# Central limit theorems for stochastic wave equations in dimension three

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Asymptotic behavior of spatial averages of stochastic partial differential equations has recently attracted much interest, and many works had devoted to establishing Gaussian fluctuations under various settings in the last few years. For stochastic heat equations, a central limit theorem (CLT) and its functional version are first proved in [2] in a one-dimensional case using the Malliavin-Stein method. After that, the authors of [3] show that similar results also hold in arbitrary spatial dimensions $d \geqslant 1$. As for stochastic wave equations, however, such proof has only been known for $d \leqslant 2$. See *e.g.* [4] and the reference therein.

In this talk, we consider three-dimensional stochastic wave equations driven by a Gaussian noise that is white in time and has some spatial correlations and prove CLT and functional CLT for the spatial average of the solution. If time allows, we will also discuss the difficulties and possibilities in obtaining CLTs for cases $d > 3$. This talk is based on [1].

## Setting and main results

Let us consider the following stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u(t,x) + \sigma(u(t,x))\dot{W}(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^3, \\ u(0,x) = 1, \quad \dfrac{\partial u}{\partial t}(0,x) = 0, \end{cases} \tag{1}$$

where $T > 0$ is fixed, $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with bounded derivative, and $\dot{W}(t,x)$ is a centered Gaussian noise with covariance (formally) given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)\gamma(x-y). \tag{2}$$

In (2), $\delta_0$ denotes the Dirac delta function and $\gamma$ satisfies one of the two conditions below:

(i) $\gamma$ is a positive integrable function such that $\gamma(x)dx$ is a nonnnegative definite tempered measure and its Fourier transform $\mu$ (in $\mathcal{S}'(\mathbb{R}^3)$) satisfies

$$\int_{\mathbb{R}^3} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty.$$

(ii) $\gamma(x) = |x|^{-\beta}$ for some $\beta \in (0,2)$.

Let $F_R(t)$ denote the following centered spatial integral of the solution to (1):

$$F_R(t) = \int_{B_R} (u(t,x) - 1)dx,$$

where $B_R \coloneqq \{x \in \mathbb{R}^3 \mid |x| \leqslant R\}$. Define $\sigma_R(t) = \sqrt{\mathrm{Var}(F_R(t))}$. Recall that the Wasserstein distance $d_{\mathrm{W}}$ between the laws of two integrable real-valued random variables $X$ and $Y$ is defined by

$$d_{\mathrm{W}}(X,Y) = \sup_{h \in \mathscr{H}} |\mathbb{E}[h(X)] - \mathbb{E}[h(Y)]|, \quad \mathscr{H} := \{h : \mathbb{R} \to \mathbb{R} \mid \|h\|_{\mathrm{Lip}} \leqslant 1\}.$$

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The first main result is the CLT for $F_R(t)$ as $R \to \infty$.

**Theorem 1.** *Assume that $\sigma(1) \neq 0$. Then, for any fixed $t \in (0, T]$, we have $\sigma_R(t) > 0$ for every $R > 0$ and*

$$\lim_{R\to\infty} d_{\mathrm{W}} \left( \frac{F_R(t)}{\sigma_R(t)}, Z \right) = 0, \tag{3}$$

*where $Z \sim \mathcal{N}(0, 1)$.*

Let $C([0, T])$ denote the space of continuous functions on $[0, T]$, and set

$$\tau_\beta = \int_{B_1^2} |x - y|^{-\beta} dx dy, \quad \eta(r) = \mathbb{E}[\sigma(u(r, 0))]. \tag{4}$$

The functional CLT is the second main result.

**Theorem 2.** (1) *Let $\gamma \in L^1(\mathbb{R}^3)$. Then, as $R \to \infty$, the process $\{R^{-\frac{3}{2}} F_R(t) \mid t \in [0, T]\}$ converges weakly in $C([0, T])$, and the limiting process is a centered Gaussian process $\{\mathcal{G}_1(t) \mid t \in [0, T]\}$ with covariance function*

$$\mathbb{E}[\mathcal{G}_1(t)\mathcal{G}_1(s)] = |B_1| \int_{\mathbb{R}^3} \mathrm{Cov}(u(t, x), u(s, 0)) dx.$$

(2) *Let $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$. Then , as $R \to \infty$, the process $\{R^{\frac{\beta}{2}-3} F_R(t) \mid t \in [0, T]\}$ converges weakly in $C([0, T])$, and the limiting process is a centered Gaussian process $\{\mathcal{G}_2(t) \mid t \in [0, T]\}$ with covariance function*

$$\mathbb{E}[\mathcal{G}_2(t)\mathcal{G}_2(s)] = \tau_\beta \int_0^{t\wedge s} (t - r)(s - r)\eta^2(r) dr,$$

*where $t \wedge s := \min\{t, s\}$.*

# References


[1] Masahisa Ebina. Central limit theorems for nonlinear stochastic wave equations in dimension three, 2022. Preprint, arXiv:2206.12957.

[2] Jingyu Huang, David Nualart, and Lauri Viitasaari. A central limit theorem for the stochastic heat equation. *Stochastic Process. Appl.*, 130(12):7170–7184, 2020.

[3] Jingyu Huang, David Nualart, Lauri Viitasaari, and Guangqu Zheng. Gaussian fluctuations for the stochastic heat equation with colored noise. *Stoch. Partial Differ. Equ. Anal. Comput.*, 8(2):402–421, 2020.

[4] David Nualart and Guangqu Zheng. Central limit theorems for stochastic wave equations in dimensions one and two. *Stoch. Partial Differ. Equ. Anal. Comput.*, 10(2):392–418, 2022.