Global solvability of a quasilinear SPDE

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This talk is based on the joint work [2] with T. Funaki, Waseda University. Let $g \in C^4(\mathbb{R})$ such that $c_- \le g'(v) \le c_+$ holds for some $c_-, c_+ > 0$ and $a = g'$. We mainly study the global-in-time solvability and establish the convergence of the solution to the stationary solution as $t \to \infty$ for the following quasilinear stochastic partial differential equation (SPDE) defined in paracontrolled sense:

$$\partial_t u = a(\nabla u)\Delta u + g(\nabla u) \cdot \xi, \tag{1.1}$$

on one dimensional torus $\mathbb{T} \simeq [0, 1)$ having the spatial noise $\xi \in C^{\alpha-2}, \alpha \in (\frac{13}{9}, \frac{3}{2})$, where $\nabla = \partial_x$, $\Delta = \partial_x^2$ and $C^\alpha \equiv C^\alpha(\mathbb{T})$ denotes the Hölder-Besov space on $\mathbb{T}$ with regularity exponent $\alpha \in \mathbb{R}$ equipped with the norm $\| \cdot \|_{C^\alpha}$. The SPDE (1.1) can be derived from a microscopic particle system in random environment on 1-d discrete lattice and the local-in-time solvability of (1.1) without restriction $a = g'$ was recently studied in [1].

For our purpose, it turns out to be more convenient to study the SPDE in the form of

$$\partial_t v = \Delta\{\varphi(v)\} + \nabla\{\varphi(v)\xi\}, \tag{1.2}$$

on $\mathbb{T}$, where $\varphi \in C^4(\mathbb{R})$ such that $c_- \le \varphi'(v) \le c_+$ for some $c_-, c_+ > 0$. In fact, roughly speaking, we have $v = \nabla u$ for $\varphi = g$. Note that (1.2) has a mass conservation law:

$$\int_{\mathbb{T}} v(t, x)dx = m \in \mathbb{R}. \tag{1.3}$$

To describe stationary solutions of (1.2), for a given ξ, we define its integral $\eta(x) := \langle \xi, 1_{[0,x]} \rangle \equiv \int_0^x \xi(y)dy, x \in \mathbb{T}$. Note that η is not periodic, but $\tilde{\eta}(x) := \langle \xi - \sigma, 1_{[0,x]} \rangle = \eta(x) - \sigma x$ is periodic, where

$$\sigma \equiv \sigma_\xi := \xi(\mathbb{T}) = \langle \xi, 1 \rangle = \eta(1) \in \mathbb{R}.$$

It is known that $\eta \in C^{\alpha-1}(\mathbb{R})$. Typically, we can take $\xi = \dot{w}(x) + \sigma$ with a periodic Brownian motion $w(x)$, $x \in \mathbb{T}$ and $\sigma \in \mathbb{R}$. The most interesting noise is $\xi = \dot{w}$ and, in this case, $\sigma_\xi = 0$ holds.

Then, from η, we define a function $\theta(x) = \theta_\xi(x)$ on $\mathbb{T}$ and a constant $\mu_\xi \in \mathbb{R}$, respectively, by

$$\theta(x) \equiv \theta(x) := e^{-\eta(x)} \Big\{ \mu \int_0^x e^{\eta(y)} dy + 1 \Big\}, \quad x \in \mathbb{T},$$

$$\mu_\xi := \frac{e^{\eta(1)} - 1}{\int_0^1 e^{\eta(y)} dy}.$$

By the definition of $\theta(x)$, one has that $\Delta(z\theta) + \nabla(z\theta \cdot \xi) = 0$ holds for every $z \in \mathbb{R}$ at least if $\xi \in C(\mathbb{T})$. In particular, $v = \varphi^{-1}(z\theta)$ are stationary solutions of (1.2), where φ^{-1} is the inverse function of φ.

For each conserved mass $m \in \mathbb{R}$ given as in (1.3), determine $z = z_m \in \mathbb{R}$ uniquely by the relation

$$m = \int_{\mathbb{T}} \varphi^{-1}(z\theta(x))dx. \tag{1.4}$$

Since φ is strictly increasing, (1.4) determines a one to one relation between z and m. Then, one has that

$$\bar{v}_m(x) := \varphi^{-1}(z_m\theta(x))$$

is a stationary solution of (1.2) satisfying $\int_{\mathbb{T}} v dx = m$ in distributional sense.

Let us now state the first main result.

Theorem 1.1. *For every initial value $v_0 \in C^{\alpha-1}$, the SPDE* (1.2) *has a global-in-time solution $v(t) \in C^{\alpha-1}$ for all $t \geq 0$. Moreover, if $|\mu_\xi|$ is sufficiently small, $v(t)$ converges exponentially fast to $\bar{v}_m$ in $C^{\alpha-1}$ as $t \to \infty$:*

$$\|v(t) - \bar{v}_m\|_{C^{\alpha-1}} \leq Ce^{-ct},$$

for some $c, C > 0$, where m is determined from v_0 as $m = \int_{\mathbb{T}} v_0(x)dx$.

This theorem for the slope $v(t) = \nabla u(t)$ of $u(t)$ implies the following result for $u(t)$ itself.

Theorem 1.2. *The SPDE* (1.1) *has a global-in-time solution $u(t) \in C^\alpha$ for all $t \geq 0$. Moreover, if $|\mu_\xi|$ is sufficiently small as in Theorem 1.1, $u(t)$ has the following uniform bound in t:*

$$\sup_{t\geq 0} \|u(t) - z_0\mu_\xi t\|_{C^\alpha} < \infty,$$

where z_0 is defined by (1.4) *with $m = 0$. In particular, we have $\frac{1}{t}u(t,x)$ converges uniformly on $\mathbb{T}$ to $z_0\mu_\xi$ as $t \to \infty$.*

References

[1] T. Funaki, M. Hoshino, S. Sethuraman and B. Xie, *Asymptotics of PDE in random environment by paracontrolled calculus*, Annales de l'Institut Henri Poincaré-Probabilités et Statistiques 57 (2021), No. 3, 1702–1735. doi.org/10.1214/20-AIHP1129.

[2] T. Funaki and B. Xie, *Global solvability and convergence to stationary solutions in singular quasilinear stochastic PDEs*, arXiv:2106.01102.