State-dependent random walks on crystal lattices and Ornstein–Uhlenbeck processes

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Let $d \in \mathbb{N}$ and $\Gamma \cong \mathbb{Z}^d$. A Γ-crystal lattice $X = (V, E)$ is a covering graph of a finite graph $X_0 = (V_0, E_0)$ whose covering transformation group is Γ. Random walks on crystal lattices have been extensively studied from both probabilistic and geometric points of view. In this talk, we introduce a state-dependent random walk on X and discuss its convergence to an Ornstein–Uhlenbeck process.

Let $p\colon E \to (0,1)$ be a positive, Γ-invariant, and non-symmetric transition probability and $m\colon V_0 \to (0,1]$ be its invariant probability measure on V_0. Denote by $o(e)$ and $t(e)$ the origin and terminus of $e \in E$, respectively. We also put $E_x = \{e \in E \mid o(e) = x\}$ for $x \in V$. Let $\mathrm{H}_1(X_0, \mathbb{R})$ be the first homology group of X_0 and define $\gamma_p = \sum_{e \in E_0} p(e) m\big(o(e)\big) e \in \mathrm{H}_1(X_0, \mathbb{R})$. Notice that the random walk is symmetric if and only if $\gamma_p = 0$. Let $\rho_{\mathbb{R}}\colon \mathrm{H}_1(X_0, \mathbb{R}) \to \Gamma \otimes \mathbb{R} \cong \mathbb{R}^d$ be the surjective linear map defined thorugh the covering map $\pi\colon X \to X_0$. We define the *modified harmonic realization* $\Phi_0\colon X \to \Gamma \otimes \mathbb{R}$ by

$$\sum_{e \in E_x} p(e) \left\{ \Phi_0\big(t(e)\big) - \Phi_0\big(o(e)\big) \right\} = \rho_{\mathbb{R}}(\gamma_p), \qquad x \in V.$$

On the other hand, we can introduce a flat metric g_0 called the *Albanese metric* on $\Gamma \otimes \mathbb{R}$ associated with p from the inner product defined on the space of harmonic one-forms.

Let $B \in \mathrm{End}(\Gamma \otimes \mathbb{R})$ and $\varepsilon > 0$. For $(x, \mathbf{z}) \in V \times \mathrm{H}_1(X_0, \mathbb{R})$, we define a state-dependent transition probability $p_\varepsilon^B(x, \mathbf{z}; \cdot)\colon E_x \to (0,1)$ by

$$p_\varepsilon^B(x, \mathbf{z}; e) := \frac{1}{Z_\varepsilon^B(x, \mathbf{z})} p(e) \exp\left\{ -\varepsilon \left\langle B(\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z})), \Phi_0\big(t(e)\big) - \Phi_0\big(o(e)\big) - \rho_{\mathbb{R}}(\gamma_p) \right\rangle_{g_0} \right\}$$

for $e \in E_x$, where $Z_\varepsilon^B(x, \mathbf{z})$ is the normalizing constant. We fix a reference point $(x_*, \mathbf{z}_*) \in V \times \mathrm{H}_1(X_0, \mathbb{R})$. Then, the corresponding Markov chain with values in $V \times \mathrm{H}_1(X_0, \mathbb{R})$ is $\{(w_k, \mathbf{z}_* + k\gamma_p)\}_{k=0}^\infty$, where $\{w_n\}_{k=0}^\infty$ is the random walk on X starting from x_* with transition probability $p_\varepsilon^B(x, \mathbf{z}; \cdot)$. Its transition operator, acting on $C_\infty\big(X \times \mathrm{H}_1(X_0, \mathbb{R})\big)$, is defined by

$$\mathcal{L}_{\varepsilon, B, p} f(x, \mathbf{z}) := \sum_{e \in E_x} p_\varepsilon^B(x, \mathbf{z}; e) f\big(t(e), \mathbf{z} + \gamma_p\big), \qquad x \in V,\ \mathbf{z} \in \mathrm{H}_1(X_0, \mathbb{R}).$$

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We also define a map $\mathcal{P}_\varepsilon \colon C_\infty(\Gamma \otimes \mathbb{R}) \to C_b\big(X \times \mathrm{H}_1(X_0, \mathbb{R})\big)$ by

$$\mathcal{P}_\varepsilon f(x, \mathbf{z}) := f\big(\varepsilon\{\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z})\}\big), \qquad x \in V,\, \mathbf{z} \in \mathrm{H}_1(X_0, \mathbb{R}).$$

Let $(\mathsf{T}_t^B)_{t \geq 0}$ be the Ornstein–Uhlenbeck semigroup on $C_\infty(\Gamma \otimes \mathbb{R})$ defined by

$$\mathsf{T}_t^B f(\boldsymbol{x}) := \mathbb{E}[f(Y_t^{\boldsymbol{x}})], \qquad t \geq 0,\, \boldsymbol{x} \in \Gamma \otimes \mathbb{R},$$

where $(Y_t^{\boldsymbol{x}})_{t \geq 0}$ solves

$$\mathrm{d}Y_t^{\boldsymbol{x}} = -BY_t^{\boldsymbol{x}}\,\mathrm{d}t + \mathrm{d}W_t, \qquad Y_0^{\boldsymbol{x}} = \boldsymbol{x}.$$

Here, W is standard Brownian motion on $(\Gamma \otimes \mathbb{R}, g_0)$.

Theorem 1 *For $t \geq 0$ and $f \in C_\infty(\Gamma \otimes \mathbb{R})$, we have*

$$\lim_{n \to \infty} \Big\| \mathcal{L}_{n^{-1}, B, p}^{\lfloor nt \rfloor} \mathcal{P}_{n^{-1/2}} f - \mathcal{P}_{n^{-1/2}} \mathsf{T}_t^B f \Big\|_\infty = 0.$$

Suppose that $B \in \mathrm{End}(\Gamma \otimes \mathbb{R})$ is symmetric and positive definite with respect to g_0, and $Q_{\Phi_0}(x) \geq \kappa I$ for $x \in V$ and some $\kappa > 0$, where $Q_{\Phi_0} \colon V \to \mathrm{End}(\Gamma \otimes \mathbb{R})$ is given by

$$Q_{\Phi_0}(x) := \sum_{e \in E_x} p(e) \left\{ \Phi_0\big(t(e)\big) - \Phi_0\big(o(e)\big) - \rho_{\mathbb{R}}(\gamma_p) \right\}^{\otimes 2}, \qquad x \in V.$$

For $n \in \mathbb{N}$, we define

$$J_n(x, \mathbf{z}) := \left(\pi(x), \frac{1}{\sqrt{n}} \{\Phi_0(x) - \rho_{\mathbb{R}}(\mathbf{z})\} \right), \qquad x \in V_0,\, \mathbf{z} \in \mathrm{H}_1(X_0, \mathbb{R}).$$

The Markov operator K_n on $V_0 \times (\Gamma \otimes \mathbb{R})$ is then characterized by

$$\mathcal{L}_{n^{-1}, B, \gamma_p}(F \circ J_n) = (K_n F) \circ J_n$$

for every bounded measurable function F on $V_0 \times (\Gamma \otimes \mathbb{R})$. This is well-defined by the periodicity of the underlying graph and the modified harmonic realization Φ_0.

Theorem 2 *For $n \in \mathbb{N}$, the transition operator K_n admits the invariant probability measures ν_n on $V_0 \times (\Gamma \otimes \mathbb{R})$ that can be chosen so that their spatial marginals $\overline{\nu}_n := (\mathrm{pr}_2)_* \nu_n$ on $\Gamma \otimes \mathbb{R}$ converge weakly to the Gaussian measure*

$$\mu_B(\mathrm{d}\boldsymbol{y}) := \frac{(\det B)^{1/2}}{\pi^{d/2}} \exp\Big(- \langle B\boldsymbol{y}, \boldsymbol{y} \rangle_{g_0} \Big)\, \mathrm{d}\boldsymbol{y},$$

where $\mathrm{d}y$ *is the Lebesgue measure on* $(\Gamma \otimes \mathbb{R}, g_0)$.

If the underlying random walk is m-symmetric, we also obtain the explicit invariant probability measures directly on V, whose rescaled push-forward measures on $\Gamma \otimes \mathbb{R}$ converge weakly to μ_B. Several examples of state-dependent random walks on crystal lattices, together with their explicit invariant measures, are discussed as well.