l \in \pi_1(S^4 \setminus 4 \text{ pts}) \), then

$$\text{length } f^N_t[l] \sim A(f)^N$$

word metric in \( \pi_1(S^4) \)

Bowen's inequality says

$$h_{\text{top}}(f_R) \geq \log(A(f))$$

Since the action of \( f \) on \( \pi_1(S^D(R)) \) does not depend on \( D > 4 \) and is the same as the action of \( \pi_1(S^4) \), we get

$$\forall D > 4 \quad h_{\text{top}}(f_R) \geq \log(A(f))$$

- In particular, \( K(f) \subset S_D(R) \)
- \( \text{Per}(f) \subset S_D(R) \)
- \( \text{WS} \wedge \text{WM} \subset S_D(R) \)
Sketch of the proof of the theorem III.

What we want to show is that the bifurcation after a small perturbation, or even a large perturbation, with \( D > 4 \), gives rise to the following local picture:

![Diagram](image)

and not something like

![Diagram](image)

Theorem (Bedford, Smillie)

Assume \( D > 4 \). If the dynamics of \( f \) on \( K(f) \) is not uniformly hyperbolic then

\[ \exists p, q \text{ saddle fixed points such that} \]

(i) \( W^u(p) \) intersects \( W^s(q) \) tangentially (with order 2)

(ii) \( p \) is s-one sided, \( q \) is u-one sided.

DOES NOT INTERSECT \( K(f) \)
Sketch of the proof of the theorem IV.

1. Assume $D_0 > 4$, not uniformly hyperbolic

2. Deform $D_0$:

   $D_0$ typically $\rightarrow$ $D_0 - \varepsilon$

   $\rightarrow$ $D_0 + \varepsilon$

   This "typical deformation" is not possible because for $D = D_0 + \varepsilon$, $W^u(p) \cap W^s(q) \neq S_D(\mathbb{R})$

   Consequence: The tangency persists when one deforms $D$ between $D_0$ and 4, up to $D = 4$

   Conclusion: Get a contradiction at $D = 4$!

   (Not so easy but it does work)