Quasi-ergodic theorems for Feynman-Kac semigroups and large deviation for additive functionals

Takara Tagawa*
(joint work with D. Kim and A. Velleret)

1 Introduction

This talk is based on [2, 3]. Let E be a locally compact separable metric space and $\mathfrak{m}$ be a positive Radon measure on E with full topological support. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \mathbb{P}_x, \zeta)$ be a strong Markov process on E with the lifetime ζ. We use $\mathbb{E}_x$ to denote the expectation with respect to $\mathbb{P}_x$ for any $x \in E$. Denote by $\{p_t\}_{t\geq 0}$ and $\{R_\alpha\}_{\alpha\geq 0}$ the transition semigroup and resolvent of X, respectively.

Insightful properties of an almost surely killed Markov process X (i.e., $\mathbb{P}_x(\zeta < \infty) = 1$ for any $x \in E$) when conditioned to survive have been derived by the study of the long-time behavior of the mean-ratio of a continuous additive functional. In [1, 4], the authors established the following quasi-ergodic limit theorem: there exists a (probability) measure η on E such that for any $f \in L^1(E; \eta)$ and $x \in E$,

$$\lim_{t\to\infty} \mathbb{E}_x \left[\frac{1}{t} \int_0^t f(X_s) \mathrm{d}s \;\middle|\; t < \zeta \right] = \int_E f(x) \eta(\mathrm{d}x). \tag{1.1}$$

The measure η is often called a quasi-ergodic distribution of X. Let G be a measurable function on $E \times E$ vanishing on the diagonal. Define a purely discontinuous additive functional of X by $A_t^G := \sum_{0<s\leq t} G(X_{s-}, X_s)$. This additive functional often appears when considering the pure jump effects in Markov processes and, in a particular case, is thought to express the number of jumps in pure jump processes. In [3], we propose to replace the integration in (1.1) over the full path of the processes by a summation over all jumps of the process up to time t (by taking into account both the position before and the position after the jump) and established a quasi-ergodic theorem for moments of the mean-ratio of A_t^G caused by the pure jump effects of symmetric Markov processes.

Let μ be a positive smooth measure on E whose associated positive continuous additive functional of X is denoted by A_t^μ. Let F be a positive symmetric bounded measurable function on $E \times E$ vanishing on the diagonal, and put $A_t^{\mu,F} := A_t^\mu + A_t^F$. In combination with the extinction that plays a similar role, we are led to consider the following family of non-local Feynman-Kac transforms, indexed by a time t which can be thought of as a final time for the biased action: for $x \in E$ and $\Lambda \in \mathcal{M}$,

$$\mathbb{P}_{x:t}^{\mu,F}(\Lambda) := \mathbb{P}_x\left(\Lambda \cdot W_t^{\mu,F}\right) \quad \text{where} \quad W_t^{\mu,F} := \exp\left(A_t^{\mu,F}\right) \mathbf{1}_{\{t<\zeta\}}. \tag{1.2}$$

With (1.2), we define a probability measure by $\mathbb{P}_{x|t}^{\mu,F}(\Lambda) := \mathbb{P}_{x:t}^{\mu,F}(\Lambda)/\mathbb{P}_{x:t}^{\mu,F}(\Omega)$ for $x \in E$ and $\Lambda \in \mathcal{M}$. We use $\mathbb{E}_{x|t}^{\mu,F}$ to denote the expectation with respect to $\mathbb{P}_{x|t}^{\mu,F}$.

The main subject of this talk is to discuss a quasi-ergodic limit theorem for the additive functional

$$A_t^{f,G} := \int_0^t f(X_s)\mathrm{d}s + \sum_{0<s\leq t} G(X_{s-}, X_s)$$

under the symmetric Markov process X driven by the *non-local Feynman–Kac transform* (1.2).

*Department of Mathematics and Engineering, Graduate School of Science and Technology, Kumamoto University

2 Main results

Let X is an $\mathfrak{m}$-symmetric irreducible explosive Markov process on E satisfying the strong Feller property (that is, $p_t(\mathfrak{B}_b(E)) \subset C_b(E)$ for any $t > 0$) and the tightness property (that is, for any $\varepsilon > 0$ there exists a compact $K \subset E$ such that $\sup_{x\in E} R_1 1_{K^c}(x) \le \varepsilon$). Let $(N(x, \mathrm{d}y), H_t)$ be a Lévy system for X. We assume that $\mu_H = \mathfrak{m}$ (so that $H_t = t$ for $t < \zeta$), where μ_H is the Revuz measure of the positive continuous additive functional H_t.

A positive smooth measure μ on E (resp. a positive measurable function F on $E \times E$) is said to be in the Kato class associated to X, if it satisfies

$$\lim_{t\to 0} \sup_{x\in E} \mathbb{E}_x \left[A_t^{\mu}\right] = 0 \quad \left(\text{resp. } \lim_{t\to 0} \sup_{x\in E} \mathbb{E}_x \left[\int_0^t \int_E F(X_s, y) N(X_s, \mathrm{d}y)\mathrm{d}s\right] = 0\right).$$

Under the Kato class conditions on μ and F, there is the ground state $\phi_0^{(\theta)} := \phi_0^{\theta\mu,\theta F}$ of the Feynman-Kac semigroup $p_t^{\theta\mu,\theta F} f(x) := \mathbb{E}_x[W_t^{\theta\mu,\theta F} f(X_t)]$, $\theta \in \mathbb{R}$. We say that $p_t^{\theta\mu,\theta F}$ is *θ-intrinsically ultracontractive* ($(\mathbf{IUC})_\theta$ in short), if for any $\theta \in \mathbb{R}$ there exists a constant $c_t(\theta) > 0$ such that $p_t^{\theta\mu,\theta F}(x, y) \le c_t(\theta)\phi_0^{(\theta)}(x)\phi_0^{(\theta)}(y)$ for all $t > 0$ and $x, y \in E$. Many examples are satisfying $(\mathbf{IUC})_\theta$ (cf. [2, 3]). Write $\phi_0 := \phi_0^{(1)}$. Define a measure $\mathcal{J}_{\phi_0}$ on $E \times E$ by

$$\mathcal{J}_{\phi_0}(\mathrm{d}x\mathrm{d}y) := \phi_0(x)\phi_0(y) \exp(F(x, y))N(x, \mathrm{d}y)\mathfrak{m}(\mathrm{d}x)$$

One of our main results is as follows:

Theorem 2.1. *Let μ and F be of the Kato class one associated to X. Assume $(\mathbf{IUC})_1$. Then, for any $f \in L^1(E; \phi_0^2\mathfrak{m})$, $G \in \mathfrak{B}(E \times E \setminus \mathsf{diag})$ satisfying $N^{\phi_0}[G] \in L^1(E; \phi_0^2\mathfrak{m})$ and any $x \in E$,*

$$\lim_{t\to\infty} \mathbb{E}_{x|t}^{\mu,F} \left[\frac{1}{t} A_t^{f,G}\right] = \int_E f\phi_0^2 \mathrm{d}\mathfrak{m} + \iint_{E\times E} G(x, y)\mathcal{J}_{\phi_0}(\mathrm{d}x\mathrm{d}y).$$

For a positive $V \in L^1(E; \phi_0^2\mathfrak{m})$, let $(\mathcal{E}^{V,F}, \mathcal{D}(\mathcal{E}))$ be the quadratic form given by

$$\mathcal{E}^{V,F}(u, v) := \mathcal{E}(u, v) - \int_E u(x)v(x)V(x)\mathfrak{m}(\mathrm{d}x) - \iint_{E\times E} u(x)v(y) \left(\exp(F(x, y)) - 1\right) N(x, \mathrm{d}y)\mathfrak{m}(\mathrm{d}x),$$

where $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the Dirichlet form associated with X. For $\theta > 0$, define the spectral function

$$C_{V,F}(\theta) := -\inf \left\{\mathcal{E}^{\theta V,\theta F}(u, u) \;\middle|\; u \in \mathcal{D}(\mathcal{E}), \int_E u^2 \mathrm{d}\mathfrak{m} = 1\right\}.$$

By the analytic perturbation theory, $C_{V,F}(\theta)$ is differentiable in θ and its derivative $C'_{V,F}(\theta)$ is strictly increasing. Put $\Psi_{V,F}(\theta) := C'_{V,F}(\theta)$. As an application of Theorem 2.1, we have the following LDP for $A_t^{V,F}$:

Theorem 2.2. *Let V and F be of the Kato class one associated to X. Assume $(\mathbf{IUC})_\theta$ for any $\theta \in \mathbb{R}$. Then, for any $\gamma \in \Psi_{V,F}(\mathbb{R}_+)^o := \{\Psi_{V,F}(\theta) : \theta \in \mathbb{R}_+\}^o$ and any $x \in E$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}_x \left(\frac{1}{t} A_t^{V,F} \in [\gamma, \infty),\ t < \zeta\right) = C_{V,F}(\theta_\gamma) - \theta_\gamma C'_{V,F}(\theta_\gamma), \quad \textit{where } \theta_\gamma = \Psi_{V,F}^{-1}(\gamma).$$

References

[1] L. A. Breyer and G. O. Roberts, *A quasi-ergodic theorem for evanescent processes*, Stochastic Process. Appl. **84**, no. 2, 177–186 (1999).

[2] D. Kim, T. Tagawa and A. Velleret, *Quasi-ergodic theorems for Feynman-Kac semigroups and large deviation for additive functionals*, preprint, arXiv:2401.17997 (2024).

[3] D. Kim and T. Tagawa, *Quasi-ergodic limits for moments of jumps under absorbing stable processes*, to appear in J. Theoret. Probab. (2024).

[4] J. Zhang, S. Li and R. Song, *Quasi-stationarity and quasi-ergodicity of general Markov processes*, Sci China Math. **57**, no.10, 2013–2024 (2014).