Approximation of irregular functionals of stochastic processes

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Abstract

Let X be a real-valued random variable on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a bounded density function $p_X : \mathbb{R} \to [0, \infty)$ with respect to Lebesgue measure. Then Avikainen [1] proved that for any real-valued random variable Y, $a \in \mathbb{R}$, $p \in [1, \infty)$ and $q \in (0, \infty)$, it holds that

$$\mathbb{E}[|\mathbf{1}_{(-\infty,a]}(X) - \mathbf{1}_{(-\infty,a]}(Y)|^p] \leq 3\|p_X\|_\infty^{\frac{q}{q+1}} \mathbb{E}[|X - Y|^q]^{\frac{1}{q+1}},$$

([1, Lemma 3.4], see also Giles, Higham and Mao [3], Giles and Xia [4]). This estimate hold true even if the test function $\mathbf{1}_{(-\infty,a]}$ is replaced by a function of bounded variation on $\mathbb{R}$ ([1, Theorem 2.4]). The idea of the proof is based on Skorokhod representation for X. This estimate can be applied to the numerical analysis on irregular functionals of stochastic differential equations (SDEs) based on the Euler–Maruyama scheme and the multilevel Monte Carlo method [2] whose computational cost is much lower than that of classical (single level) Monte Carlo method. Recently, Taguchi, Tanaka and Yuasa [8] generalized this estimate to multi-dimensional random variables. They prove that for $\mathbb{R}^d$-valued random variables X and Y, if both X and Y admit bounded density functions, then it holds that for any $p \in [1, \infty)$, $q \in (0, \infty)$ and compact set $D \subset \mathbb{R}^d$ with C^2-boundary,

$$\mathbb{E}[|\mathbf{1}_D(X) - \mathbf{1}_D(Y)|^p] \leq C\mathbb{E}[|X - Y|^q]^{\frac{1}{q+1}},$$

([8, Lemma 2.15]). This estimate hold true even if the test function $\mathbf{1}_D$ is replaced by a function of bounded variation on $\mathbb{R}^d$ ([8, Theorem 2.11]). However, we need to assume both X and Y admit bounded density functions. On the other hand, for one-dimensional setting, Giles, Higham and Mao [3] give a simple proof for the estimate (see also [4]). The idea of [3, 4] is based on the following condition: there exists a constant $C_{(-\infty,a]} > 0$ such that for any $\varepsilon \in (0, \infty)$,

$$\mathbb{P}(X \in (a - \varepsilon, a + \varepsilon)) \leq C_{(-\infty,a]}\varepsilon.$$

In this talk, we generalize the approach of [3, 4] to the multi-dimensional random variables. In particular, we prove that if a $\mathbb{R}^d$-valued random variable X and $D \in \mathscr{B}(\mathbb{R}^d)$ satisfy the following surface area condition: there exists a constant $C_D > 0$ such that for any $\varepsilon \in (0, \infty)$,

$$\mathbb{P}(d(X, \partial D) < \varepsilon) \leq C_D\varepsilon,$$

then for any $\mathbb{R}^d$-valued random variable Y, $p \in [1, \infty)$, $q \in (0, \infty)$ and bounded Lipschitz continuous function $f : \mathbb{R}^d \to \mathbb{R}$, it holds that

$$\mathbb{E}[|f(X)\mathbf{1}_D(X) - f(Y)\mathbf{1}_D(Y)|^p] \leq C_{p,q,f,D}\mathbb{E}[|X - Y|^q]^{\frac{1}{q+1}},$$

where the constant $C_{p,q,f,D}$ depend only on p, q, f and C_D (independent from Y).

The surface area condition $\mathrm{Leb}(\{x \in \mathbb{R}^d \mid d(x, \partial D) < \varepsilon\}) \leq L_D \varepsilon$ is related to Minkowski content in Geometric measure theory (see also Steiner's formula and Weyl's tube formula). In particular, if X admits a bounded density function and D is a compact set with Lipschitz boundary or convex body, then the surface area condition holds true. Moreover, if X is a d-dimensional Gaussian random variable, then the surface area condition $\sup_{D:\mathrm{convex}} \mathbb{P}(d(X, \partial D) < \varepsilon) \leq C d^{1/4} \varepsilon$ holds true [6, 7]. This estimate is called Nazarov's inequality [6] which can be applied to study Berry–Esseen type estimate. For example, Zhai [9] proved that there exists a constant $C > 0$ such that

$$|\mathbb{P}(X \in D) - \mathbb{P}(Y \in D)| \leq C \mathbb{E}[|X - Y|^2]^{1/3},$$

and then proved Berry–Esseen type estimate for convex set D by using the result of the central limit theorem in Wasserstein-2 distance. From our main result, we can also derive the above estimate, so it can be regarded as a generalization with respect to the L^p-norm for any $p \in [1, \infty)$.

We apply our main result to numerical analysis for some irregular functionals of several stochastic processes. To be specific, we pick up the following four stochastic processes:

(i) SDEs driven by Brownian motion

(ii) SDEs and its maximum

(iii) SDEs driven by α-stable process

(iv) BSDEs and its L^2-time regularity

For the first three examples, we can use the multilevel Monte Carlo method [2]. The last example can be applied to numerical schemes for BSDEs based on machine learning (e.g. [5]).

References

[1] Avikainen, R. On irregular functionals of SDEs and the Euler scheme. *Finance Stoch.* **13**(3) 381–401 (2009).

[2] Giles, M. B. Multilevel Monte Carlo path simulation. *Oper. Res.* **56**(3) 607–617 (2008).

[3] Giles, M. B., Higham, D. J. and Mao, X. Analysing multi-level Monte Carlo for options with non-globally Lipschitz payoff. *Finance Stoch.* **13**(3) 403–413 (2009).

[4] Giles, M. B. and Xia, Y. Multilevel Monte Carlo for exponential Lévy models. *Finance Stoch.* **21**(4) 995–1026 (2017).

[5] Huré, C., Pham, H. and Warin, X. Deep backward schemes for high-dimensional nonlinear PDEs. *Math. Comp.* **89** 1547–1579 (2020).

[6] Nazarov, F. On the maximal perimeter of a convex set in $\mathbb{R}^n$ with respect to a Gaussian measure. *Geometric Aspects of Functional Analysis (2001-2002), Lecture notes in Math.* 169–187 (2003).

[7] Raič, M. A multivariate Berry–Esseen theorem with explicit constants. *Bernoulli* **25**(4A) 2824–2853 (2019).

[8] Taguchi D., Tanaka, A. and Yuasa. T. L^q-error estimates for approximation of irregular functionals of random vectors. *IMA J. Numer. Anal.* **42**(1) 840–873 (2022).

[9] Zhai, A. A high-dimensional CLT in $\mathcal{W}_2$ distance with near optimal convergence rate. *Probab. Theory Related Fields* **170**(3) 821-845 (2018).