# Large deviation principle for stochastic differential equations driven by stochastic integrals

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This talk is based on [4]. Proving large deviation principle (LDP) for stochastic differential equations is important from the point of view of its application to the field of mathematical finance. In this talk, we will mainly consider the family stochastic differential equations

$$\mathrm{d}Y_t^\epsilon = \sigma(Y_t^\epsilon) f(\hat{X}_t^\epsilon, t)\mathrm{d}X_t^\epsilon - \frac{1}{2}\sigma^2(Y_t^\epsilon) f^2(\hat{X}_t^\epsilon, t)\mathrm{d}t, \quad t \in [0,1] \quad \epsilon > 0, \tag{0.1}$$

and will show how to prove the LDP for the solution of (0.1) on path spaces when $\epsilon \searrow 0$. Here $\sigma \in C_b^3$, $f$ is in $C^1$ or $\alpha$-Hölder spaces ($\alpha \in (0,1)$), $(X^\epsilon, \hat{X}^\epsilon) := \sqrt{\epsilon}(X, \hat{X})$, and $X$ is a Brownian motion. $\hat{X}$ is a stochastic process defined by

$$\hat{X}_t := \mathcal{K}W_t, \quad t \geq 0,$$

where for $\zeta, \gamma \in (0,1)$, $\mathcal{K} : C^{\gamma\text{-Hld}}([0,1]) \to C^{\zeta\text{-Hld}}([0,1])$ is a generalized fractional operator defined by

$$\mathcal{K}f(t) := \kappa(t)(f(t) - f(0)) + \int_0^t (f(s) - f(t))\kappa'(t-s)\mathrm{d}s, \quad f \in C^{\gamma\text{-Hld}}([0,1]),$$

$\kappa : (0,1] \to \mathbb{R}$ is given by

$$\kappa(t) := g(t)t^{\zeta-\gamma}, \quad g \text{ is a Lipschitz continuous function,} \tag{0.2}$$

and $W$ is a Brownian motion correlated to $X$. It is well-known that the Hölder regularity of $\hat{X}$ is $\zeta$. So if $\zeta \in (0, 1/3)$, then $\hat{X}$ is not controlled by $Y^\epsilon$ in the sense of controlled paths theory, which means that one cannot apply the usual rough path theory.

In [1], the authors proved the LDP for (0.1) when $\sigma = 1$, $f \in C^\infty$ and $\kappa(t) = t^{H-1/2}$ ($H \in (0, 1/2)$) by using the theory of regularity structures. In [2], the result obtained in [1] was generalized in the sense that $\sigma \in C_b^3$ and $\kappa$ is given by (0.2) by using a partial rough path approach. However, it is essential to assume that $f$ is a smooth function in these works, and so one can not apply the result when $f(s,t) = \sqrt{|s|}$ (see (0.3), for example).

Although $f(\hat{X}^\epsilon, \cdot)$ is not controlled by $Y^\epsilon$ as mention above, one can prove that the Itô integral $f(\hat{X}^\epsilon, \cdot) \cdot X^\epsilon$ is $\alpha$-Hölder continuous for $\alpha \in (1/3, 1/3)$. Therefore it is possible to use the usual rough path theory if we regard (0.1) as the equation driven by $f(\hat{X}^\epsilon, \cdot) \cdot X^\epsilon$. This idea is actually useful because the proof of the LDP for (0.1) comes down to that of the LDP for $f(\hat{X}^\epsilon, \cdot) \cdot X^\epsilon$ on Hölder spaces. Now let us note that how to prove the LDP for (0.1). We will first prove the LDP for stochastic integrals $\{Z^\epsilon\}_{\epsilon>0} := \{(f(\hat{X}^\epsilon, \cdot) \cdot X^\epsilon, f^2(\hat{X}^\epsilon, \cdot) \cdot \Lambda)\}_{\epsilon>0}$ on Hölder spaces, where $\Lambda(t) := t$. Then we lift $Z^\epsilon$ to rough path spaces continuously by using the Young pairing, and prove the LDP for (0.1) by using the contraction principle.

There are mainly two mathematical contributions for our work. Firstly, our approach does not use advanced theory like a partial rough path approach or the theory of regularity structures, just use the usual rough path theory and we are able to obtain more elementary and direct proof. Moreover, our method allows for a unified treatment of pathwise LDP for (0.1). There is the restriction for the assumption of $f$ in [1, 2]. Nevertheless, the assumption for $f$ can be weakened to the case of $C^1$ or Hölder. For example, one can treat the LDP for a kind of rough Heston models discussed in [3]

$$\begin{aligned} \mathrm{d}Y_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}X_t, \quad Y_0 = 0, \\ V_t &= |(\mathcal{K}\tilde{Y})_t|, \end{aligned} \tag{0.3}$$

in our setting, while it does not in the setting in [1, 2]. To the best of the author's knowledge, no such pathwise LDP for these models is known in the literature. Furthermore, our approach also works to prove the LDP for the solution of stochastic differential equation

$$\mathrm{d}Y_t^\epsilon = \sigma(Y_t^\epsilon)A_t^\epsilon \mathrm{d}X_t^\epsilon - \frac{1}{2}\sigma^2(Y_t^\epsilon)(A_t^\epsilon)^2\mathrm{d}t, \quad \epsilon > 0,$$

under the suitable assumption, where $A^\epsilon$ is an adapted continuous process.

# References


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