Geometric L-packets of Howe-unramified toral supercuspidal representations I

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(Automorphic Project)

Main Result (in this week)

- F: a non-archimedean local field.
- G: a connected reductive group over F.

Theorem (Chan-O., arXiv:2105.06341)

For any pair (S, θ) of

- *$S \subset G$: an unramified elliptic maximal torus over F,*
- *$\theta\colon S(F) \to \mathbb{C}^\times$: a toral character,*

there uniquely exists a regular supercuspidal representation π of $G(F)$ whose Harish-Chandra character Θ_π is given by

$$\Theta_\pi(\gamma) = c \cdot \sum_{w \in W_{G(F)}(S)} \theta({}^w\gamma) \qquad ({}^w\gamma := w\gamma w^{-1})$$

at any unramified very regular element $\gamma \in S(F)$ ($c \in \mathbb{C}^\times$ is a constant independent of γ).

In fact, "the" representation π can be explicitly given via Yu–Kaletha theory.

Why important?

In principle, any π is determined by its Harish-Chandra character Θ_π.

- Θ_π is a $\mathbb{C}$-valued function on (the regular semisimple locus of) $G(F)$.
- It is often very difficult to compute $\Theta_\pi(\gamma)$ at all $\gamma \in G(F)$.
- It is sometimes possible to write $\Theta_\pi(\gamma)$ in a simple form at some special γ.

In practice, we want π to be recovered by the behavior of Θ_π on a small set of such special elements.

Application (next week)

- characterization of L-packets/LLC

  (supplement to Kaletha's construction for regular supercuspidals)
- comparison of two representations having different origins

  (Yu's supercuspidal representations vs. Chan–Ivanov representations)

Plan of today

(1) Review Yu's supercuspidal representations and regular supercuspidal representations.

- J.-K. Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), no. 3, 579–622.
- T. Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32** (2019), no. 4, 1071–1170.

(2) Explain a character formula in a special case.

- J. D. Adler and L. Spice, *Supercuspidal characters of reductive p-adic groups*, Amer. J. Math. **131** (2009), no. 4, 1137–1210.
- S. DeBacker and L. Spice, *Stability of character sums for positive-depth, supercuspidal representations*, J. Reine Angew. Math. **742** (2018), 47–78.

(3) Explain the outline of the proof of our result.

- C. Chan and M. Oi, *Geometric L-packets of Howe-unramified toral supercuspidal representations*, preprint, `arXiv:2105.06341`, 2021.

Yu's supercuspidal representations

$$\{\text{supercuspidal representations}\}/\sim$$
$$\cup$$
$$\{\text{Yu's supercuspidal representations}\}/\sim$$
$$\cup$$
$$\{\text{regular supercuspidal representations}\}/\sim$$

Yu's construction: to each Yu-datum $(\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi})$, associate a pair (K, ρ) of an open compact-mod-center subgroup $K \subset G(F)$ and an irreducible smooth representation of (ρ, V) such that

$$\text{c-Ind}_K^{G(F)} \rho := \left\{ f\colon G \to V \;\middle|\; \begin{array}{l} f \text{ is smooth, compact-mod-center supported,} \\ f(kg) = \rho(k) f(g) \text{ for every } k \in K,\, g \in G \end{array} \right\}$$

is an irreducible supercuspidal representation of $G(F)$.

- A key is Yu's precise study of the structure of p-adic reductive groups (based on Bruhat–Tits theory).

Moy–Prasad filtrations of parahoric subgroups

- $\mathcal{B}(G, F)$: (reduced) Bruhat–Tits building (a simplicial set) $\curvearrowleft G(F)$
- $\mathbf{x} \in \mathcal{B}(G, F) \rightsquigarrow G_{\mathbf{x},0} \subset G(F)$: a parahoric subgroup (open compact)
- $G_{\mathbf{x},0} \subset G_{\mathbf{x}} := \mathrm{Stab}_{G(F)}(\{\mathbf{x}\})$
- $G_{\mathbf{x},0}$ has a descending filtration $\{G_{\mathbf{x},r}\}_{r \in \mathbb{R}_{\geq 0}}$ (Moy–Prasad filtration)

Example: hyperspecial case

- $G = \mathrm{GL}_N$, Z_G:= center of G
- $\mathbf{x}$: (hyperspecial) vertex, i.e., facet of minimal dimension

$$G_{\mathbf{x},0} = \mathrm{GL}_N(\mathcal{O}_F) \quad \subset \quad G_{\mathbf{x}} = Z_G(F) \cdot \mathrm{GL}_N(\mathcal{O}_F)$$

$$G_{\mathbf{x},0+} = \cdots = G_{\mathbf{x},1} = \begin{pmatrix} 1 + \mathfrak{p}_F & & \mathfrak{p}_F \\ & \ddots & \\ \mathfrak{p}_F & & 1 + \mathfrak{p}_F \end{pmatrix},$$

$$G_{\mathbf{x},1+} = \cdots = G_{\mathbf{x},2} = \begin{pmatrix} 1 + \mathfrak{p}_F^2 & & \mathfrak{p}_F^2 \\ & \ddots & \\ \mathfrak{p}_F^2 & & 1 + \mathfrak{p}_F^2 \end{pmatrix}, \ldots$$

Moy–Prasad filtrations of parahoric subgroups

Example: Iwahori case

- $G = \mathrm{GL}_N$
- $\mathbf{x}$: barycenter of an alcove (facet of maximal dimension)

$$G_{\mathbf{x},0} = \begin{pmatrix} \mathcal{O}_F^\times & & \mathcal{O}_F \\ & \ddots & \\ \mathfrak{p}_F & & \mathcal{O}_F^\times \end{pmatrix}$$

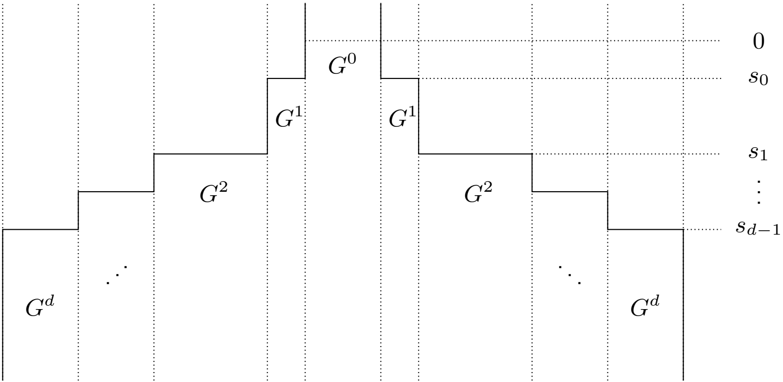
$$G_{\mathbf{x},0+} = \cdots = G_{\mathbf{x},\frac{1}{N}} = \begin{pmatrix} 1+\mathfrak{p}_F & & \mathcal{O}_F \\ & \ddots & \\ \mathfrak{p}_F & & 1+\mathfrak{p}_F \end{pmatrix},$$

$$G_{\mathbf{x},\frac{1}{N}+} = \cdots = G_{\mathbf{x},\frac{2}{N}} = \begin{pmatrix} 1+\mathfrak{p}_F & \mathfrak{p}_F & & \mathcal{O}_F \\ & \ddots & \ddots & \\ \mathfrak{p}_F & & \ddots & \mathfrak{p}_F \\ \mathfrak{p}_F^2 & \mathfrak{p}_F & & 1+\mathfrak{p}_F \end{pmatrix}, \ldots$$

Yu-datum and Yu's pyramid

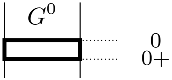
- In the following, assume $p \neq 2$.
- First three parts of a Yu-datum $(\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi})$:
  - $\vec{G} = (G^0 \subsetneq G^1 \subsetneq \cdots \subsetneq G^d = G)$: G^i tame Levi, Z_{G^0}/Z_G is anisotropic,
  - $\vec{r} = (0 \leq r_0 < \cdots < r_{d-1} \leq r_d)$: real numbers ($0 < r_0$ when $d > 0$),
  - $\mathbf{x} \in \mathcal{B}(G^0, F)$: a vertex.
- Put

$$K^i := G^0_{\mathbf{x}} \cdot (G^0, G^1, \ldots, G^i)_{\mathbf{x},(s_0,\ldots,s_{i-1})} \quad (s_i := r_i/2).$$

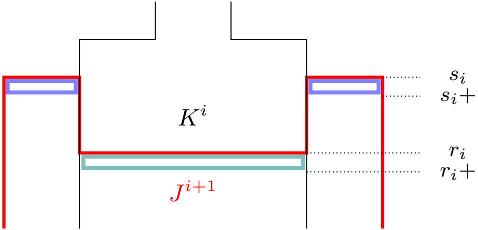


Two particular properties of Yu's pyramid

- A finite connected reductive group $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q) = G^0_{\mathbf{x},0}/G^0_{\mathbf{x},0+}$ lives in K^0.



- A Heisenberg group appears in each difference between K^i and K^{i+1}.
  - Put $J^{i+1} := (G^i, G^{i+1})_{\mathbf{x},(r_i,s_i)}$.
  - We have $K^{i+1} = K^i J^{i+1}$.
  - Each J^{i+1} has a Heisenberg group H^i (= "$V^i \boxtimes \mathbb{F}_p$") as its quotient.

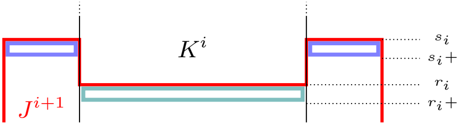


□: symplectic space part ("V^i") of the Heisenberg group

□: center part ("$\mathbb{F}_p$") of the Heisenberg group

Construction of a supercuspidal representation

- Remaining part of a Yu-datum $(\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi})$:
  - ρ_0: irrep. of $K^0 = G^0_{\mathbf{x}}$ such that $\rho_0|_{G^0_{\mathbf{x},0}} \supset \kappa$: a cuspidal irrep. of $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q)$
  - $\vec{\phi} = (\phi_0, \ldots, \phi_d)$: "generic" characters $\phi_i \colon G^i(F) \to \mathbb{C}^\times$;
    - ϕ_i is "G^{i+1}-generic" of depth r_i at $\mathbf{x}$ for $0 \leq i < d$,
    - $\mathrm{depth}_{\mathbf{x}}(\phi_d) = r_d$ when $r_{d-1} < r_d$, and $\phi_d = \mathbb{1}$ when $r_{d-1} = r_d$.
- Construct a representation ρ_{i+1} of K^{i+1} from ρ_i of K^i inductively.



  - $\phi_i \rightsquigarrow H^i \supset \mathbb{F}_p \to \mathbb{C}^\times$: a non-trivial character
  - Stone–von Neumann theorem: $\exists!$ irrep. τ_{ϕ_i} of H^i with cent. char. ϕ_i
  - τ_{ϕ_i} extends to a rep. $\tilde{\tau}_{\phi_i}$ of $\mathrm{Sp}(V^i) \ltimes H^i$ (Heisenberg–Weil representation)
  - $\tilde{\phi}_i :=$ pull back of $\tilde{\tau}_{\phi_i}$ via conjugate action map $K^i \ltimes J^{i+1} \to \mathrm{Sp}(V^i) \ltimes H^i$

$$\tilde{\phi}_i \otimes \big((\rho_i \otimes \phi_i|_{K^i}) \ltimes \mathbb{1}\big) \rightsquigarrow \rho_{i+1} \qquad\qquad K^i \ltimes J^{i+1} \twoheadrightarrow K^i J^{i+1} = K^{i+1}$$

Regular supercuspidal representations

Fact (Yu + Fintzen): $\mathrm{c\text{-}Ind}_{K^d}^{G(F)} \rho_d \otimes \phi_d$ is irreducible supercuspidal.

- Hakim–Murnaghan: studied the fibers of Yu's construction ("G-equiv.").
- Kim, Fintzen: studied the image (surjective if $p \gg 0$).

- Kaletha defined the notion of *regularity* for Yu-data/Yu-supercuspidals and parametrized them via *tame elliptic regular pairs*.

$$\begin{array}{ccc}
\{\text{Yu-data}\}/G\text{-eq.} & \xrightarrow{1:1} & \{\text{Yu-s.c. rep'ns of } G\}/\sim \\
\cup & & \cup \\
\{\text{regular Yu-data}\}/G\text{-eq.} & \xrightarrow{1:1} & \{\text{regular s.c. rep'ns of } G\}/\sim \\
1:1 \updownarrow \text{(Kaletha)} & \nearrow & \\
\{\text{"tame elliptic regular pairs"}\}/G\text{-conj.} & &
\end{array}$$

- Deligne–Lusztig theory plays a key role in this.

Regular supercuspidal representations

Deligne–Lusztig theory: Let $\mathbb{G}$ be a connected reductive group over $\mathbb{F}_q$. Each pair $(\mathbb{S}, \bar{\phi})$ of a maximal torus $\mathbb{S}$ of $\mathbb{G}$ and its character $\bar{\phi}\colon \mathbb{S}(\mathbb{F}_q) \to \mathbb{C}^\times (\cong \overline{\mathbb{Q}}_\ell^\times)$ gives rise to a virtual representation $R_{\mathbb{S}}^{\mathbb{G}}(\bar{\phi})$ of $\mathbb{G}(\mathbb{F}_q)$:

$$R_{\mathbb{S}}^{\mathbb{G}}(\bar{\phi}) := \sum_i (-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)[\bar{\phi}].$$

- $(\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi})$: a Yu-datum ($\rightsquigarrow \rho_0|_{G^0_{\mathbf{x},0}} \supset \kappa$: a cusp. irrep. of $\mathbb{G}^0_{\mathbf{x}}(\mathbb{F}_q)$).
- DL theory $\implies \kappa \subset R_{\mathbb{S}}^{\mathbb{G}}(\bar{\phi})$ for some $(\mathbb{S}, \bar{\phi})$ with elliptic $\mathbb{S} \subset \mathbb{G}^0_{\mathbf{x}}$.
- $\exists$ a "maximally unramified" elliptic maximal torus $S \subset G^0$ s.t. $S_0/S_{0+} \cong \mathbb{S}$.
- We say ρ_0 is *regular* if the stabilizer of $\bar{\phi}$ in $W_{G^0(F)}(S)$ is trivial.
- Regular ρ_0's are parametrized by pairs (S, ϕ_{-1}) of
  - $S \subset G^0$: a "maximally unramified" elliptic maximal torus,
  - $\phi_{-1}\colon S(F) \to \mathbb{C}^\times$: a depth-zero character whose $\phi_{-1}|_{S_0}$ realizes $\bar{\phi}$.

Kaletha's parametrizing theorem

- $(\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi})$: a regular Yu-datum $\rightsquigarrow \phi := \prod_{i=-1}^{d} \phi_i|_S$.

Fact (Kaletha)

When $p \gg 0$, the association $(\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi}) \mapsto (S, \phi)$ gives a bijection

$$\{\text{regular Yu-data}\}/G\text{-eq.} \xrightarrow{1:1} \{\text{tame elliptic regular pairs}\}/G\text{-conj.}$$

Hence we get a bijection $(S, \phi) \mapsto \pi_{(S,\phi)}$;

$$\{\text{tame elliptic regular pairs}\}/G\text{-conj.} \xrightarrow{1:1} \{\text{regular s.c. rep'ns of } G(F)\}/\sim$$

- Converse procedure $[(S, \phi) \rightsquigarrow (\vec{G}, \vec{r}, \mathbf{x}, \rho_0, \vec{\phi})]$ is called *Howe factorization*.
- Group part of Howe factorization $[(S, \phi) \rightsquigarrow \vec{G}]$ works for any character ϕ.
  - ϕ is *toral* $\iff$ ϕ gives rise to a sequence $\vec{G}$ with $G^0 = S$.
  - ϕ is 0-*toral* $\iff$ ϕ gives rise to a sequence $\vec{G} = (G^0 = S \subsetneq G^1 = G)$.
- $\pi_{(S,\phi)}$: a regular supercuspidal associated to (S, ϕ).
  - $\pi_{(S,\phi)}$ is *Howe-unramified* $\iff$ S is unramified
  - $\pi_{(S,\phi)}$ is *toral* $\iff$ ϕ is toral
  - $\pi_{(S,\phi)}$ is 0-*toral* $\iff$ ϕ is 0-toral

Character of Howe-unramified regular supercuspidal

- $\pi_{(S,\phi)}$: Howe-unramified regular supercuspidal of $G(F)$.
- $\gamma \in G_{\mathbf{x},0}$ is *unramified very regular* $\iff$
  - regular semisimple (hence the connected centralizer T_γ is a maximal torus),
  - T_γ: unramified whose "apartment" $\mathcal{A}(T_\gamma, F) \subset \mathcal{B}(G, F)$ contains $\mathbf{x}$, and
  - for any absolute root $\alpha \in R(T_\gamma, G)$, $\alpha(\gamma) \not\equiv 1 \pmod{\mathfrak{p}}$.

Note. $\gamma \in G_{\mathbf{x},0}$: unramified very regular $\implies$ $\bar{\gamma} \in \mathbb{G}_{\mathbf{x}}(\mathbb{F}_q)$: regular semisimple

Proposition (variant of a formula of Adler–DeBacker–Spice–Kaletha)

Let $\gamma \in G_{\mathbf{x},0}$ unramified very regular.

- *If γ is not G-conjugate to an element of S, $\Theta_{\pi_{(S,\phi)}}(\gamma) = 0$.*
- *If $\gamma \in S$, then we have*

$$\Theta_{\pi_{(S,\phi)}}(\gamma) = (-1)^{\bullet} \sum_{w \in W_{G(F)}(S)} (\phi \cdot \varepsilon[\phi])({}^{w}\gamma),$$

where $(-1)^{\bullet} \in \{\pm 1\}$ and $\varepsilon[\phi] \colon S(F) \to \mathbb{C}^{\times}$ is a sign character.

Proof of character formula

- (Consider only the case $\gamma \in S$)
- Recall: $\pi_{(S,\phi)} = \text{c-Ind}_{K^d}^{G(F)} \rho_d \otimes \phi_d$.
- Harish–Chandra integral formula:

$$\Theta_{\pi_{(S,\phi)}}(\gamma) = \frac{\deg \pi_{(S,\phi)}}{\dim \rho_d} \phi_d(\gamma) \int_{G(F)/Z_G(F)} \int_{\mathcal{K}} \dot{\Theta}_{\rho_d}({}^{gc}\gamma)\, dc\, dg$$

  - $\deg \pi_{(S,\phi)}$: formal degree of $\pi_{(S,\phi)}$ (w.r.t. a Haar measure dg),
  - $\mathcal{K} \subset G(F)$: an open compact subgroup with $dc(\mathcal{K}) = 1$,
  - $\dot{\Theta}_{\rho_d}$: zero extension of Θ_{ρ_d} from K^d to $G(F)$.
- Kaletha's trick on the support of $G(F)/Z_G(F) \to \mathbb{C}$: $g \mapsto \dot{\Theta}_{\rho_d}({}^g\gamma)$:

$$\gamma \text{ is unram. very reg. } \implies [g \mapsto \dot{\Theta}_{\rho_d}({}^g\gamma)] \text{ is supported on } G_{\mathbf{x}}/Z_G(F).$$

- standard argument via Fubini's theorem implies

$$\Theta_{\pi_{(S,\phi)}} = \phi_d(\gamma) \sum_{g \in K^d \backslash G_{\mathbf{x}}} \dot{\Theta}_{\rho_d}({}^g\gamma).$$

Proof of character formula

$$\Theta_{\pi_{(S,\phi)}} = \phi_d(\gamma) \sum_{g \in K^d \backslash G_{\mathbf{x}}} \dot{\Theta}_{\rho_d}({}^g\gamma).$$

- investigate the support of Θ_{ρ_d} (Adler–Spice + Deligne–Lusztig):
  $\rightsquigarrow$ unram. very reg. γ satisfies $\dot{\Theta}_{\rho_d}({}^g\gamma) \neq 0 \implies {}^g\gamma \in S$.
  $\rightsquigarrow$ index set $= W_{G^0(F)}(S) \backslash W_{G(F)}(S)$.
- Recall: $\rho_d = \tilde{\phi}_{d-1} \otimes \big((\rho_{d-1} \otimes \phi_{d-1}|_{K^{d-1}}) \ltimes \mathbb{1}\big)$.

$$\rightsquigarrow \quad \Theta_{\rho_d}({}^g\gamma) = \Theta_{\tilde{\phi}_{d-1}}({}^g\gamma) \cdot \Theta_{\rho_{d-1}}({}^g\gamma) \cdot \phi_{d-1}({}^g\gamma)$$

- Do this inductively:

$$\phi_d(\gamma)\dot{\Theta}_{\rho_d}({}^g\gamma) = \prod_{i=0}^{d} \Theta_{\tilde{\phi}_i}({}^g\gamma) \cdot \Theta_{\rho_0}({}^g\gamma) \cdot \prod_{i=0}^{d} \phi_i({}^g\gamma)$$

  - Deligne–Lusztig: $\Theta_{\rho_0}({}^g\gamma) = \sum_{w \in W_{G^0(F)}(S)} \phi_{-1}({}^{wg}\gamma)$.
  - Adler–DeBacker–Spice: $\Theta_{\tilde{\phi}_i}({}^g\gamma) = (-1)^{\bullet} \varepsilon[\phi_i]({}^g\gamma)$.
- Get

$$\Theta_{\pi_{(S,\phi)}}(\gamma) = (-1)^{\bullet} \sum_{w \in W_{G(F)}(S)} (\phi \cdot \varepsilon[\phi])({}^w\gamma), \quad \varepsilon[\phi] := \prod_{i=0}^{d} \varepsilon[\phi_i].$$

Proof of main theorem

Theorem (Chan-O., `arXiv:2105.06341`)

For any unramified toral pair (S,θ), $\exists!$ a regular supercuspidal π such that

$$\Theta_\pi(\gamma) = c \cdot \sum_{w \in W_{G(F)}(S)} \theta({}^w\gamma)$$

at any unramified very regular element $\gamma \in S(F)$.

- $\exists$-part: guaranteed by the previous character formula (even for non-toral).
    - $\pi = \pi_{(S,\phi)}$ for ϕ such that $\theta = \phi \cdot \varepsilon[\phi]$ ($\phi := \theta \cdot \varepsilon[\theta]^{-1}$).
    - $c = (-1)^{\bullet}$: an explicit sign.
- Need the torality assumption on θ for !-part.
- Suppose: $\pi' = \pi_{(S',\phi')}$: another such regular supercuspidal of $G(F)$.

Note: At this point we do not know whether S' is unramified (or also ϕ' is toral). (However we suppose π' is regular supercuspidal from the beginning.)

Proof of main theorem

Key lemma

For any toral character $\theta\colon S(F) \to \mathbb{C}^\times$, there exists $\gamma \in S_{\mathrm{vreg}}(F)$ such that

$$\sum_{w \in W_{G(F)}(S)} \theta({}^w\gamma) \neq 0.$$

Proof.

- Suppose (for a contradiction): $\sum_{w \in W_{G(F)}(S)} \theta({}^w -) \equiv 0$ on $\gamma S_{0+} \subset S_{\mathrm{vreg}}$.
- Since $\theta(\gamma) \cdot \theta|_{S_{0+}} = -\sum_{w \neq 1} \theta({}^w\gamma) \cdot \theta^w|_{S_{0+}}$, we have

$$\theta(\gamma) \cdot \langle \theta|_{S_{0+}}, \theta|_{S_{0+}} \rangle_{S_{0+}} = -\sum_{w \neq 1} \theta({}^w\gamma) \cdot \langle \theta^w|_{S_{0+}}, \theta|_{S_{0+}} \rangle_{S_{0+}}.$$

- $\mathrm{LHS} = \theta(\gamma) \neq 0.$
- Kaletha: $\mathrm{Stab}_{W_{G(F)}(S)}(\theta|_{S_{0+}}) = \mathrm{Stab}_{W_{G^0(F)}(S)}(\theta_{-1}|_{S_{0+}})$ (when $p \gg 0$).
- θ: toral $\implies \mathrm{Stab}_{W_{G(F)}(S)}(\theta|_{S_{0+}}) = \{1\} \implies \mathrm{RHS} = 0$ (contradiction). □

Proof of main theorem

- By Key lemma, take γ such that $\Theta_{\pi'}(\gamma) = c \cdot \sum_{w \in W_{G(F)}(S)} \theta({}^w\gamma) \neq 0$.
- Again by the "support part" of the character formula, we must have $\gamma \in S'(F)$ (up to $G(F)$-conjugate). $\implies$ $S' = S$
- Again by the character formula, for any $S_{\mathrm{vreg}}(F)$, we have

$$(-1)^{\bullet} \sum_{w \in W_{G(F)}(S)} (\phi' \cdot \varepsilon[\phi'])({}^w\gamma) = c \cdot \sum_{w \in W_{G(F)}(S)} \theta({}^w\gamma).$$

Q. Does this implies: $\phi' \cdot \varepsilon[\phi'] = \theta^w$ for some w?

- By a similar argument to Key lemma, we get $\phi' \cdot \varepsilon[\phi'] \equiv \theta^w$ on $S_{\mathrm{vreg}}(F)$.
- So that this holds on $S(F)$, we need enough many unram. very reg. elements.
- $\mathbb{G}_{\mathbf{x}}(\mathbb{F}_q) \supset \mathbb{S}_{\mathrm{vreg}}(\mathbb{F}_q) :=$ the image of $S_{\mathrm{vreg}}(F)$ under the reduction map.
  $\rightsquigarrow$ a sufficient condition is:

$$(\star) \qquad \frac{|\mathbb{S}_{\mathrm{vreg}}(\mathbb{F}_q)|}{|\mathbb{S}(\mathbb{F}_q)|} \geq \frac{1}{2}.$$

Remark on the inequality $(\star)$

Claim. inequality $(\star)$ holds whenever $q \gg 0$.

- well-known: this is true for $\mathbb{S}_{\mathrm{reg}}(\mathbb{F}_q)$ instead of $\mathbb{S}_{\mathrm{vreg}}(\mathbb{F}_q)$.
- However, in general, we might have $\mathbb{S}_{\mathrm{vreg}}(\mathbb{F}_q) \subsetneq \mathbb{S}_{\mathrm{reg}}(\mathbb{F}_q)$.
- $\mathbf{x} \in \mathcal{B}(G, F)$: characterized as the unique point in $\mathcal{A}(S, F)$.
  $\rightsquigarrow$ $\mathbf{x}$: Chevalley valuation $\implies$ $\mathbb{S}_{\mathrm{vreg}}(\mathbb{F}_q) = \mathbb{S}_{\mathrm{reg}}(\mathbb{F}_q)$.
- $\exists$ an inner twist $\psi : G \to G^*$ transferring S to S^* whose associated point $\mathbf{x}^* \in \mathcal{B}(G^*, F)$ is Chevalley (Kaletha's study of "superspecial" points).

$$\begin{array}{ccccccc}
S_{\mathrm{vreg}} & \twoheadrightarrow & \mathbb{S}_{\mathrm{vreg}}(\mathbb{F}_q) & \hookrightarrow & \mathbb{S}_{\mathrm{reg}}(\mathbb{F}_q) := \mathbb{G}_{\mathbf{x},\mathrm{reg}}(\mathbb{F}_q) \cap \mathbb{S}(\mathbb{F}_q) & \hookrightarrow & \mathbb{G}_{\mathbf{x}}(\mathbb{F}_q) \\
{\scriptstyle\psi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\psi}\downarrow{\scriptstyle\cong} & & & & \\
S^*_{\mathrm{vreg}} & \twoheadrightarrow & \mathbb{S}^*_{\mathrm{vreg}}(\mathbb{F}_q) & = & \mathbb{S}^*_{\mathrm{reg}}(\mathbb{F}_q) := \mathbb{G}^*_{\mathbf{x},\mathrm{reg}}(\mathbb{F}_q) \cap \mathbb{S}^*(\mathbb{F}_q) & \hookrightarrow & \mathbb{G}^*_{\mathbf{x}}(\mathbb{F}_q)
\end{array}$$

- We get the claim by estimating $\mathbb{S}^*_{\mathrm{reg}}(\mathbb{F}_q) \subset \mathbb{S}^*(\mathbb{F}_q)$.