Fluctuation theory of Markov additive processes and self-similar Markov processes.

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By a self-similar process we mean a stochastic process $X = (X_t : t \geq 0)$ having the scaling property, that is, there is an index $\alpha > 0$, such that for every $c > 0$,

$$((cX_{c^{-\alpha}t} : t \geq 0), \mathbb{P}_x) \stackrel{d}{=} ((X_t : t \geq 0), \mathbb{P}_{cx}) \qquad x \in \mathbb{R}^d.$$

Self- similar processes often arise in various parts of probability theory as limit of re-scaled processes. Among the class of self-similar processes there are several important sub-families that permit a better understanding of these. Some of them are the self-similar Gaussian processes; the class of additive self-similar processes, that is those with independent increments, those with homogeneous increments, and, those which are of particular interest to us, which have the strong Markov property.

In this course we will be mainly interested by $\mathbb{R}^d$-valued self-similar strong Markov process (ssMp). These processes are involved for instance in branching processes, Lévy processes, coalescent processes and fragmentation theory. Some particularly well known examples are Brownian motion, Bessel processes, stable subordinators, stable processes, stable Lévy processes conditioned to stay positive, etc. Our main purpose in this course is to give a panorama of properties of ssMp that have been obtained since the early sixties under the impulse of Lamperti's work, where the study of the case of positive valued self-similar Markov processes is initiated. The main result in Lamperti's work establishes that there is an explicit bijection between positive valued self-similar Markov processes and real valued Lévy processes. Recently it has been proved by Alili et al. that $\mathbb{R}^d$-valued ssMp are in a bijection with a generalization of Lévy processes, namely Markov additive processes.

Suppose $((\xi, \Theta), \mathbf{P}) = \left((\xi_t, \Theta_t)_{t\geq 0}, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \{\mathbf{P}_{x,\theta} : (x, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}\}\right)$ is a (possibly killed) Markov process with $\mathbf{P}_{x,\theta}(\xi_0 = x, \Theta_0 = \theta) = 1$, taking values in $\mathbb{R} \times \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the d-dimensional sphere. Here $(\mathcal{F}_t)_{t\geq 0}$ is the minimal augmented admissible filtration and $\mathcal{F}_\infty = \bigvee_{t=0}^{+\infty} \mathcal{F}_t$. The process $((\xi, \Theta), \mathbf{P})$ is called a Markov additive process (MAP) on $\mathbb{R} \times \mathbb{S}^{d-1}$ if, for any $t \geq 0$, given $\{(\xi_s, \Theta_s), s \leq t\}$, the process $(\xi_{s+t} - \xi_t, \Theta_{s+t})_{s\geq 0}$ has the same law as $(\xi_s, \Theta_s)_{s\geq 0}$ under $\mathbf{P}_{0,v}$ with $v = \Theta_t$. Notice that when $d = 1$, and Θ is a constant process, then ξ is a a Lévy processes, so MAPS include Lévy processes and in fact they bear similar properties as Lévy processes, most of them are

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extensions of properties known for Lévy processes that are valid conditionally on Θ.

The connection between self-similar processes and Markov additive processes reads as follows.

Theorem 1. *[Lamperti transform] Let $X = (X_t : t \geq 0)$ be a (possibly killed) self–similar Markov process on $\mathbb{R}^d$, then there exists a Markov additive process (ξ, Θ) on $\mathbb{R} \times \mathbb{S}^{d-1}$ such that*

$$(X, \mathbb{P}_x) \stackrel{d}{=} ((\Theta_{\varphi(\|x\|^{-\alpha}t)} e^{\xi_{\varphi(\|x\|^{-\alpha}t)}} : t \geq 0), \mathbb{P}_{\log\|x\|, x/\|x\|}) \tag{0.1}$$

where

$$\varphi(t) = \inf\left\{ s > 0 : \int_0^s e^{\xi_u}\, \mathrm{d}u > t \right\}.$$

Conversely any Markov additive process (ξ, Θ), $(X, \mathbb{P}_x)$ defined by (0.1) is a self–similar Markov process.

In this course we will mainly focus in the study of self-similar Markov processes making a systematic application of the fluctuation theory of Lévy and Markov additive processes. So, we will start by giving a review of some key results in the fluctuation theory of Lévy processes and random walks, and then extending some of those results to Markov additive processes. We will study some particular examples, most of them are self-similar Markov processes obtained as a path transformation of stable processes.

We aim to cover the following topics in the course:

- Preliminaries on Lévy processes: Subordinators, Renewal Theory for Subordinators, First passage problems, non-monotone paths. Fluctuation theory of Lévy processes.
- General self-similar processes. Self-similar Lévy processes. Positive self-similar Markov processes (pssMp). Some pssMp obtained as transformations of stable processes.
- Lamperti's transformation: a useful bijection between positive self-similar Markov processes and real valued self-similar Markov processes. Explicit examples and a detailed study of its associated Lévy process: transformations the stable process and extremal processes. Wiener-Hopf factorization of Lamperti-stable processes
- Exponential functionals of Lévy processes and its connections to pssMp. Some asymptotic results for the tail distribution of the exponential functional of a Lévy process.
- Potentials of stable processes. Some radial excursion theory for self-similar Markov processes. Fluctuation theory for Markov additive processes and its applications to self-similar Markov processes.

A detailed bibliography will be provided throughout the course.