Bounds for density function of solutions to stochastic functional differential equations

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Abstract

In this talk, we shall consider the following stochastic functional differential equations (SFDEs)

$$X(t) = \begin{cases} x_0 + \int_0^t \sigma(s, \{X(u)\}_{-\infty<u\leq s-r})dw(s) + \int_0^t b(s, \{X(u)\}_{-\infty<u\leq s-r})ds, t \geq 0 \\ \eta(t), t < 0 \end{cases} \tag{1}$$

where $r \geq 0$, $\eta(\cdot) \in C((-\infty, 0]; \mathbb{R}^D)$ is deterministic, $\eta(0) = x_0 \in \mathbb{R}^D$, $[0,\infty) \times C((-\infty, 0]; \mathbb{R}^D) \times C([0, s - r]; \mathbb{R}^D) \ni (s, \psi, \phi) \mapsto \sigma(s, \psi, \phi) \in \mathbb{R}^D \otimes \mathbb{R}^d$, $[0, \infty) \times C((-\infty, 0]; \mathbb{R}^D) \times C([0, s-r]; \mathbb{R}^D) \ni (s, \psi, \phi) \mapsto b(s, \psi, \phi) \in \mathbb{R}^D$ are measurable and $\{w(t), t \in [0, \infty)\}$ denotes a d-dimensional standard Brownian motion.

The smoothness of the density function of $X(t)$ denoted as $p_t(x_0, x)$ has been studied by some authors. For example, Bell-Mohammed [1, 2] prove the smoothness of $p_t(x_0, x)$ under some conditions on the coefficients. Takeuchi [5] proves the smoothness of the density function $p_t(x_0, x)$ for more general coefficients.

The goal of this talk is to study on bounds of $p_t(x_0, x)$ under certain smoothness, boundedness and ellipticity conditions for the coefficients of (1). In the talk, we first obtain the following upper bound for the density function,

$$p_t(x_0, x) \leq M_1(t)e^{M_2(t)|x-x_0|^2}$$

in the case that $r = 0$. Then, we show the following lower bound,

$$p_t(x_0, x) \geq m_1(t, r)e^{m_2(t,r)|x-x_0|^2}$$

in the case that $r > 0$. For the proof, we shall use the Malliavin calculus developed in Kusuoka-Stroock [4]. The proof of the upper bound is based on an integration by parts formula derived from the Malliavin calculus. In order to prove the lower bound, we will use Watanabe distribution theory which allows us to express $p_t(x_0, x)$ as generalized expectation of $\delta_x(X(t))$ where δ_x denotes the Dirac δ-function at $x \in \mathbb{R}^D$(e.g. Chap. V of [3]).

References

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