Invariance of Brownian motion associated with exponential functionals

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It is well known that Brownian motion enjoys several distributional invariances such as the scaling property and the time reversal. In this talk, we provide another invariance of Brownian motion that is compatible with the time reversal. The invariance, which seems to be new to our best knowledge, is described in terms of an anticipative path transformation involving exponential functionals as anticipating factors.

Let $B = \{B_t\}_{t\geq 0}$ be a one-dimensional standard Brownian motion. Fix $t > 0$ below and denote by $C([0,t];\mathbb{R})$ the space of real-valued continuous functions over $[0,t]$, on which we define an anticipative path transformation $\mathcal{T}$ by

$$\mathcal{T}(\phi)(s) := \phi_s - \log\left\{1 + \frac{A_s(\phi)}{A_t(\phi)}\left(e^{2\phi_t} - 1\right)\right\}, \quad 0 \leq s \leq t,\ \phi \in C([0,t];\mathbb{R}).$$

Here $A_s(\phi) := \int_0^s e^{2\phi_u}\, du$. One of the main results of this talk is

Theorem 1 ([4, Theorem 1.1 and Corollary 1.1])**.** *It holds that*

$$\left\{\left(\mathcal{T}(B)(s),\, B_s\right)\right\}_{0\leq s\leq t} \overset{(d)}{=} \left\{\left(B_s,\, \mathcal{T}(B)(s)\right)\right\}_{0\leq s\leq t}.$$

In particular, the Wiener measure on $C([0,t];\mathbb{R})$ is invariant under $\mathcal{T}$.

Related results such as an invariance of Brownian motion with drift will also be discussed. This talk is based on [4] and [5]; other references will be referred to in the talk.

References

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