TRANSPORTATION COST INEQUALITIES FOR SINGULAR SPDES

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Transportation cost inequalities (TCIs) have been extensively studied due to the connection with the concentration of measure phenomenon, exponential integrability, and Sanov's type deviation estimate. Usually we say that, a (Borel) probability measure μ on a metric space $(\mathcal{X}, d)$ satisfies the p-TCI for $p \geq 1$ if

$$W_p(\nu, \mu) \lesssim \sqrt{H(\nu|\mu)}$$

for any other probability measure ν on $(\mathcal{X}, d)$, where $W_p(\nu, \mu)$ is the p-Wasserstein distance between ν and μ, and $H(\nu|\mu)$ is the relative entropy of ν with respect to μ. In particular, 2-TCIs have a deep connection with Gaussian measures, as Talagrand [9] showed that Gaussian measures on $\mathbb{R}^n$ satisfy a 2-TCI with a dimension-free proportional constant. Feyel and Üstünel [3] extended Talagrand's 2-TCI into abstract Wiener spaces, where the 2-Wasserstein distance is defined with respect to the Cameron–Martin distance. Moreover, Otto and Villani [7] showed that the 2-TCI is derived from the log-Sobolev inequality and also derives the Poincaré inequality, in complete connected Riemannian manifolds.

TCIs for the laws of solutions to stochastic differential equations (SDEs) have been also studied. First, Djellout, Guillin, and Wu [2] showed a 1-TCI with respect to the uniform distance for SDEs driven by a standard Brownian motion. Moreover, Riedel [8] showed a $(2-\epsilon)$-TCI with p-variation distance for $\epsilon > 0$ and $p \leq 2$, for SDEs driven by a wide class of Gaussian processes with finite p-variation. In the proof, Riedel exploited the rough path theory. An advantage of this theory is that the solution map of an SDE is decomposed into two parts: the lifting map from the driving noise to the rough path, and the solution map from the rough path to the solution of the SDE. Riedel showed that the TCI of the noise propagates through to each object by the local Lipschitz continuity of each map, even though the former map is continuous only with respect to Cameron–Martin directions.

Concerning TCIs for solutions to (especially, singular) stochastic partial differential equations (SPDEs), the celebrated theory of regularity structures by Hairer [5] would be a powerful tool. This theory inherits the basic story from the rough path theory. By an argument similar to Riedel [8], Gasteratos and Jacquier [4] showed TCIs for some stochastic problems to which the regularity structure can be applied, including 2-dimensional parabolic Anderson model. However, there is an essential change that rough paths are replaced by a family of more complicated stochastic objects called *models*. As the SPDE becomes more complex, the corresponding model also becomes more complex, making it difficult to prove the convergence of models through direct calculations. This is why the work [4] focused only on a particular SPDE.

In this talk, we refine the proof of general convergence result by Bailleul and the speaker [1] to establish TCIs for a wide class of models. Moreover we show that, if the SPDE under study admits a global-in-time solution with a suitable a priori estimate, the solution satisfy a certain TCI. For example, our result can be applied to the $\phi^4_{4-\epsilon}$ model with $\epsilon > 0$.

Following [4], we define a general type TCI to consider noise functionals that do not satisfy 1-TCI.

Definition 1. *Let $\mathcal{X}$ be a topological space and denote by $\mathcal{P}(\mathcal{X})$ the set of all Borel probability measures on $\mathcal{X}$. For any measurable function $c:\mathcal{X}\times\mathcal{X}\to[0,\infty]$ vanishing on the diagonal, the* **transportation cost** *between $\mu,\nu\in\mathcal{P}(\mathcal{X})$ is defined by*

$$W_c(\mu,\nu)=\inf_{\pi\in\Pi(\mu,\nu)}\iint_{\mathcal{X}\times\mathcal{X}}c(x,y)\pi(dxdy),$$

where $\Pi(\mu,\nu)$ is the collection of all couplings between μ and ν. For any increasing function $\alpha:[0,\infty]\to[0,\infty]$ such that $\alpha(0)=0$, we say that $\mu\in\mathcal{P}(\mathcal{X})$ satisfies the **(α,c)-TCI** *if*

$$\alpha\big(W_c(\mu,\nu)\big)\le H(\nu\,|\,\mu)$$

holds for all $\nu\in\mathcal{P}(\mathcal{X})$.

Theorem 1. *Let $\xi=\xi(t,x)$ be a spacetime noise on $\mathbb{R}\times\mathbb{T}^d$ taking values in a Schwartz distribution space and satisfying $(C(\cdot),\|\cdot-\cdot\|_H^2)$-TCI for some constant $C>0$ and some Sobolev space $H=H^{s_0}(\mathbb{R}\times\mathbb{T}^d)$ with $s_0\in\mathbb{R}$. We consider a parabolic type singular SPDE*

$$(\partial_t-\Delta)u=F(u,\nabla u,\xi),\qquad t>0,\ x\in\mathbb{T}^d, \tag{1}$$

where u is an unknown function/distribution and F is a smooth function which is affine with respect to ξ. If the SPDE (1) *is locally subcritical in the sense of* [5] *and satisfies the assumption of* [1] *(it is an assumption to avoid the "variance blowup" – see* [6]*), then the law of the BPHZ model associated with ξ satisfies an (α,c)-TCI, where α is an inverse of $\alpha^{-1}(s)=C(s^{1/2}+s^{1/(2m)})$ for some $C,m>0$ and $c(\cdot,\cdot)$ is a homogeneous distance in the space of models. Moreover, if the equation* (1) *admits a global-in-time solution u with a polynomial type a priori bounds in terms of the model norm, then u satisfies a certain $(C(\cdot)^2,\|\cdot-\cdot\|_{\mathcal{X}}^a)$-TCI for some $C,a>0$ and a spacetime function/distribution space $\mathcal{X}$.*

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References

[1] I. Bailleul and M. Hoshino, *Random models on regularity-integrability structures.* arXiv:2310.10202.
[2] H. Djellout, A. Guillin, and L. Wu, *Transportation cost-information inequalities and applications to random dynamical systems and diffusions*, The Annals of Probability **32** (2004), pp. 2702–2732.
[3] D. Feyel and A. S. Üstünel, *Measure transport on Wiener space and the Girsanov theorem*, Comptes Rendus Mathematique **334** (2002), pp. 1025–1028.
[4] I. Gasteratos and A. Jacquier, *Transportation-cost inequalities for non-linear Gaussian functionals.* arXiv:2310.05750.
[5] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), 269–504.
[6] M. Hairer, *Renormalisation in the presence of variance blowup*, arXiv:2401.10868.
[7] F. Otto and C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality.* Journal of Functional Analysis **173** (2000), pp. 361–400.
[8] S. Riedel, *Transportation-cost inequalities for diffusions driven by Gaussian processes.* Electronic Journal of Probability, **22** (2017), pp. 1–26.
[9] M. Talagrand, *Transportation cost for Gaussian and other product measures.* Geometric & Functional Analysis **6** (1996), pp. 587–600.

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