Estimates of the local spectral dimension of the Sierpinski gasket*

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Let K be the two-dimensional standard Sierpinski gasket and λ the normalized Hausdorff measure. The transition density $p_t(x,y)$ of Brownian motion on K—which is associated with the canonical Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\lambda)$—was extensively studied by Barlow and Perkins [1]. In particular, the following sub-Gaussian estimate is known:

$$
\begin{aligned}
c_1 t^{-d_\mathrm{s}/2}\exp\left(-c_2\left(\frac{|x-y|_{\mathbb{R}^2}^{d_\mathrm{w}}}{t}\right)^{-1/(d_\mathrm{w}-1)}\right) &\le p(t,x,y)\\
\le c_3 t^{-d_\mathrm{s}/2}\exp\left(-c_4\left(\frac{|x-y|_{\mathbb{R}^2}^{d_\mathrm{w}}}{t}\right)^{-1/(d_\mathrm{w}-1)}\right), &\qquad x,y\in K,\ t\in(0,1],
\end{aligned}
$$

where c_j $(j=1,2,3,4)$ are positive constants, $d_\mathrm{s}=2\log_5 3=1.36521\cdots$ is the *spectral dimension*, and $d_\mathrm{w}=\log_2 5=2.32192\cdots>2$ is the *walk dimension*. On the other hand, the transition density $q_t(x,y)$ of the time-changed Brownian motion by the Kusuoka measure ν—which is associated with the Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\nu)$—was studied in [7, 5, 4]. The behavior of $q_t(x,y)$ is quite different from that of $p_t(x,y)$ and is somewhat Gaussian-like. Concerning the short-time asymptotics of the on-diagonal $q_t(x,x)$, in particular, the following result is known.

Theorem 1 ([4, Theorem 1.3 (2) and Proposition 6.6])**.** *There exists a constant* $d_\mathrm{s}^\mathrm{loc}\in(1,2\log_{25/3}5]$ *such that*

$$
\lim_{t\downarrow 0}\frac{2\log q_t(x,x)}{-\log t}=d_\mathrm{s}^\mathrm{loc},\quad \nu\text{-}a.e.\ x.
$$

Moreover, $d_\mathrm{s}^\mathrm{loc}$ *is described as*

$$
d_\mathrm{s}^\mathrm{loc}=2-\frac{2\log(5/3)}{\log(5/3)-\rho}, \tag{0.1}
$$

where $\rho=\lim_{m\to\infty}\rho_m=\inf_{m\in\mathbb{N}}\rho_m$ *with*

$$
\rho_m=\frac{1}{m}\sum_{w\in W_m}\nu(K_w)\log\nu(K_w). \tag{0.2}
$$

Here, $W_m=\{1,2,3\}^m$ *is the totality of words consisting of letters* $1,2,3$ *with length* m*;* K_w *is a cell of* K *corresponding the word* $w\in W_m$*.*

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We call $d_{\mathrm{s}}^{\mathrm{loc}}$ the *local spectral dimension* of K. From numerical computation of ρ_m with $m = 16$, a quantitative estimate of $d_{\mathrm{s}}^{\mathrm{loc}}$ is given in [4, Remark 6.7 (1)] as

$$\left(2 - \frac{2\log(5/3)}{\log(5/3) - \rho_{16}} = \right) 1.27874\cdots \le d_{\mathrm{s}}^{\mathrm{loc}} \le 1.51814\cdots \bigl(= 2\log_{25/3} 5\bigr).$$

It seems difficult to obtain a substantially sharper estimate of $d_{\mathrm{s}}^{\mathrm{loc}}$ by using only the above equations. Here, we discuss quantitative estimates of $d_{\mathrm{s}}^{\mathrm{loc}}$ by another approach. The following are main results.

Theorem 2. *It holds that*

$$\begin{aligned}(1.271650\cdots =)\, \frac{15\log 3 + 15\log 5 - 14\log 7}{15\log 5 - 7\log 7} &\le d_{\mathrm{s}}^{\mathrm{loc}} \\ &\le \frac{5\log 5 - 3\log 3}{5\log 5 - 4\log 3}\,(= 1.300763\cdots). \end{aligned} \tag{0.3}$$

In particular, $d_{\mathrm{s}}^{\mathrm{loc}} < d_{\mathrm{s}}$.

Numerical result. With the help of numerical calculation by *Mathematica* [8],

$$1.291008\cdots \le d_{\mathrm{s}}^{\mathrm{loc}} \le 1.291026\cdots.$$

In particular, the first few digits of $d_{\mathrm{s}}^{\mathrm{loc}}$ are $1.2910\cdots$, a value that happens to be close to $\sqrt{5/3} = 1.290994\cdots$.

The ingredients for the arguments are a result about a bias of the distribution ratios of ν in [3], which was firstly studied in [2], and an integral representation of ρ by an invariant measure of some Markov chain on the space of distribution ratios of ν.

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