On a comparison of algebraic/geometric constructions of supercuspidal representations
超尖点表現の代数的・幾何的構成の比較について

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Today's theme

- Let F be a p-adic field.
- Let G be a connected reductive group over F.

Problem

Want to compare two representations of $G(F)$;

- "algebraic" representation $\pi^{\mathrm{KY}}_{(S,\eta)}$ constructed by Kaletha–Yu and
- "geometric" representation $\pi^{\mathrm{CI}}_{(S,\eta)}$ constructed by Chan–Ivanov.

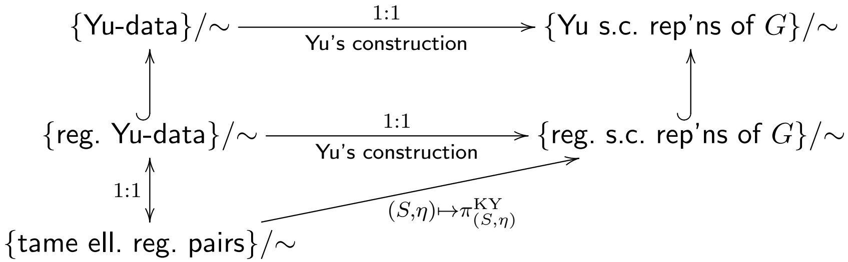
Yu supercuspidal representations

- (Assume: G is tamely ramified.)
- supercuspidal representations = "building blocks" in representation theory of p-adic reductive groups.
- In 2001, Yu gave an explicit construction of a certain wide class of supercuspidal representations.

$$\begin{array}{ccc} & & \{\text{irred. smooth rep'ns of } G\}/\sim \\ & & \big\uparrow \\ & & \{\text{irred. s.c. rep'ns of } G\}/\sim \\ & & \big\uparrow \ {=}\ \text{if } p \gg 0 \\ \{\text{Yu-data}\}/\sim & \xrightarrow[\text{Yu's construction}]{1:1} & \{\text{Yu s.c. rep'ns of } G\}/\sim \end{array}$$

Kaletha–Yu representation $\pi^{\mathrm{KY}}_{(S,\eta)}$

- In 2019, Kaletha introduced the notion of "regularity" of Yu-data and proved that regular Yu-data are parametrized by much simpler data.
- much simpler data = "tame elliptic regular pairs" (S, η), where
  - a tamely ramified elliptic maximal torus S of G defined over F, and
  - a "regular" character $\eta \colon S(F) \to \mathbb{C}^{\times}$.



- In 2019, Chan and Ivanov (`arXiv:1903.06153`) constructed a representation of $G(F)$ by considering a "p-adic version" of Deligne–Lusztig theory.
- Deligne–Lusztig theory: constructs representations of finite reductive groups via geometry.
  - Let $\mathbb{G}$ be a connected reductive group over $\mathbb{F}_q$.
  - Take a maximal torus $\mathbb{T}$ of $\mathbb{G}$ defined over $\mathbb{F}_q$.

Example. When $\mathbb{G} = \mathrm{GL}_n$, there are two typical maximal tori:

split: $(\mathbb{F}_q^\times)^n \subset \mathrm{GL}_n(\mathbb{F}_q)$

elliptic: $\mathbb{F}_{q^n}^\times \subset \mathrm{GL}_n(\mathbb{F}_q)$

  - Take a Borel subgroup $\mathbb{B} = \mathbb{T}\mathbb{U}$ of $\mathbb{G}$ containing $\mathbb{T}$ (not necessarily defined over $\mathbb{F}_q$).

Deligne–Lusztig theory

- Put
$$X := \{g \in \mathbb{G}(\overline{\mathbb{F}}_q) \mid g^{-1}\,\mathrm{Frob}(g) \in \mathbb{U}\}.$$
- Then the groups $\mathbb{G}(\mathbb{F}_q)$ and $\mathbb{T}(\mathbb{F}_q)$ act on this variety via left and right multiplication, respectively.
- Thus the alternating sum of the ℓ-adic cohomology
$$H_c^*(X, \overline{\mathbb{Q}}_\ell) := \sum_i (-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)$$
gives a virtual representation of $\mathbb{G}(\mathbb{F}_q) \times \mathbb{T}(\mathbb{F}_q)$. ($\ell \neq p$, fix $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$)
- Finally, by taking a character θ of $\mathbb{T}(\mathbb{F}_q)$, we get a representation of $\mathbb{G}(\mathbb{F}_q)$:
$$R_{\mathbb{T}}^{\theta} := H_c^*(X, \overline{\mathbb{Q}}_\ell)[\theta].$$
- Summary: each $(\mathbb{T}, \theta)$ produces a virtual representation $R_{\mathbb{T}}^{\theta}$ of $\mathbb{G}(\mathbb{F}_q)$.

Chan–Ivanov representation $\pi^{\mathrm{CI}}_{(S,\eta)}$

- Chan–Ivanov considered this story in the p-adic setting.
- Let G be a connected reductive group over a p-adic field F.
- For each pair (S, η) of
  - an unramified maximal torus S of G defined over F, and
  - a character $\eta\colon S(F) \to \mathbb{C}^\times$,

  they defined a variety "X" and get a virtual representation

$$\rho^{\mathrm{CI}}_{(S,\eta)} := H^*_c(X, \overline{\mathbb{Q}}_\ell)[\eta]$$

of a certain open compact (mod center) subgroup G_x.

Our comparison problem: Kaletha–Yu vs. Chan–Ivanov

- In fact, $\pi^{\mathrm{KY}}_{(S,\eta)}$ is obtained by the compact induction from G_x:

$$\pi^{\mathrm{KY}}_{(S,\eta)} := \text{c-Ind}_{G_x}^{G(F)} \rho^{\mathrm{KY}}_{(S,\eta)}.$$

Q. For an unramified elliptic regular pair (S,η), compare $\rho^{\mathrm{KY}}_{(S,\eta)}$ with $\rho^{\mathrm{CI}}_{(S,\eta)}$.

Main result (joint work with Charlotte Chan)

Let q be the order of the residue field of F. Assume that

- $q \gg 0$ (an explicit condition depending on G and S), and
- $\rho^{\mathrm{CI}}_{(S,\eta)}$ is irreducible (up to sign ε).

Then we have $\rho^{\mathrm{KY}}_{(S,\eta)} \cong \varepsilon \cdot \rho^{\mathrm{CI}}_{(S,\eta)}$.

$\rightsquigarrow$ In particular, we get $\pi^{\mathrm{KY}}_{(S,\eta)} \cong \pi^{\mathrm{CI}}_{(S,\eta)}$ by putting

$$\pi^{\mathrm{CI}}_{(S,\eta)} := \text{c-Ind}_{G_x}^{G(F)} (\varepsilon \cdot \rho^{\mathrm{CI}}_{(S,\eta)}).$$

Characters tell all

- For a finite-dimensional representation (π, V) of G_x, we can define its "character" by taking its trace:

$$\Theta_\pi \colon G_x \to \mathbb{C}; \quad g \mapsto \mathrm{tr}(\pi(g)).$$

- Two finite-dimensional representations π_1 and π_2 are isomorphic if and only if $\Theta_{\pi_1} \equiv \Theta_{\pi_2}$ as a function on G_x.

  $\rightsquigarrow$ We try to compare the characters $\Theta_{\rho^{\mathrm{KY}}_{(S,\eta)}}$ and $\Theta_{\rho^{\mathrm{CI}}_{(S,\eta)}}$.

- Adler–DeBacker–Spice character formula describes $\Theta_{\rho^{\mathrm{KY}}_{(S,\eta)}}$ explicitly.
  (algebraic; based on their precise structure theory of p-adic reductive groups)
- Chan–Ivanov character formula describes $\Theta_{\rho^{\mathrm{CI}}_{(S,\eta)}}$ explicitly.
  (geometric; based on the argument of Deligne–Lusztig via geometric tools such as the Lefschetz trace formula)

Character comparison on unramified very regular elements

Point: If $g \in G_x$ is "unramified very regular", these formulas are drastically simplified and comparable.
(Roughly, "unramified very regular" $=$ has a regular semisimple reduction)

Proposition (special case of ADS/CI character formulas)

Let $g \in G_x$ be an unramified very regular element.

- When g is not G_x-conjugate to an element of $S(F)$, we have
$$\Theta_{\rho^{\mathrm{KY}}_{(S,\eta)}}(g) = \Theta_{\rho^{\mathrm{CI}}_{(S,\eta)}}(g) = 0.$$
- When g is G_x-conjugate to an element of $S(F)$, we may assume that g itself belongs to $S(F)$ by taking conjugation. Then there exists $\varepsilon \in \{\pm 1\}$ (independent of g) such that we have
$$\varepsilon \cdot \Theta_{\rho^{\mathrm{KY}}_{(S,\eta)}}(g) = \Theta_{\rho^{\mathrm{CI}}_{(S,\eta)}}(g) = \sum_{w \in W_{G_x}(S)} \eta^w(g)$$

Recover a representation from its character

- Put $G_{x,\mathrm{reg}}$ to be the set of unramified very regular elements of G_x.

Corollary

There exists a sign $\varepsilon \in \{\pm 1\}$ such that

$$\varepsilon \cdot \Theta_{\rho^{\mathrm{KY}}_{(S,\eta)}}(g) = \Theta_{\rho^{\mathrm{CI}}_{(S,\eta)}}(g) \quad \text{for any } g \in G_{x,\mathrm{reg}}.$$

$\rightsquigarrow$ Can we recover a representation from its characters only on $G_{x,\mathrm{reg}}$?

Q. Suppose that ρ is an irreducible representation of G_x satisfying

$$\varepsilon \cdot \Theta_{\rho}(g) = \Theta_{\rho^{\mathrm{CI}}_{(S,\eta)}}(g) \quad \text{for any } g \in G_{x,\mathrm{reg}}.$$

Then can we deduce $\varepsilon \cdot \rho \cong \rho^{\mathrm{CI}}_{(S,\eta)}$?

Flavor of the proof

Let me explain our proof via a toy model (DL representations).

- Let $\mathbb{G}$ be a connected reductive group over $\mathbb{F}_q$.
- Take (fix) a pair $(\mathbb{T}, \theta)$ of
  - a maximal torus $\mathbb{T} \subset \mathbb{G}$ defined over $\mathbb{F}_q$, and
  - a character $\theta\colon \mathbb{T}(\mathbb{F}_q) \to \mathbb{C}^\times$.

  $\rightsquigarrow$ get a Deligne–Lusztig virtual representation $R_{\mathbb{T}}^{\theta}$ of $\mathbb{G}(\mathbb{F}_q)$
- NOTE: if θ is regular, then $R_{\mathbb{T}}^{\theta} = \pm$(irr. rep'n)

Q. Suppose that ρ is an irreducible representation of $\mathbb{G}(\mathbb{F}_q)$ satisfying

$$\varepsilon \cdot \Theta_\rho(g) = \Theta_{R_{\mathbb{T}}^{\theta}}(g) \quad \text{for any } g \in \mathbb{G}(\mathbb{F}_q)_{\mathrm{reg}}$$

Then do we have $\varepsilon \cdot \rho \cong R_{\mathbb{T}}^{\theta}$?

Flavor of the proof

- $\mathbb{G}(\mathbb{F}_q)_{\mathrm{reg}} := \{g \in \mathbb{G}(\mathbb{F}_q) \mid g$ is regular semisimple$\}$
  - semisimple = diagonalizable
  - regular semisimple = having distinct "eigenvalues" (root values)
- $\mathbb{G}(\mathbb{F}_q)_{\mathrm{nreg}} := \mathbb{G}(\mathbb{F}_q) \setminus \mathbb{G}(\mathbb{F}_q)_{\mathrm{reg}}$
- For any two irreducible representations ρ_1 and ρ_2 of $\mathbb{G}(\mathbb{F}_q)$, we put

$$\langle \rho_1, \rho_2 \rangle_\bullet := \frac{1}{|\mathbb{G}(\mathbb{F}_q)|} \sum_{g \in \mathbb{G}(\mathbb{F}_q)_\bullet} \Theta_{\rho_1}(g)\overline{\Theta_{\rho_2}(g)} \quad (\bullet \in \{\emptyset, \mathrm{reg}, \mathrm{nreg}\}).$$

$\rightsquigarrow$ Note that we have

$$\langle \rho_1, \rho_2 \rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \ncong \rho_2. \end{cases}$$

Flavor of the proof

Task: to show $\langle \rho, R_{\mathbb{T}}^{\theta} \rangle \neq 0$. (Then we get $\varepsilon \cdot \rho \cong R_{\mathbb{T}}^{\theta}$.)

$$\langle \rho, \rho \rangle = \langle \rho, \rho \rangle_{\mathrm{reg}} + \langle \rho, \rho \rangle_{\mathrm{nreg}}$$
$$\langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle = \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{reg}} + \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}$$

- We have $\langle \rho, \rho \rangle = 1 = \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle$ by the irreducibility of ρ and $\varepsilon R_{\mathbb{T}}^{\theta}$.
- We have $\langle \rho, \rho \rangle_{\mathrm{reg}} = \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{reg}}$ by the assumption on the characters.
- Consequently, we get $\langle \rho, \rho \rangle_{\mathrm{nreg}} = \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}$.
- We put

$$X_{\mathrm{reg}} := \langle \rho, \rho \rangle_{\mathrm{reg}} = \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{reg}} > 0,$$
$$X_{\mathrm{nreg}} := \langle \rho, \rho \rangle_{\mathrm{nreg}} = \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}} > 0.$$

$\rightsquigarrow$ Note that $X_{\mathrm{reg}} + X_{\mathrm{nreg}} = 1$.

Flavor of the proof

$$\langle \rho, R_{\mathbb{T}}^{\theta} \rangle = \langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{reg}} + \langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}$$

- We have $\langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{reg}} = \varepsilon X_{\mathrm{reg}}$ by the assumption on the characters.
- We use the Cauchy–Schwarz inequality to evaluate $\langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}$:

$$|\langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}| \leq \langle \rho, \rho \rangle_{\mathrm{nreg}}^{\frac{1}{2}} \cdot \langle R_{\mathbb{T}}^{\theta}, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}^{\frac{1}{2}} = X_{\mathrm{nreg}}.$$

- Therefore, if we have $X_{\mathrm{nreg}} < X_{\mathrm{reg}}$, then the $\triangle$-inequality gives:

$$\begin{aligned} |\langle \rho, R_{\mathbb{T}}^{\theta} \rangle| &\geq |\langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{reg}}| - |\langle \rho, R_{\mathbb{T}}^{\theta} \rangle_{\mathrm{nreg}}| \\ &\geq X_{\mathrm{reg}} - X_{\mathrm{nreg}} \\ &> 0. \end{aligned}$$

Flavor of the proof

Q. When do we have $X_{\mathrm{nreg}} < X_{\mathrm{reg}}$?

- Recall: $X_{\mathrm{reg}} + X_{\mathrm{nreg}} = 1$.
  $\rightsquigarrow$ $X_{\mathrm{nreg}} < X_{\mathrm{reg}}$ if and only if $X_{\mathrm{nreg}} < \frac{1}{2}$
- A simple application of the Deligne–Lusztig character formula gives:

$$X_{\mathrm{nreg}} = \frac{1}{|\mathbb{T}(\mathbb{F}_q)|} \sum_{w \in W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)} \sum_{t \in \mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)} \theta(t) \cdot \overline{\theta(t^w)}.$$

($\mathbb{T}_{\mathrm{nreg}} := \mathbb{T} \cap \mathbb{G}_{\mathrm{nreg}}$ and $W_{\mathbb{T}}^{\mathbb{G}}$ is the Weyl group of $\mathbb{T}$ in $\mathbb{G}$)

$$\rightsquigarrow \qquad X_{\mathrm{nreg}} \leq \frac{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|}{|\mathbb{T}(\mathbb{F}_q)|} \cdot |W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|.$$

Conclusion. If $\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|$, then we get $\varepsilon \cdot \rho \cong R_{\mathbb{T}}^{\theta}$.

Flavor of the proof

- We carry out the same argument in the p-adic setting by using the character formula of Chan–Ivanov and Chan instead of DL.
- Then we finally arrive at the same inequality:
  - The reductions of $S \subset G_x$ gives $\mathbb{T} \subset \mathbb{G}$ over $\mathbb{F}_q$.
  - Then the required assumption is given by

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|. \qquad (\star)$$

  - $|\mathbb{T}(\mathbb{F}_q)|$ and $|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|$ are polynomials in q.
  - $\deg(|\mathbb{T}(\mathbb{F}_q)|) > \deg(|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|)$
  - $2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|$: independent of q.

$\rightsquigarrow$ $(\star)$ is satisfied whenever q is large enough

How large q must be? –the case of GL_n

We need

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|. \qquad (\star)$$

Let $\mathbb{G} := \mathrm{GL}_n$ (assume: n is a prime for simplicity).

- When $\mathbb{T}$ is elliptic: $\mathbb{T}(\mathbb{F}_q) = \mathbb{F}_{q^n}^{\times}$,
  - $\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q) = \mathbb{F}_q^{\times}$;
  - $W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q) \cong \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$.

  $\rightsquigarrow$ $(\star)$ is

$$\frac{q^n - 1}{q - 1} > 2n$$

- This fails only for $(q, n) = (2, 2), (3, 2)$.

How large q must be? –the case of GL_n

We need

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|. \qquad (\star)$$

Let $\mathbb{G} := \mathrm{GL}_n$ (assume: n is a prime for simplicity).

- When $\mathbb{T}$ is split: $\mathbb{T}(\mathbb{F}_q) = (\mathbb{F}_q^{\times})^n$,
  - $\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q) =$ some two entries are the same;
  - $W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q) \cong \mathfrak{S}_n$.

  $\rightsquigarrow$ $(\star)$ is

$$\frac{(q-1)^n}{(\text{polynomial in } q \text{ of degree } n-1)} > 2n!$$

- This fails for many (q, n)!

How large q must be? –the case of exceptional group E_6

We need

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|. \qquad (\star)$$

Let $\mathbb{G} := E_6$.

- When $\mathbb{T}$ is "Coxeter";
  - $|\mathbb{T}(\mathbb{F}_q)| = (q^4 - q^2 + 1)(q^2 + q + 1)$;
  - $|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)| = (q^2 + q + 1)$;
  - $|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)| = 12$.

  $\rightsquigarrow$ $(\star)$ is

$$q^4 - q^2 + 1 > 24$$

- This fails only for $q = 2$.

Better estimate in the toral case

- When η is a particularly special regular character (called "toral"), we can get a better result by another proof (due to Charlotte).
- The necessary assumption is:

$$\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2.$$

- In the toral case, we also know the irreducibility of $\rho^{\mathrm{CI}}_{(S,\eta)}$ (up to sign) by Chan–Ivanov.
- Thus our result in this special case is as follows:

Main result (the toral case)

Assume that η is toral and

- q is sufficiently large so that $\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2$.

Then we have $\rho^{\mathrm{KY}}_{(S,\eta)} \cong \varepsilon \cdot \rho^{\mathrm{CI}}_{(S,\eta)}$. Hence, in particular, $\pi^{\mathrm{KY}}_{(S,\eta)} \cong \pi^{\mathrm{CI}}_{(S,\eta)}$.

On the irreducibility of $\rho^{\mathrm{CI}}_{(S,\eta)}$

Main result (joint work with Charlotte Chan)

Assume that

- $q \gg 0$ (an explicit condition depending on G and S), and
- $\rho^{\mathrm{CI}}_{(S,\eta)}$ is irreducible (up to sign ε).

Then we have $\rho^{\mathrm{KY}}_{(S,\eta)} \cong \varepsilon \cdot \rho^{\mathrm{CI}}_{(S,\eta)}$. Hence, in particular, $\pi^{\mathrm{KY}}_{(S,\eta)} \cong \pi^{\mathrm{CI}}_{(S,\eta)}$.

- It should be reasonable to expect that $\rho^{\mathrm{CI}}_{(S,\eta)}$ is always irreducible (as long as $\eta\colon S(F) \to \mathbb{C}^{\times}$ is regular).
  - OK if η is depth zero (follows from Deligne–Lusztig theory).
  - OK if η is toral (follows from Chan–Ivanov).
  - Recently Dudas and Ivanov announced a result for some specific S called "Coxeter type" (Dudas–Ivanov; `arXiv:2010.15489`).

Lusztig's recent post

- In fact, the toy version of our result is not new for large q.
- Recently Lusztig posted a short article on arXiv (`2011.01824`):
  - "I have proved Proposition 4 in 1977 (it was part of a talk I gave at the LMS Symposium on Representations of Lie groups, Oxford, June-July 1977). Recently C. Chan amd M. Oi [Ch] proved that when q is large, an irreducible cuspidal representation of G^F of the form $\pm R_T^\theta$ can be recovered from the knowledge of its character at regular semisimple elements. This can be deduced from proposition 4."
- Lusztig's assumption: $\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2^{2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|-1}$.
  - Roughly, show that the contribution of $\mathbb{T}_{\mathrm{reg}}(\mathbb{F}_q)$ is big enough. ("linear independence of characters")
  - So it works even if we do not know the character values on $\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)$.
- Our assumption: $\frac{|\mathbb{T}(\mathbb{F}_q)|}{|\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)|} > 2|W_{\mathbb{T}}^{\mathbb{G}}(\mathbb{F}_q)|$.
  - Roughly, show that the contribution of $\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)$ is small enough.
  - So we also need to know the character values on $\mathbb{T}_{\mathrm{nreg}}(\mathbb{F}_q)$.