Error distribution of general numerical solution of 1-dim SDE driven by fBm with arbitrary Hurst index

UEDA Kento

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In this talk, we consider an ODE

$$y_t = \xi + \int_0^t \sigma(y_s)dB_s + \int_0^t b(y_s)ds \tag{1}$$

driven by the fractional Brownian motion(fBm) B with Hurst exponent H and numerical schemes $\hat{y}^{(m)}_{\cdot}$ of this equation. Then, we aim to calculate asymptotic error distributions G, defined by

$$\lim_{m\to\infty} 2^{mR}\left(\hat{y}^{(m)}_{\cdot} - y_{\cdot}\right) = {}^{\exists}G_{\cdot}\ ,$$

of these schemes.

Equation (1) is defined in the context of a rough path of arbitrary dimension but can be defined using symmetric integral[1] in one dimension.

Studies of error distributions belong to the central limit theorem, and Brownian motion independent from the solution may appear in the error distributions.

The (k-)Milstein method

$$\hat{y}^{(m)}_t = \begin{cases} \hat{y}^{(m)}_{\tau^m_r} + \sum_{j=1}^k \frac{1}{j!}(\mathcal{D}^{j-1}\sigma)(\hat{y}^{(m)}_{\tau^m_r})B^j_{\tau^m_r,t} + b(\hat{y}^{(m)}_{\tau^m_r})(t-\tau^m_r) & \\ +\frac{1}{2}b'b(\hat{y}^{(m)}_{\tau^m_r})(t-\tau^m_r)^2 + \frac{1}{2}(\sigma b' + b\sigma')B^j_{\tau^m_r,t}(t-\tau^m_r) & \tau^m_r < t \le \tau^m_{r+1} \\ \xi & t = 0 \end{cases} \tag{2}$$

and the Crank-Nicholson method

$$\hat{y}^{(m)}_t = \begin{cases} \hat{y}^{(m)}_{\tau^m_r} + \frac{1}{2}(\sigma(\hat{y}^{(m)}_{\tau^m_r})B_{\tau^m_r,t} + \sigma(\hat{y}^{(m)}_t)) + \frac{1}{2}(b(\hat{y}^{(m)}_{\tau^m_r}) + b(\hat{y}^{(m)}_t))(t-\tau^m_r) & \tau^m_r < t \le \tau^m_{r+1} \\ \xi & t = 0 \end{cases} \tag{3}$$

have been particularly well studied. The Milstein method is the most standard numerical solution method with good behavior, while the Milstein method is a numerical solution method with a simple definition but special behavior.

In the one-dimensional case, for equations without drift terms, the error distributions for the Milstein method for arbitrary Hurst exponents in [3], the Crank-Nicholson method for specific diffusion coefficients and $H > 1/6$ in [2], and the Crank-Nicholson method for $H > 1/3$ in [4] have been calculated. On the other hand, [6] obtains error distributions in the range of the Hurst exponent $H > 1/3$ for both the Milstein and Crank-Nicholson schemes with drift coefficients. By contrast, in the multidimensional case, [5][7] justifies the error distribution of the Milstein scheme with $H > 1/3$. This talk is an extension of the method of [7].

We show the following result about one-dimensional SODE and numerical schemes.

- We have found and justified a method for computing error distributions in general iterative schemes, except when the numerical solutions converge at an exceptionally low Hurst indices.. For example, the Crank-Nicholson method can calculate the error distribution when $H > 1/4$. Also, the (k-)Milstein method can compute the numerical solution when $H > 1/(2\lfloor k/2 \rfloor)$.
- We proved the existence of schemes that converges faster than previously known numerical solution methods and calculated their error distribution.
- We classified numerical schemes defined independently from the Hurst index, including Runge-Kutta schemes, Milstein schemes, and Crank-Nicolson scheme, according to the qualitative form of the error distribution.

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