CAPACITY OF THE RANGE OF RANDOM WALK

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1. Introduction

This talk is based on the joint work with Amir Dembo (Stanford university).

We study the capacity of the range of a simple random walk in three and higher dimensions. It is known that the order of the capacity of the random walk range in n dimensions is similar to that of the volume of the random walk range in $n-2$ dimensions. We show that this correspondence breaks down for the law of the iterated logarithm for the capacity of the random walk range in three dimensions. We also prove the law of the iterated logarithm in higher dimensions.

2. Main Results

Let τ_A denote the first positive hitting time of a finite set A by a simple random walk on $\mathbb{Z}^d$, denoted by $(S_m)_{m\geq 0}$. We define the corresponding (Newtonian) capacity for $d \geq 3$, by

$$\mathrm{Cap}(A) := \sum_{x\in A} P^x(\tau_A = \infty).$$

We consider the asymptotics of the capacity $\mathrm{Cap}(\mathcal{R}_n)$ of the random walk range $\mathcal{R}_n := \{S_1, \dots, S_n\}$.

For $d = 3$, we fully determine the almost sure fluctuations of $n \mapsto \mathrm{Cap}(\mathcal{R}_n)$ for the simple random walk in the form of the law of the iterated logarithm (possibly after centering $\mathrm{Cap}(\mathcal{R}_n)$ when $d \geq 4$). Specifically, we use $\log_k a = \log(\log_{k-1} a)$ for $k \geq 2$, with $\log_1 a$ for the usual logarithm.

Theorem 2.1. *For $d = 3$, almost surely,*

$$\limsup_{n\to\infty} \frac{\mathrm{Cap}(\mathcal{R}_n)}{h_3(n)} = 1\,, \qquad \liminf_{n\to\infty} \frac{\mathrm{Cap}(\mathcal{R}_n)}{\hat{h}_3(n)} = 1,$$

where

$$h_3(n) := \frac{\sqrt{6}\pi}{9}(\log_3 n)^{-1}\sqrt{n\log_2 n}, \quad \hat{h}_3(n) := \frac{\sqrt{6}\pi^2}{9}\sqrt{n(\log_2 n)^{-1}}.$$

Theorem 2.2. *For $d = 4$, almost surely,*

$$\limsup_{n\to\infty} \frac{\mathrm{Cap}(\mathcal{R}_n) - E\mathrm{Cap}(\mathcal{R}_n)}{h_d(n)} = 1, \quad \liminf_{n\to\infty} \frac{\mathrm{Cap}(\mathcal{R}_n) - E\mathrm{Cap}(\mathcal{R}_n)}{\hat{h}_d(n)} = -1, \tag{2.1}$$

where for some non-random $0 < c_\star < \infty$,

$$h_4(n) := \frac{\pi^2}{8}\frac{n\log_3 n}{(\log n)^2}\,, \qquad \hat{h}_4(n) := c_\star \frac{n\log_2 n}{(\log n)^2}\,.$$

For any $d \geq 5$, (2.1) hold almost surely, now with

$$h_d(n) = \hat{h}_d(n) := \sigma_d\sqrt{2n(1 + 1_{\{d=5\}}\log n)\log_2 n}\,, \quad d \geq 5,$$

where the non-random, finite $\sigma_d^2 > 0$ are given by the leading asymptotic of $\mathrm{var}(\mathrm{Cap}(\mathcal{R}_n))$.