RANDOM MODELS ON REGULARITY-INTEGRABILITY STRUCTURES

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This talk is based on a joint work [1] with Ismaël Bailleul (Univ Brest).

In the past decade, **singular stochastic PDEs (SSPDEs)** have been one of the hottest topics in the probability theory. An important example is the dynamical Φ^4_3 model, which is a nonlinear heat equation of the form

$$\partial_t \Phi(t,x) = \Delta\Phi(t,x) - \Phi^3(t,x) + \xi(t,x), \qquad t>0,\ x\in\mathbb{R}^3$$

with a spacetime Gaussian white noise $\xi(t,x)$. The main difficulty of this equation is the irregularity of ξ. Because of this, the solution Φ cannot be realized as a measurable function, but as a tempered distribution. Hence the cubic term Φ^3 cannot be defined in the classical sense. However, if we perform a transform involving divergence for the nonlinear term, we happen to get a meaningful solution in some cases. (Of course, the additional term should be of less degree than the original nonlinear term.) Indeed, given an approximation $\{\xi_n\}_{n=1}^\infty$ of the noise ξ by smooth functions, there is a deterministic constant C_n which diverges as $n\to\infty$ such that, the solution Φ_n to

$$\partial_t \Phi_n(t,x) = \Delta\Phi_n(t,x) - \Phi_n^3(t,x) + C_n\Phi_n(t,x) + \xi_n(t,x)$$

(with the same initial value for all n) converges as $n\to\infty$ in the space of distributions. Such a transform is called a **renormalization**. The theory of **regularity structures** [4] has developed as a basic tool for understanding renormalizations of SSPDEs.

The perturbative approach provides a heuristic picture of renormalization. Once we regard ξ as an a "smooth" input function and perform a usual Picard iteration, we can express the "solution" Φ as an infinite series of multilinear functionals of ξ. However, since ξ is actually a distribution, we have to define each multilinear functional of ξ via renormalization. This makes the work much easier than trying to renormalize the original nonlinear term Φ^3 of a completely unknown distribution Φ directly. **BPHZ theorem** (named after Bogoliubov, Parasiuk, Hepp, and Zimmerman) is a statement that ensures the convergences of such renormalized multilinear functionals of ξ under some assumptions.

In the study of SSPDEs, it has been a challenging problem to obtain a simple proof of BPHZ theorem. Chandra and Hairer [3] proved BPHZ theorem in the most general setting so far, but their proof is quite long (more than 120 pages) and tough reading. On the other hand, Linares, Otto, Tempelmayr, and Tsatsoulis [6] recently proposed an inductive proof based on the spectral gap inequality of ξ in the study of quasilinear SSPDEs. Inspired by their approach, Hairer and Steele [5] also obtained an alternative inductive proof of [3]. Their theorems are less general than [3], but relatively compact and sufficient for Gaussian cases. In this talk, we introduce an extension of the regularity structure including integrability exponents and provide a much simpler proof (less than 30 pages) than [6, 5].

The following is a rough explanation of our theorem. Let $\xi(t,x)$ be a spacetime Gaussian white noise on $\mathbb{R}\times\mathbb{R}^d$. Starting from the symbol Ξ expressing ξ, we recursively construct abstract symbols by acting two kinds of operators: (i) multiplications of symbols $(\tau,\sigma)\mapsto\tau\sigma$ and (ii) linear operators $\tau\mapsto\mathcal{I}_k\tau$ ($k\in\mathbb{N}^{1+d}$) expressing the convolution with k-th derivative of heat kernel. To each symbol τ, we give an "expected regularity" $r(\tau)\in\mathbb{R}$ by the following rule.

$$r(\Xi)=-\frac{d+2}{2}-\varepsilon,\qquad r(\tau\sigma)=r(\tau)+r(\sigma),\qquad r(\mathcal{I}_k\tau)=r(\tau)+2-|k|_s,$$

where $\varepsilon>0$ is a fixed small constant and $|(k_i)_{i=0}^d|_s:=2k_0+\sum_{i=1}^d k_i$ is the "parabolic scale" of multiindex. For any smooth approximation $\{\xi_n\}$ of ξ, we can naively define the "interpreter" $\{\Pi_x^n\}_{x\in\mathbb{R}^{1+d}}$ which translates the symbols τ into the smooth functions $(\Pi_x^n\tau)(\cdot)$ by

$$\begin{aligned}&(\Pi_x^n\Xi)(\cdot)=\xi_n(\cdot),\qquad \Pi_x^n(\tau\sigma)=(\Pi_x^n\tau)(\Pi_x^n\sigma),\\ &(\Pi_x^n\mathcal{I}_k\tau)(\cdot)=\partial^k I(\Pi_x^n\tau)(\cdot)-\sum_{\ell\in\mathbb{N}^{d+1},\, r(\mathcal{I}_{k+\ell}\tau)>0}\frac{(\cdot-x)^\ell}{\ell!}\partial^{k+\ell}I(\Pi_x^n\tau)(x)\quad (k\in\mathbb{N}^{d+1}),\end{aligned}$$

where $I(F):=\int_{-\infty}^t e^{(t-s)(\Delta-1)}F(s)ds$ denotes the convolution with heat kernel. In general, the family of operators $\{\Pi_x\}_{x\in\mathbb{R}^{1+d}}$ which interpret symbols to some distributions is called a **model**. As explained at the beginning of this abstract, we cannot expect the convergence of naive models $\{\Pi_x^n\}$ as $n\to\infty$ without renormalization. The algebraic procedure to get the most appropriate renormalized version $\{\hat{\Pi}_x^n\}$ (called the **BPHZ model**) is already established in [2] without convergence results.

Theorem 1 ([5, 1])**.** *If $r(\tau)>-\frac{d+2}{2}$ for any symbol τ except Ξ, then the BPHZ model $\{\hat{\Pi}_x^n\}$ converges to a distribution-valued model as $n\to\infty$.*

References

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