Stochastic quantization of the weighted exponential QFT on the two-dimensional torus

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This talk is based on a preprint [2], which is joint work with Seiichiro Kusuoka (Kyoto University). We consider the equation

$$\partial_t \Phi_t = \frac{1}{2}(\Delta - 1)\Phi_t - \frac{1}{2}\int_{[-\alpha_0,\alpha_0]} \alpha \exp^{\diamond}(\alpha\Phi_t)\nu(d\alpha) + \dot{W}_t \tag{1}$$

on the 2-dimensional torus $\Lambda := \mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$, where ν is a finite nonnegative Borel measure on $[-\alpha_0, \alpha_0]$, $\alpha_0 \in (0, \sqrt{8\pi})$ and $\dot{W}$ is space-time white noise. This is the stochastic quantization equation of the so-called weighted $\exp(\Phi)_2$-quantum field model, and its invariant probability measure is given by

$$\mu^{(\nu)}(d\phi) = \frac{1}{Z^{(\nu)}} \exp\left(- \int_{[-\alpha_0,\alpha_0]} \left(\int_\Lambda \exp^{\diamond}(\alpha\phi)(x)dx \right) \nu(d\alpha) \right) \mu_0(d\phi), \tag{2}$$

where μ_0 is the massive Gaussian free field defined as a centered Gaussian probability measure on the space of periodic distributions $\mathcal{D}'(\Lambda)$ with covariance $(1-\Delta)^{-1}$, $Z^{(\nu)}$ is the normalization constant and $\exp^{\diamond}(\alpha\phi)$ denotes the Wick exponential formally given by

$$\exp^{\diamond}(\alpha\phi)(x) = \exp\left(\alpha\phi(x) - \frac{\alpha^2}{2} \int |\phi^2(x)|\mu_0(d\phi) \right). \tag{3}$$

Note that the diverging term $\int |\phi^2(x)|\mu_0(d\phi)$ plays the role of Wick renormalization. Some particular examples of $\mu^{(\nu)}$ are the $\exp(\Phi)_2$-model (or (massive) Liouville model)

$$\mu_L^{(\alpha)}(d\phi) = \frac{1}{Z_L^{(\alpha)}} \exp\Big(- \int_\Lambda \exp^{\diamond}(\alpha\phi)(x)dx \Big) \mu_0(d\phi),$$

and the $\cosh(\Phi)_2$-model (or sinh-Gordon model)

$$\mu_S^{(\alpha)}(d\phi) = \frac{1}{Z_S^{(\alpha)}} \exp\Big(- \int_\Lambda \cosh^{\diamond}(\alpha\phi)(x)dx \Big) \mu_0(d\phi).$$

These models correspond to the cases $\nu = \delta_\alpha$ and $\nu = \frac{\delta_\alpha + \delta_{-\alpha}}{2}$, respectively.

In this talk, we are mainly concerned with the (time-global) well-posedness of (1). Since equation (1) needs to be interpreted using a suitable renormalization, we consider an approximation operator P_N, $N \in \mathbb{N}$ defined by

$$P_N f := \sum_{l \in \mathbb{Z}^2, |l| \le A^N} \hat{f}(l) e_l, \tag{4}$$

where $A > 0$ is a suitably chosen constant, $e_l(x) \coloneqq \frac{1}{2\pi} e^{il\cdot x}$, and

$$\hat{f}(l) \coloneqq \int_\Lambda f(x) e_l(x) dx, \quad l \in \mathbb{Z}^2 \tag{5}$$

is the Fourier coefficient of $f \in \mathcal{D}(\Lambda) \coloneqq C^\infty(\Lambda)$. Let $H^s(\Lambda)$, $s \in \mathbb{R}$ be the (L^2-based) Sobolev spaces.

Our main result in this talk is the following.

Theorem. *Let $\alpha_0^2 < 4\pi$, $\beta \in (\frac{\alpha_0^2}{4\pi}, 1)$ and let Φ^N be the solution to the equation*

$$\begin{cases} \partial_t \Phi_t^N = \frac{1}{2}(\Delta - 1)\Phi_t^N - \frac{1}{2}\int_{[-\alpha_0, \alpha_0]} \alpha \exp\left(\alpha \Phi_t^N - \frac{\alpha^2}{2} C_N\right) \nu(d\alpha) + P_N \dot{W}_t \\ \Phi_0^N = P_N \xi + \eta, \end{cases} \tag{6}$$

where ξ is distributed according to the massive Gaussian free field measure μ_0 with covariance $(1-\Delta)^{-1}$ and is independent of $\{W_t\}_{t\geq 0}$, C_N is the renormalization constant and $\eta \in H^{2-\beta}(\Lambda)$ is non-random. Then, $\{\Phi^N\}_{N\in\mathbb{N}}$ converges to some Φ in $C([0,T]; H^{-\beta}(\Lambda))$ almost surely as $N \to \infty$.

Remarks.

- *The case $\nu = \delta_\alpha$ (the unweighted $\exp(\Phi)_2$-model) has already been considered in previous works, and the well-posedness result for the entire L^1-regime $\alpha^2 < 8\pi$ is known (cf. [1]). In the case of the general weighted $\exp(\Phi)_2$-model, the drift term in (1) can be both positive and negative, which makes the equation more difficult to treat. Extending the claim of the above theorem to the full regime $\alpha_0^2 < 8\pi$ is an open problem.*

- *The Dirichlet form approach to equation (1) will also be discussed in the talk. Unlike the direct PDE approach in the above theorem, this approach covers the full L^1-regime $\alpha_0^2 < 8\pi$.*

References

[1] M. Hoshino, H. Kawabi, and S. Kusuoka. Stochastic quantization associated with the $\exp(\Phi)_2$-quantum field model driven by space-time white noise on the torus in the full L^1-regime. Probab. Theory Related Fields, 185(1-2):391-447, 2023.

[2] S. Kusuoka and H. Nagoji, Stochastic quantization of the weighted exponential QFT, arXiv:2512.18927.