Infinite dimensional Markovian lifts of stochastic Volterra equations*

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1 Introduction

We consider the following stochastic Volterra equation (SVE for short):

$$X_t = x(t) + \int_0^t K(t-s) b(X_s) \, \mathrm{d}s + \int_0^t K(t-s) \sigma(X_s) \, \mathrm{d}W_s, \quad t > 0, \tag{1}$$

associated with a d-dimensional Brownian motion W. Here, the drift coefficient $b : \mathbb{R}^n \to \mathbb{R}^n$, the diffusion coefficient $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$, the kernel $K : (0, \infty) \to [0, \infty)$ and the forcing term $x : (0, \infty) \to \mathbb{R}^n$ are given. If the kernel K and the forcing term x are constants, say $K(t) = 1$ and $x(t) = x_0$ for some $x_0 \in \mathbb{R}^n$, then the SVE (1) becomes a standard stochastic differential equation (SDE for short). More generally, if K is the fractional kernel, i.e. $K(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$ with exponent $\alpha \in (\frac{1}{2}, 1]$, and if the forcing term x is of the form $x(t) = \frac{1}{\Gamma(\alpha)} t^{a-1} x_0$ for some $x_0 \in \mathbb{R}^n$, then the SVE (1) corresponds to a kind of time-fractional SDE. SVEs provide suitable models of dynamics with hereditary properties, memory effects and roughness of the path, which cannot be described by standard SDEs.

The analysis of SVEs is, however, much more difficult than that of standard SDEs since the solutions are no longer Markovian or semimartingales in general. For these reasons, we cannot apply fundamental tools established in the literature of Markov or semimartingale theory to SVEs directly, and many important issues such as well-posedness of SVEs with non-Lipschitz coefficients are remained open. A new framework to analyse SVEs is highly demanded in views of theory and applications.

2 Main results

This talk is based on the preprints [1, 2]. In this talk, we introduce an infinite dimensional framework which captures Markov and semimartingale structures behind the SVE (1). Assume that the kernel $K : (0, \infty) \to [0, \infty)$ is completely monotone, i.e., K is infinitely differentiable and satisfies $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} K(t) \geq 0$ for any $t \in (0, \infty)$ and any nonnegative integers k. By Bernstein's theorem, there exists a unique Radon measure μ on $[0, \infty)$ such that $K(t) = \int_{[0,\infty)} e^{-\theta t} \, \mu(\mathrm{d}\theta)$, $t > 0$. Taking into account this representation, we reformulate the SVE (1) into the following equation:

$$\begin{cases} \mathrm{d}Y_t(\theta) = -\theta Y_t(\theta) \, \mathrm{d}t + b(\mu[Y_t]) \, \mathrm{d}t + \sigma(\mu[Y_t]) \, \mathrm{d}W_t, & \theta \in \operatorname{supp} \mu, \ t > 0, \\ Y_0(\theta) = y(\theta), & \theta \in \operatorname{supp} \mu. \end{cases} \tag{2}$$

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Here, $\operatorname{supp}\mu \subset [0,\infty)$ is the support of the measure μ, and $\mu[y] := \int_{\operatorname{supp}\mu} y(\theta)\,\mu(\mathrm{d}\theta)$ for suitable functions $y : \operatorname{supp}\mu \to \mathbb{R}^n$. The above equation can be seen as a stochastic evolution equation (SEE for short) on an infinite dimensional space consisting of functions $y : \operatorname{supp}\mu \to \mathbb{R}^n$. We formulate a state space of the SEE (2) as a separable Hilbert space $\mathcal{H}_\mu$ (or a Gelfand triplet of Hilbert spaces $\mathcal{V}_\mu \hookrightarrow \mathcal{H}_\mu \hookrightarrow \mathcal{V}_\mu^*$), incorporating the singularity of the kernel K. The SEE (2) turns out to be an "infinite dimensional Markovian lift" of the SVE (1).

In this talk, we explain the following three main results in details:

1. *There is a* one-to-one correspondence *between solutions X of the SVE (1) and (mild) solutions Y of the SEE (2); the correspondence is given by the representation formula $X_t = \mu[Y_t]$.*

2. *Assume that the coefficients b and σ are uniformly continuous and that σ is non-degenerate (in the sense that $\sigma(x)\sigma(x)^\top \geq c_{\mathrm{UE}} I_{n\times n}$ for any $x \in \mathbb{R}^n$ for some constant $c_{\mathrm{UE}} > 0$). Then, under a "balance condition" between the modulus of continuity of σ and the singularity of the kernel K,* weak existence *and* uniqueness in law *hold for the SVE (1) and the SEE (2). Furthermore, every weak solution Y of the SEE (2) (which is unique in law) is a time-homogeneous Markov process on the Hilbert space $\mathcal{H}_\mu$ and satisfies the Feller property.*

3. *Assume that the coefficients b and σ are Lipschitz continuous and that σ is non-degenerate. Then, the Markov semigroup $\{P_t\}_{t\geq 0}$ (defined on the space $\mathcal{B}_{\mathrm{b}}(\mathcal{H}_\mu)$ of real-valued bounded Borel measurable functions on the Hilbert space $\mathcal{H}_\mu$) associated with the SEE (2) satisfies the following* asymptotic log-Harnack inequality*:*
$$P_t \log f(\bar{y}) \leq \log P_t f(y) + \frac{c_{\mathrm{Lip}}}{2c_{\mathrm{UE}}}\|y-\bar{y}\|^2_{\mathcal{H}_\mu} + c_{\mathrm{Lip}} e^{-\delta t/2}\|y-\bar{y}\|_{\mathcal{H}_\mu}\|\nabla \log f\|_\infty$$
for any $t \geq 0$, any $y, \bar{y} \in \mathcal{H}_\mu$ and any $f \in \mathcal{B}_{\mathrm{b}}(\mathcal{H}_\mu)$ with $f \geq 1$ and $\|\nabla \log f\|_\infty < \infty$. Here, $\delta := \inf \operatorname{supp}\mu\,(\geq 0)$, and $c_{\mathrm{Lip}} > 1$ is a constant depending on Lipschitz constants of b and σ.

Let us make some remarks on the above results.

- Concerning the second result, the idea of the proof is to show the weak well-posedness of the SEE (2) using stochastic calculus in Hilbert spaces, and then to translate it into the original SVE (1). Note that no additional conditions are assumed on the drift coefficient b other than the uniform continuity. For example, although the (one-dimensional) deterministic fractional Volterra equation $X_t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|X_s|^\beta \mathrm{sign}(X_s)\,\mathrm{d}s$ with $\alpha \in (\frac{1}{2},1]$ and $\beta \in (0,1)$ has infinitely many solutions, the corresponding SVE with non-degenerate "Volterra noise" is weakly well-posed in the sense of uniqueness in law. This reveals the regularization-by-noise effect for Volterra equations.

- Concerning the third result, assuming in addition that $\delta := \inf \operatorname{supp}\mu > 0$, the asymptotic log-Harnack inequality implies asymptotic properties for the Markov semigroup $\{P_t\}_{t\geq 0}$ such as asymptotic gradient estimate (hence, asymptotic strong Feller property), asymptotic heat kernel estimate, asymptotic irreducibility, and uniqueness of the invariant probability measure.

References

[1] Y. Hamaguchi, Markovian lifting and asymptotic log-Harnack inequality for stochastic Volterra integral equations, preprint, arXiv:2304.06683, 2023.

[2] Y. Hamaguchi, Weak well-posedness of stochastic Volterra equations with completely monotone kernels and non-degenerate noise, preprint, arXiv:2310.16030, 2023.