A REGULARITY STRUCTURE FOR THE QUASILINEAR GENERALIZED KPZ EQUATION

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This talk is based on [2], a joint work with Ismaël Bailleul (Université Rennes 1) and Seiichiro Kusuoka (Kyoto University).

1. Introduction

Let $\mathbb{T}$ be the one dimensional torus. Fix smooth functions f, g, and a taking values in a compact subset of $(0,\infty)$. We consider the quasilinear generalized KPZ equation

$$\big(\partial_t - a(u)\partial_x^2\big)u = f(u)\xi + g(u)(\partial_x u)^2 \tag{1}$$

in $(t,x)\in(0,\infty)\times\mathbb{T}$, where ξ is a random spacetime distribution with regularity $\beta\in(-2,0)$. The spacetime white noise is contained in the case $\beta=-\frac{3}{2}-$. One of the difficulties for (1) is that the regularity β is too rough to define the multiplications $a(u)\partial_x^2 u$, $f(u)\xi$, and $g(u)(\partial_x u)^2$ in classical senses. Hence we need renormalizations of these terms.

There are some previous researches on the renormalization of quasilinear equations based on the theory of regularity structures. For example,

- In [7, 5], the equation (1) with $g=0$ was considered in the spacetime periodic setting. In [6], a priori Hölder estimate was proved for the solutions to the (renormalized) equation (1) with $(f,g)=(1,0)$.
- In [4], the local well-posedness was proved for the (renormalized) equation (1), where ξ is a noise with regularity $\beta=-\frac{3}{2}+$. In [3], a similar result was proved for the case $(f,g)=(1,0)$ and $\beta\in(-2,0)$.

Our aim is to prove the local well-posedness for the equation (1) in the full regime $\beta\in(-2,0)$.

2. Main results

Our approach is based on [1], where the two dimensional quasilinear parabolic Anderson model was studied. It is related to the equation (1) with $g=0$ and $\beta=-1-$.

Let $u_0\in C^{0+}(\mathbb{T})$ be the initial condition of (1). Following [1], we set $v=e^{t_0\partial_x^2}u_0$ with small $t_0>0$ and rewrite the equation (1) as

$$\big(\partial_t - a(v)\partial_x^2\big)u = f(u)\xi + g(u)(\partial_x u)^2 + \big(a(u)-a(v)\big)\partial_x^2 u. \tag{2}$$

Our approach is to establish the theory of regularity structures for the equation (2), which can be seen as a semilinear equation with a rough coefficient $a(v)$ and with the perturbation term $\big(a(u)-a(v)\big)\partial_x^2 u$.

Theorem 1 ([2, Theorem 1]). *Let $\{\xi^\varepsilon\}_{\varepsilon\in(0,1]}$ be a family of smooth noises converging to ξ as $\varepsilon\to 0$. We assume that the "BPHZ" model associated with* (2) *converges as $\varepsilon\to 0$. Then the solution u_ε to*

$$\big(\partial_t - a(u_\varepsilon)\partial_x^2\big)u_\varepsilon = f(u_\varepsilon)\xi_\varepsilon + g(u_\varepsilon)(\partial_x u_\varepsilon)^2 - \sum_{\tau^{\boldsymbol{p}}} \frac{C_\varepsilon^v[\tau^{\boldsymbol{p}}](x)}{S(\tau^{\boldsymbol{p}})}\Upsilon^v[\tau^{\boldsymbol{p}}](u_\varepsilon,\partial_x u_\varepsilon)$$

with $u_\varepsilon(0,\cdot)=u_0$ converges in probability (locally in time), where

- *$\tau^{\boldsymbol{p}}$ runs over an* infinite *number of rooted decorated trees,*
- *$S(\tau^{\boldsymbol{p}})$ is an integer coming from the symmetry of the tree,*
- *C_ε^v is a continuous function of x,*
- *$\Upsilon^v[\tau^{\boldsymbol{p}}]$ is a smooth function at most linear with respect to $\partial_x u_\varepsilon$.*

The above theorem is general but has some uncomfortable points in the renormalization term: an infinite summation and the dependence on v. In particular, C_ε^v and Υ^v depend on v non-locally in general. Nevertheless, we consider the next assumption.

Assumption 1. *The main part of the function $C_\varepsilon^v[\tau^{\boldsymbol{p}}]$ is a local function of $a(v)$, that is, there is a function $c_\varepsilon[\tau^{\boldsymbol{p}}]$ with the property*

$$C_\varepsilon^v[\tau^{\boldsymbol{p}}](x) = c_\varepsilon[\tau^{\boldsymbol{p}}]\big\{a(v(x))\big\} + O(1).$$

This assumption is too strong to be expected to hold in general, but it does hold for some settings studied in the previous researches [1, 4, 3].

Theorem 2 ([2, Theorem 2]). *Under the above assumption, the renormalization term takes the form*

$$\sum_{\tau^{\boldsymbol{p}}} \frac{C_\varepsilon^v[\tau^{\boldsymbol{p}}](x)}{S(\tau^{\boldsymbol{p}})}\Upsilon^v[\tau^{\boldsymbol{p}}](u_\varepsilon,\partial_x u_\varepsilon) = \sum_{\tau} \frac{c_\varepsilon[\tau](a(u_\varepsilon))}{S(\tau)}\Upsilon[\tau](u_\varepsilon,\partial_x u_\varepsilon) + O(1),$$

where τ runs over a finite *number of trees, Υ does not depend on v, and $O(1)$ is uniform in ε.*

References

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