Numerical solution and error distribution of 1-dimentional SDE driven by fBm

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1 Introduction

We consider the solution $y_\cdot$ and some numerical solution $\hat{y}_\cdot^{(m)}$ of the following stochastic differential equation (SDE) driven by fractional Brownian motion with Hurst index H.

$$\begin{cases} y_t = y_0 + \int_0^t \sigma(y_u)dB_u + \int_0^t b(y_u)du \\ y_0 = \alpha \end{cases}$$

If it follows that $y_\cdot - \hat{y}_\cdot^{(m)} = O(R(m))$ and it exists that $\lim_{m\to\infty} R(m)^{-1}(\hat{y}_\cdot^{(m)} - y_\cdot)$, then it is called that error distribution. In this talk, we calculate error distributions of some numerical scheme.

The calculation of the error distribution is a kind of limit theorem, and since the error includes parts that take the form of Riemann integrals, stochastic integrals with B, and Ito integrals, it behaves in a complicated way depending on H.

This problem was studied in one-dimentional and multidimentional case. for instance, the latest papers are Aida-Naganuma [1] and Liu-Tindel [2]. the former calculated the error disribution of Crank-Nicolson scheme in one-dimenitional case and the latter calculated the error distribution of modified Euler scheme in multidimenitonal case.

The stochastic integration by fBm is defined pathwisely in rough path theory, and while rough paths are trivially constructed in the one-dimensional case, it is essential that rough paths can be approximated by a dyadic approximation in order for numerical approximation methods to converge in the multidimensional case. It is known that this is only true for $H > 1/4$, which indicates that ordinally numerical methods diverge in the multidimensional case and for $H \le 1/4$. In addition, the rate of convergence of the approximated rough path is a limit of the rate of convergence of numerical schemes.

In other words, in multidimentional case, calculation of error distribution is unsolved only if $1/4 < H \le 1/3$, but in one dimentional case and if $0 < H \le 1/3$, error distribution of any numerical scheme is left as unsolved problem.

2 Main result

Definition 1. *In one-dimentional case, we set* $\sigma, b \in C_b^\infty(\mathbb{R}), B_{s,t} = B_t - B_s, B_{s,t}^{(q)} = (q!)^{-1}B_{s,t}^q, \mathcal{D}f = (\sigma f)', p \ge 2$ *and we define* $\hat{y}_\cdot^{(m)}$ *as follows.*

$$\begin{cases} \hat{y}_t^{(m)} = \hat{y}_{\tau_r^m}^{(m)} + \sum_{l=1}^p \mathcal{D}^{l-1}\sigma(\hat{y}_{\tau_r^m}^{(m)})B_{\tau_r^m,t}^{(q)} + b(t-\tau_r^m) + \frac{1}{2}(\sigma b' + b\sigma')(t-\tau_r^m)B_{\tau_r^m,t} \ (\tau_r^m < t \le \tau_{r+1}^m) \\ \hat{y}_0^{(m)} = \alpha \end{cases}$$

Then we got following result.

Result.

- *With any Hurst index $0 < H < 1$, if $p > 1/H - 1$, then we have already proved that $\hat{y}^{(m)}_{\cdot}$ converges true solution $y_{\cdot}$ as means on $D[0,\infty)$, so we calculated error distribution in all case.*

$$\lim_{m\to\infty} 2^{mq}(\hat{y}^{(m)}_{\cdot} - y_{\cdot}) = \int_0^{\cdot} d\phi \text{ in } D[0,\infty) \text{ in distribution}$$

Condition	q	$d\phi$
$b \equiv 0$;q:odd;$1/(q+1) < H < 1/2$	$(q+1)H - 1$	$d\phi_1$
$b \equiv 0$;q:even;$1/(q+1) < H < 1/2$	$(q+2)H - 1$	$d\phi_2$
$b \equiv 0$;q:even;$H = 1/2$	$q/2$	$d(\phi_2 + \phi_3 + \phi_4)$
$b \equiv 0$;q:even;$1/2 < H < 1$	qH	$d\phi_4$
$b \not\equiv 0$;q:odd;$1/(q+1) < H < 1/(q-1)$	$(q+1)H - 1$	$d\phi_1$
$b \not\equiv 0$;q:even;$1/(q+1) < H < 1/q$	$(q+2)H - 1$	$d\phi_2$
$b \not\equiv 0$;q:odd;$H = 1/(q-1) < 1/2$	$2H$	$d(\phi_1 + \phi_5 + \phi_6)$
$b \not\equiv 0$;q:even;$H = 1/q < 1/2$	$2H$	$d(\phi_2 + \phi_5 + \phi_6)$
$b \not\equiv 0$;q:odd;$1/(q-1) < H < 1/2$	$2H$	$d(\phi_5 + \phi_6)$
$b \not\equiv 0$;q:even;$1/q < H < 1/2$	$2H$	$d(\phi_5 + \phi_6)$
$b \not\equiv 0$;$q \geq 4$;$H = 1/2$	1	$d(\phi_6 + \phi_7)$
$b \not\equiv 0$;$q = 3$;$H = 1/2$	1	$d(\phi_1 + \phi_6 + \phi_7)$
$b \not\equiv 0$;$q = 2$;$H = 1/2$	1	$d(\phi_2 + \phi_6 + \phi_7)$
$b\sigma' \neq \sigma b'$;$q \geq 2$;$1/2 < H < 1$	$H + 1/2$	$d\phi_7$
$b\sigma' = \sigma b' \neq 0$;$q = 2$;$1/2 < H < 1$	$2H$	$d(\phi_4 + \phi_6)$
$b\sigma' = \sigma b' \neq 0$;$q \geq 3$;$1/2 < H < 1$	$2H$	$d\phi_6$

$$d\phi_1 = \frac{\mu_{q+1}}{(q+1)!}\mathcal{D}^q\sigma(y_u)du, \qquad d\phi_2 = -\frac{\mu_{q+2}}{2(q+2)!}((q+2)\sigma'\mathcal{D}^q\sigma + q\mathcal{D}^{q+1}\sigma)(y_u)du$$

$$d\phi_3 = \frac{\sqrt{\mu_{2q+2}}}{(q+1)!}(\mathcal{D}^q\sigma)(y_u)dW_u \qquad d\phi_4 = \frac{\mu_{q+2}}{(q+1)!}\mathcal{D}^q\sigma(y_u)\circ dB_u, \qquad d\phi_5 = \frac{2H-1}{2+4H}((\sigma b' - \sigma' b)\sigma)'$$

$$d\phi_6 = -\frac{1}{2+4H}((\sigma'\sigma)'b + (2H-1)(\sigma' b)'\sigma + (b'\sigma)'\sigma)(y_u)du \qquad d\phi_7 = \frac{1}{\sqrt{\tilde{\mu}}}(\sigma b' - \sigma' b)(y_u)d\tilde{W}_u$$

- *Numerical solutions satisfying a certain condition, for instance Crank-Nicolson scheme, behave differently from the Milsiten scheme if $H < 1/2$.*
- *We modified the Milstein scheme dependently on H and inproved convergence rate. Also, we calculated the error distributons of modified schemes.*

We used the technique in Aida-Naganuma [4], the preprint calculated the error distribution of the Clank-Nicolson scheme in mutidimentional case and if $H > 1/3$.

References

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[3] L.Coutin., Z.Qian. : Stochastic analysis, rough path analysis and fractional Brownian motions. Probab. Theory Relat. Fields 122 (2002), 108–140

[4] S.Aida., N.Naganuma. : An interpolation of discrete rough differential equations and its applications to analysis of error distributions. preprint.