Wandering domains and Singularities

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Transcendental dynamics

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  - $U$ is a Baker domain of period 1 if $f^n |_{U} \to \infty$ loc. unif.
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- Transcendental maps may have Fatou components that are not basins of attraction nor rotation domains:
  - \( U \) is a **Baker domain** of period 1 if \( f^n|_U \to \infty \) loc. unif.
  - \( U \) is a **wandering domain** if \( f^n(U) \cap f^m(U) = \emptyset \) for all \( n \neq m \).

\[
z + a + b \sin(z) \quad \text{[Figures: Christian Henriksen]} \quad z + 2\pi + \sin(z)
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- A point $a \in \hat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma(t) \to \infty$ such that $f(\gamma(t)) \to a$. (Morally, $a$ has infinitely many preimages collapsed at infinity).

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- $f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$ is a covering map of infinite degree.

- Define the postsingular set of $f$ as

  $$P(f) = \overline{\bigcup_{s \in S} \bigcup_{n \geq 0} f^n(s)}.$$
Singular values

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What about Baker and wandering domains?
Baker domains

The best result for Baker domains is the following.

**Theorem** (Bergweiler’95, Mihaljevic-rempe’13, Baranski-F-Jarque-Karpinska’17)

Let $f$ be transcendental meromorphic, $U$ an invariant Baker domain, $U \cap S(f) = \emptyset$. Then there exists a sequence $\{p_n\} \subset P(f)$ such that

1. $|p_n| \to \infty$
2. $\left| \frac{p_{n+1}}{p_n} \right| \to 1$
3. $\frac{\text{dist}(p_n, U)}{|p_n|} \to 0$

The theorem is sharp: there exists an (ETF) example for which $\text{dist}(p_n, U) > c > 0$. 

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Some classes of maps are singled out depending on their singular values.

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\[ S = \{ f \text{ ETF (or MTF) such that } S(f) \text{ is finite} \} \]

Example: \( z \mapsto \lambda \sin(z) \)
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Example: \( z \mapsto \lambda \sin(z) \)

Maps in \( S \) have **NO WANDERING OR BAKER DOMAINS.**
Special classes

- The Eremenko-Lyubich class

\[ \mathcal{B} = \{ f \text{ ETF (or MTF) such that } S(f) \text{ is bounded} \} \]

Example: \( z \mapsto \lambda \frac{z}{\sin(z)} \).

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If \( U \) is a wandering domain, and \( L(U) \) is the set of limit functions of \( f^n \) on \( U \), then, all limit functions are constant and

\[
U \text{ is } \begin{cases} 
\text{escaping} & \text{if } L(U) = \{ \infty \} \\
\text{oscillating} & \text{if } \{ \infty, a \} \subset L(U) \text{ for some } a \in \mathbb{C}.
\end{cases}
\]

\[
\text{“bounded”} & \text{if } \infty \not\in L(U).
\]
Existence of wandering domains

Question

Can maps in class $\mathcal{B}$ have wandering domains at all?

Theorem (Bishop'15)

There exists an entire map $f \in \mathcal{B}$ such that $f$ has an (oscillating) wandering domain.

The proof is based on quasiconformal folding, a qc surgery construction.

Incidentally, $U_n \cap \mathcal{P}(f) \neq \emptyset$ for all $n$.

Open question

Does there exist a map with a "bounded" wandering domain?
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Examples of wandering domains

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- **Quasiconformal surgery** [Kisaka-Shishikura’05, Bishop’15].
Wandering domains and singularities: Motivating examples

The relation of a wandering domain with the postcritical set is not so clear.

Example 1

\( z \mapsto z + 2\pi \sin(z) \) One critical point in each WD.
Wandering domains and singularities: Motivating examples

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Example 1 (escaping):

\[ z \mapsto z + 2\pi + \sin(z) \]

One critical point in each WD.
Example 2 (escaping and Univalent, $\partial U \subset \overline{P(f)}$):

Left: Siegel disk of $g(w) = \frac{e^{2-\lambda}}{2-\lambda} w^2 e^{-w}$ with $\lambda = e^{2\pi i (1-\sqrt{5})/2}$, around $w = 2 - \lambda$. Right: Lift to a wandering domain $U$. 
Wandering domains and singularities: Examples

**Example 3** [Kisaka-Shishilkura’05, Bergweiler-Rippon-Stallard’13]. Wandering orbit of annuli such that

- \( \mathcal{U} \cap P(f) = \emptyset \)
- \( P(f) \subset F(f) \).
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Example 4 [Bishop’15]
The oscillating domain of Bishop in class $\mathcal{B}$ contains critical points of arbitrary high multiplicity, responsible for the high contraction necessary.
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**Example 4** [Bishop’15]
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**Question**

*Does there exist an oscillating wandering domain in class $\mathcal{B}$ on which $f^n$ is univalent for all $n > 0$? (In part. $\mathcal{P}(f) \cap U_n = \emptyset$?)*
Known results

Recall, for $U$ a wandering domain, the set of limit functions

$$L(U) = \{ a \in \hat{\mathbb{C}} \mid f^{n_k}|_U \Rightarrow a \text{ for some } n_k \to \infty \}.$$

**Theorem (Bergweiler et al’93, Baker’02, Zheng’03)**

Let $f$ be a MTF with a wandering domain $U$. If $a \in L(U)$ then $a \in P(f)’ \cap J(f)$.
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Theorem (Mihaljevic-Rempe’13)

If $f \in B$ and $f^n(S(f)) \Rightarrow \infty$ uniformly (+ extra geometric assumption), then $f$ has no wandering domains.
Theorem B (Baranski-F-Jarque-Karpinska’17)

Let $f$ be a MTF and $U$ be a wandering domain of $f$. Let $U_n$ be the Fatou component such that $f^n(U) \subset U_n$. Then for every $z \in U$ there exists a sequence $p_n \in P(f)$ such that

$$
\frac{\text{dist}(p_n, U_n)}{\text{dist}(f^n(z), \partial U_n)} \to 0 \quad \text{as } n \to \infty.
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In particular, if for some $d > 0$ we have $\text{dist}(f^n(z), \partial U_n) < d$ for all $n$ (for instance if the diameter of $U_n$ is uniformly bounded), then $\text{dist}(p_n, U_n) \to 0$ as $n$ tends to $\infty$. 
Wandering domains and singular orbits

**Theorem B (Baranski-F-Jarque-Karpinska’17)**

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Proof: normal families argument, hyperbolic geometry.... Based on the improvement of a technical lemma from Bergweiler on Baker domains. Compare also [Mihaljevic-Rempe’13].
Application: Topologically hyperbolic functions

- A MTF is topologically hyperbolic if

\[ \text{dist}(P(f), J(f) \cap \mathbb{C}) > 0. \]

This condition can be regarded as a kind of weak hyperbolicity in the context of transcendental meromorphic functions since \( |(f^n)'(z)| \to \infty \) for all \( z \in J(f) \) [Stallard'90, Mayer-Urnbanski'07'10].

Topologically hyperbolic maps do not possess parabolic cycles, rotation domains or wandering domains which do not tend to infinity. Examples include many Newton's methods of entire functions.
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Corollary C

Let $f$ be a MTF topologically hyperbolic. Let $U$ be a wandering domain s.t. $U_n \cap P(f) = \emptyset$ for $n > 0$. Then for every compact set $K \subset U$ and every $r > 0$ there exists $n_0$ such that for every $z \in K$ and every $n \geq n_0$,

$$D(f^n(z), r) \subset U_n.$$ 

In particular,

$$\text{diam } U_n \to \infty \quad \text{and} \quad \text{dist}(f^n(z), \partial U_n) \to \infty$$

for every $z \in U$, as $n \to \infty$.

This can be applied to show that many functions, including Newton’s method of $h(z) = ae^z + bz + c$ with $a, b, c \in \mathbb{R}$, have no wandering domains

[c.f. Bergweiler-Terglane, Kriete].
No wandering domains

Newton’s method for $F(z) = z + e^z$. 

N. Fagella (Universitat de Barcelona) Wandering domains and singularities RIMS Kyoto
Theorem A (F-Lazebnik-Jarque’17)

There exists an ETF $f \in \mathcal{B}$ such that $f$ has a wandering domain $U$ on which $f^n|_U$ is univalent for all $n \geq 0$. 

The proof is based on Bishop's quasiconformal folding construction. We substitute the high degree maps $(z - z^n)d^n$ on the disk components by $(z - z^n)d^n + \delta_n(z - z^n)$, which are univalent near $z^n$ and show that the critical values can be kept outside (but very close to) the wandering component.
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Thank you for your attention!
The technical lemma on the proof is the following.

**Lemma**

$f : TMF$, $U$ wandering domain, $U_n = f^n(U)$. Then, for all $n > n_0$, $z \in K$, $\gamma$ curve connecting $f^n(z)$ to $w \in \partial U$ with

$$\text{length}(\gamma) \leq \text{dist}(f^n(z), \partial U_n)$$

there exists

$$p \in \mathbb{D}(\gamma, \varepsilon \text{ length}(\gamma)) \cap P(f).$$