Regularity of the density function for SDEs with BV drift

Dai Taguchi (Kansai University)

Abstract

Let $X^x = (X_t^x)_{t\in[0,T]}$ be a solution to d-dimensional stochastic differential equation (SDE)

$$\mathrm{d}X_t^x = b(t, X_t^x)\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t,\ X_0^x = x \in \mathbb{R}^d,\ t \in [0,T], \tag{1}$$

where $B = (B(t))_{t\in[0,T]}$ is a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, and the drift coefficient $b : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is bounded measurable and the diffusion coefficient $\sigma : [0,T] \to \mathbb{R}^{d\times d}$ is bounded measurable and uniformly elliptic.

The existence and regularity of the density function with respect to Lebesgue measure of SDEs (1) have been widely studied by many authors. For one-dimensional case, if the drift coefficient b is bounded, continuous and Lipschitzs continuous in space and the diffusion coefficient $\sigma = 1$, Maruyama [3] prove the existence of the density function denoted by $p_t(x,\cdot)$ of X_t^x by using the Euler–Maruyama scheme. On the other hand, it is well-known that by using Levi's parametrix method, there exists the fundamental solution of parabolic type partial differential equations (Kolmogorov equations), and it is the density function of a solution of associated SDEs. The parametrix method is useful tools in order to prove the regularity for the density function. Indeed, the density function $p_t(x,\cdot)$ has a continuous version and it is β-Hölder continuous for any $\beta \in (0,1)$, that is, there exist $C > 0$ and $c > 0$ such that for any $x, y, z \in \mathbb{R}^d$ and $t \in (0,T]$,

$$|p_t(x,y) - p_t(x,z)| \le \frac{C}{t^{\beta/2}} |y-z|^{\beta} \left\{ g_{ct}(x,y) + g_{ct}(x,z) \right\}. \tag{2}$$

As a probabilistic approach, Malliavin calculus is a useful tool to study smoothness of the density function. However, if the drift coefficient is only assumed to be bounded measurable, then it might be difficult to use the Malliavin calculus to study existence and regularity of the density function. As an alternative approach, Fournier–Printems [2] use the "one-step Euler–Maruyama scheme" defined by

$$X_s^{x,\varepsilon} = X_{t-\varepsilon}^x + \int_{t-\varepsilon}^s b(u, X_{t-\varepsilon}^x)\,\mathrm{d}u + \int_{t-\varepsilon}^s \sigma_u\,\mathrm{d}B_u,\ s \in [t-\varepsilon, t].$$

Note that in [2], the authors consider SDE $\mathrm{d}X_t^x = b(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}B_t$, $X_0^x = x$ and use the one-step Euler–Maruyama scheme $X_s^{x,\varepsilon} = X_{t-\varepsilon}^x + \sigma(X_{t-\varepsilon}^x)(B_s - B_{t-\varepsilon})$ without drift coefficient. They prove the existence of the density function of one-dimensional SDEs, stochastic heat equations and Lévy driven SDEs. Moreover, Romito [4] prove the existence and Besov regularity for the density function of multi-dimensional SDEs $\mathrm{d}X_t^x = b(X_t^x)\,\mathrm{d}t + \sigma(X_t^x)\,\mathrm{d}B_t$, $X_0^x = x$. More precisely, he prove that by using the fractional integration by parts formula, if the drift coefficient is bounded

measurable and diffusion coefficient is bounded, uniformly elliptic and β-Hölder continuous with $\beta \in (0,1]$, then the density function belongs to the Besov spaces $B^s_{p,\infty}(\mathbb{R}^d)$ for any $p \in [1, d/(d-1))$ and $s \in (0, \beta)$. Moreover, if the drift coefficient is β-Hölder continuous with $\beta \in (0,1]$ and the diffusion coefficient is the identity matrix, then the density function $p_t(x, \cdot)$ belongs to the Besov space $B^s_{1,\infty}(\mathbb{R}^d)$ for any $s \in (0, 1+\beta)$. These results imply that the Besov regularity of the density function depends on the Hölder continuity of the coefficients.

In this talk, for SDEs (1) with "dis-continuous" drift coefficient, we extend the above results. More precisely, if the drift coefficient belongs to function of bounded variation $BV(\mathbb{R}^d)$, we prove that the density function belongs to the Besov spaces $B^s_{p,\infty}(\mathbb{R}^d)$ for any $p \in [1, d/(d-1))$ and $s \in (0, 1+1/p)$. The main idea of the proof is to use the following inequality. Let $X, \widehat{X}$ be $\mathbb{R}^d$ valued random variables with bounded density functions p_X and $p_{\widehat{X}}$, respectively, then for any $f \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, $p \in (0, \infty)$ and $q \in [1, \infty)$, it holds that

$$\mathbb{E}[|f(X) - f(\widehat{X})|^q] \leq C(p,q) \mathbb{E}[|X - \widehat{X}|^p]^{\frac{1}{p+1}},$$

(for example, $\mathbf{1}_E \in BV(\mathbb{R}^d)$ for a bounded subset E of $\mathbb{R}^d$ with C^2 boundary). Avikainen [1] proved this inequality for one-dimensional random variables, and then Taguchi–Tanaka–Yuasa [5] extend it for multi-dimensional case. Note that the estimate can be applied to the numerical analysis for the payoff of the binary option in mathematical finance based on the multilevel Monte Carlo methods.

It is known that, if the drift coefficient is dis-continuous, then we cannot expect the continuously differentiability of the density function. On the other hand, the Bessov regularity of the density function with $p = 1$ and $s \in [1, 2)$ leads to a Sobolev regularity of the density function $p_t(x, \cdot) \in W^{s,1}(\mathbb{R}^d)$ for any $s \in [1, 2)$. In particular, comparing with Hölder continuity (2), by using Hajłasz's characterized of Sobolev spaces, we have the following pointwise estimate; there exist $K > 0$ and a Lebesgue null set $N \in \mathscr{B}(\mathbb{R}^d)$ such that for any $y, z \in \mathbb{R}^d \setminus N$,

$$|p_t(x,y) - p_t(x,z)| \leq K|y-z| \left\{ \mathscr{M}(|Dp_t(x,\cdot)|)(y) + \mathscr{M}(|Dp_t(x,\cdot)|)(z) \right\},$$
$$|D_i p_t(x,y) - D_i p_t(x,z)| \leq K|y-z|^{s-1} \left\{ \mathscr{M}(G_{s-1,1} D_i p_t(x,\cdot))(y) + \mathscr{M}(G_{s-1,1} D_i p_t(x,\cdot))(z) \right\},$$

for $i = 1, \ldots, d$, where $\mathscr{M} f$ is the Hardy–Littlewood maximal function, and for $r \in (0,1)$ and $p \in [1, \infty)$, the operator $G_{r,p}$ are Gagliardo semi-norm.

References

[1] Avikainen, R. On irregular functionals of SDEs and the Euler scheme. *Finance Stoch.* **13**(3) 381–401 (2009).

[2] Fournier, N. and Printems, J. Absolute continuity of some one-dimensional processes. *Bernoulli* **16** 343–360 (2010).

[3] Maruyama, G. On the transition probability functions of the Markov process. *Nat. Sci. Rep. Ochanomizu Univ.* **5** 10–20 (1954).

[4] Romito, M. A simple method for the existence of a density for stochastic evolutions with rough coefficients. *Electorn. J. Probab.* **23**(113) 1–43 (2018).

[5] Taguchi D., Tanaka, A. and Yuasa. T. L^q-error estimates for approximation of irregular functionals of random vectors. *IMA J. Numer. Anal.* **42**(1) 840–873 (2022).