Normalizability of the Gibbs measures associated with multivariate version of $P(\Phi)_2$ model

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1 Abstract

In this talk, we consider the Gibbs measures associated with multivariate version of $P(\Phi)_2$ quantum field model on the torus. We observe the (non-)normalizability of the measures by the variational method introduced by Barashkov and Gubinelli in [1]. We also consider some other related models.

2 Setting

Let μ be the Gaussian measure on $\mathcal{D}'(\mathbb{T}^2)$ with the covariance $(1-\Delta)^{-1}$ and we consider the probability measure on $\mathcal{D}'(\mathbb{T}^2)^{\otimes n}$ formally written by

$$\text{“}\mu^F(d\Phi) = \frac{1}{Z} e^{\int_{\mathbb{T}^2} F(\Phi(x))dx} \mu^{\otimes n}(d\Phi)\text{”} \tag{1}$$

where Z is the normalizing constant, F is an n-variate polynomial and $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ is the 2-dimensional torus. The construction of this kind of measures is important in view of Euclidean quantum field theory. The density function of the measure μ^F in (1) is ill-defined because the Gaussian measure μ is supported in the Sobolev space $H^{-\epsilon}(\mathbb{T}^2)$ with $\epsilon > 0$, so we have to introduce renormalization to define the measure i.e. we consider the Wick renormalized density $e^{\int_{\mathbb{T}^2} :F(\Phi(x)):dx}$, where it is defined by

$$:F(\Phi) := \lim_{N\to\infty} :F(P_N^{\otimes n}\Phi) := \lim_{N\to\infty} \sum_{\beta\in A} c_\beta H_\beta\left(P_N^{\otimes n}\Phi;\sigma_N\right) \tag{2}$$

with the following notation. Let F be the polynomial written by

$$F(\boldsymbol{x}) = \sum_{\beta\in A} c_\beta \boldsymbol{x}^\beta = \sum_{\beta=(\beta_1,\cdots,\beta_n)\in A} c_\beta \prod_{i=1}^n x_i^{\beta_i}, \qquad \boldsymbol{x} = (x_1,\cdots,x_n)\in\mathbb{R}^n$$

where $c_\beta \in \mathbb{R}\backslash\{0\}$ and $A \subset \mathbb{N}^n$, let P_N be the approximation operator given by

$$P_N f := \sum_{l\in\mathbb{Z}^2,\ |l|\le N} \hat{f}(l) e^{\sqrt{-1}l\cdot x}, \quad \text{for } f \in \mathcal{D}'(\mathbb{T}^2),$$

and let σ_N be the suitably chosen renormalization constant which diverges logarithmically as $N \to \infty$. Moreover, for $\beta \in \mathbb{N}^n, \boldsymbol{x}\in\mathbb{R}^n$ and $\sigma\in\mathbb{R}$, we define the Hermite polynomial $H_\beta(\boldsymbol{x};\sigma)$ by

$$e^{\boldsymbol{t}\cdot\boldsymbol{x} - \frac{1}{2}\sigma|\boldsymbol{t}|^2} = \sum_{\beta\in\mathbb{N}^n} \frac{\boldsymbol{t}^\beta}{\beta!} H_\beta(\boldsymbol{x};\sigma) \qquad \boldsymbol{t}\in\mathbb{R}^n. \tag{3}$$

The limit (2) is known to be well-defined as a distribution-valued random variable.

It is well-known that when $n = 1$ and F is a polynomial of degree 3 or larger, μ^F is well-defined as a probability measure (after the above renormalization procedure) if and only if F is bounded above. In our main theorems, we extended it to the multivariate setting like (1).

3 Main theorem

We define

$$A^- := \{\beta \in \mathbb{N}^n \ ; \ \beta < \gamma \ \exists \gamma \in A\} \quad \text{where} \ \ \beta < \gamma \iff \gamma - \beta \in \mathbb{R}^n_+ \backslash \{\mathbf{0}\},$$

and write $\boldsymbol{a}_{\boldsymbol{r}}(x) := (a_1 x^{r_1}, \cdots, a_n x^{r_n}) \in \mathbb{R}^n$ for $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $\boldsymbol{r} = (r_1, \cdots, r_n) \in \mathbb{N}^n$ and $x \in \mathbb{R}$. We obtained the following results.

Theorem 1. *1. If there exist some $\epsilon > 0$ and $m < \frac{1}{2}$ such that F satisfies*

$$\sup_{\boldsymbol{x} \in \mathbb{R}^n} \left(F(\boldsymbol{x}) + \sum_{\beta \in A^-} |\boldsymbol{x}^\beta|^{1+\epsilon} - m|\boldsymbol{x}|^2 \right) < \infty, \tag{4}$$

there holds

$$\int e^{\int_{\mathbb{T}^2} :F(\Phi):(x)dx} \mu^{\otimes n}(d\Phi) < \infty.$$

In particular, the probability measure

$$\mu^F(d\Phi) = \frac{1}{Z} e^{\int_{\mathbb{T}^2} :F(\Phi(x))dx:} \mu^{\otimes n}(d\Phi)$$

with $Z = \int e^{\int_{\mathbb{T}^2} :F(\Phi):(x)dx} \mu^{\otimes n}(d\Phi)$ is well-defined.

2. If there exist some $\boldsymbol{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, $\boldsymbol{r} = (r_1, \cdots, r_n) \in \mathbb{N}^n$, $m > \frac{1}{2}$ and $C > 0$ such that

$$\sup_{x \in \mathbb{R}_+} \left(F(\boldsymbol{a}_{\boldsymbol{r}}(x)) + C \sum_{\beta \in A^-} |\boldsymbol{a}_{\boldsymbol{r}}(x)^\beta| - m|\boldsymbol{a}_{\boldsymbol{r}}(x)|^2 \right) = \infty, \tag{5}$$

then, there holds

$$\int e^{\int_{\mathbb{T}^d} :F(\Phi):(x)dx} \mu^{\otimes n}(d\Phi) = \infty.$$

Remark 3.1. *As you can see, there is a gap between the sufficinet condition for the normalizability and the one for the non-normaliability. But still we can determine whether μ^F is well-defined as a probability measure for a large class of polynomial F from our results.*

This result can be extended to some other related models such as the Gibbs measure tamed by Wick-ordered L^2-norm. We also consider such models in the talk.

References

[1] N. Barashkov and M. Gubinelli. A variational method for Φ^4_3. Duke Math. J. 169 (2020), no. 17, 3339-3415.

[2] H. Nagoji. Construction of the Gibbs measures associated with Euclidean quantum field theory with various polynomial interactions in the Wick renormalizable regime. arXiv:2305.19583 [math.PR]