REGULARITY-INTEGRABILITY STRUCTURE AND ITS APPLICATION

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One of the key theorems in the theory of regularity structures [4] is the **reconstruction theorem**. This theorem provides a solution to the "inverse problem of Taylor's theorem", as described below.

Problem. *Given a family of distributions $\{F_x(\cdot)\}_{x\in\mathbb{R}^d}$, under what conditions concerning their "consistency" can one construct a unique distribution $F(\cdot)$ which is close to $F_x(\cdot)$ for each $x \in \mathbb{R}^d$?*

This problem can be seen as related to the sewing lemma in the rough path theory. In the reconstruction theorem initially proved by Hairer [4], the conditions concerning the consistency of $\{F_x(\cdot)\}_{x\in\mathbb{R}^d}$ are provided using two new concepts of model and modelled distribution. Hairer's theorem was later extended by Caravenna and Zambotti [3], who presented a necessary and sufficient condition on $\{F_x(\cdot)\}_{x\in\mathbb{R}^d}$ in a more general form, without using these new concepts.

In [4] and [3], the consistency conditions are described by Hölder type norms. However, to study problems involving Malliavin calculus, it would be convenient to extend the reconstruction theorem to Sobolev or Besov type norms, since the Cameron–Martin space of Gaussian noise is typically an L^2-type Sobolev space. As an extension of Hairer's theorem, Hairer and Labbé [5] proved a reconstruction theorem for $B^{(\cdot)}_{\infty,\infty}$-type models and $B^{(\cdot)}_{p,q}$-type modelled distributions. On the other hand, by Broux and Lee [2], Caravenna and Zambotti's general reconstruction theorem was also extended to $B^{(\cdot)}_{p,q}$-type Besov norms.

In this talk, we introduce a new concept of **regularity-integrability structures (RIS)**. In these structures, different integrability exponents are assigned to models and modelled distributions. Such a concept would be more suited for Malliavin calculus, since it allows for assigning different integrability exponents, such as "∞" and "2", to elements of the Wiener space and the Cameron–Martin space, respectively.

To introduce RIS, we define an ordering on two-dimensional labels. For any generic element $\mathbf{a} \in \mathbb{R}\times[1,\infty]$, we write $\mathbf{a} = (r(\mathbf{a}), i(\mathbf{a}))$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}\times[1,\infty]$, we define

$$\mathbf{b} \prec \mathbf{a} \quad \underset{\text{def}}{\Leftrightarrow} \quad r(\mathbf{b}) < r(\mathbf{a}) \ \ \& \ \ i(\mathbf{b}) \geq i(\mathbf{a}).$$

When $\mathbf{b} \prec \mathbf{a}$, we define $\mathbf{a} \ominus \mathbf{b} = \big(r(\mathbf{a}) - r(\mathbf{b}), (i(\mathbf{a})^{-1} - i(\mathbf{b})^{-1})^{-1}\big)$.

Definition 1. *A regularity-integrability structure (RIS) $\mathcal{T} = (\mathbf{A}, \mathbf{T}, \mathbf{G})$ consists of the following objects.*

- $\mathbf{A} \subset \mathbb{R}\times[1,\infty]$ *such that* $\{\mathbf{b} \in \mathbf{A}\,;\, \mathbf{b} \prec \mathbf{a}\}$ *is finite for any* $\mathbf{a} \in \mathbb{R}\times[1,\infty]$.
- $\mathbf{T} = \bigoplus_{\mathbf{a}\in\mathbf{A}} \mathbf{T}_\mathbf{a}$ *is an algebraic sum of Banach spaces* $(\mathbf{T}_\mathbf{a}, \|\cdot\|_\mathbf{a})$.

- $\mathbf{G}$ *is a group of continuous linear operators on* $\mathbf{T}$ *such that, for any* $\Gamma \in \mathbf{G}$ *and* $\mathbf{a} \in \mathbf{A}$,
$$(\Gamma - \mathrm{id})\mathbf{T}_{\mathbf{a}} \subset \mathbf{T}_{\prec \mathbf{a}} := \bigoplus_{\mathbf{b} \in \mathbf{A},\, \mathbf{b} \prec \mathbf{a}} \mathbf{T}_{\mathbf{b}}.$$

For any $\tau \in \mathbf{T}$ and $\mathbf{a} \in \mathbf{A}$, write $\|\tau\|_{\mathbf{a}} := \|\tau_{\mathbf{a}}\|_{\mathbf{a}}$, where $\tau_{\mathbf{a}}$ is the $\mathbf{T}_{\mathbf{a}}$-component of τ.

Definition 2. *A model* $M = (\Pi, \Gamma)$ *over a RIS* $\mathcal{T} = (\mathbf{A}, \mathbf{T}, \mathbf{G})$ *is a pair of the family of continuous linear operators*
$$\Pi = \{\Pi_x : \mathbf{T} \to \mathcal{S}'(\mathbb{R}^d)\}_{x \in \mathbb{R}^d} \quad \& \quad \Gamma = \{\Gamma_{xy}\}_{x,y \in \mathbb{R}^d} \subset \mathbf{G}$$
satisfying the following properties.

- $\Pi_x \Gamma_{xy} = \Pi_y$, $\Gamma_{xx} = \mathrm{id}$, *and* $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ *for any* $x, y, z \in \mathbb{R}^d$.
- *For any* $\mathbf{a} \in \mathbf{A}$ *and* $t \in (0,1]$, $\left\| \sup_{\|\tau\|_{\mathbf{a}} \le 1} \left| (e^{t\Delta}\Pi_x \tau)(x) \right| \right\|_{L_x^{i(\mathbf{a})}} \lesssim t^{r(\mathbf{a})/2}$.
- *For any* $\mathbf{b} \prec \mathbf{a}$ *and* $y \in \mathbb{R}^d$, $\left\| \sup_{\|\tau\|_{\mathbf{a}} \le 1} \|\Gamma_{(x+y)x}\tau\|_{\mathbf{b}} \right\|_{L_x^{i(\mathbf{a} \ominus \mathbf{b})}} \lesssim |y|^{r(\mathbf{a} \ominus \mathbf{b})}$.

Now our reconstruction theorem is presented as follows.

Theorem 1 ([6, Theorem 4.1])**.** *Let* $M = (\Pi, \Gamma)$ *be a model over a RIS* $\mathcal{T} = (\mathbf{A}, \mathbf{T}, \mathbf{G})$. *Let* $\mathbf{c} \in \mathbb{R} \times [1, \infty]$ *and let* $f : \mathbb{R}^d \to \mathbf{T}_{\prec \mathbf{c}}$ *be such that, for any* $\mathbf{a} \prec \mathbf{c}$,
$$\big\| \|f(x)\|_{\mathbf{a}} \big\|_{L_x^{i(\mathbf{c} \ominus \mathbf{a})}} + \sup_{y \in \mathbb{R}^d \setminus \{0\}} \frac{\big\| \|f(x+y) - \Gamma_{(x+y)x} f(x)\|_{\mathbf{a}} \big\|_{L_x^{i(\mathbf{c} \ominus \mathbf{a})}}}{|y|^{r(\mathbf{c} \ominus \mathbf{a})}} < \infty.$$
If $r(\mathbf{c}) > 0$, *there exists a unique* $\mathcal{R}f \in \mathcal{S}'(\mathbb{R}^d)$ *such that, for any* $t \in (0,1]$,
$$\big\| e^{t\Delta}\big(\mathcal{R}f - \Pi_x f(x)\big)(x) \big\|_{L_x^{i(\mathbf{c})}} \lesssim t^{r(\mathbf{c})/2}.$$

In the latter half part of this talk, based on [1], I will report on the application of the above reconstruction theorem to the inductive proof of the BPHZ theorem.

References

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