Numerical schemes for radial Dunkl processes

Dai Taguchi (Okayama University)
joint work with
Hoang-Long Ngo (Hanoi National University of Education)

Abstract

Let R be a (reduced) root system in $\mathbb{R}^d$, that is, R is a finite set of nonzero vectors in $\mathbb{R}^d$ such that (R1) $R \cap \{c\alpha \ ; \ c \in \mathbb{R}\} = \{\alpha, -\alpha\}$, for any $\alpha \in R$; (R2) $\sigma_\alpha(R) = R$ for any $\alpha \in R$. Here σ_α is the orthogonal reflection with respect to $\alpha \in \mathbb{R}^d \setminus \{0\}$ defined by

$$\sigma_\alpha x = x - \frac{2\langle \alpha, x\rangle}{|\alpha|^2}\alpha = \left(I_d - \frac{2}{|\alpha|^2}\alpha\alpha^\top\right)x, \ x \in \mathbb{R}^d.$$

For a total ordering $<$ of $\mathbb{R}^d$, a positive subsystem of the root system R is denoted by R_+. A sub-group $W = W(R)$ of $O(d)$ is called the Weyl group generated by a root system R, if it is generated by the reflections $\{\sigma_\alpha \ ; \ \alpha \in R\}$, that is, $W = \langle \sigma_\alpha \mid \alpha \in R\rangle$.

The Dunkl operator T_i on $\mathbb{R}^d$ associated with W are introduced by Dunkl [5] and are differential-difference operators given by

$$T_i f(x) := \frac{\partial f(x)}{\partial x_i} + \sum_{\alpha \in R_+} k\alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

Dunkl operators have been widely studied in both mathematics and physics, for example, there operators play a crucial role to the study special functions associated with root systems and the Hamiltonian operators of some Calogero-Moser-Sutherland quantum mechanical systems. Moreover, Rösler [8] studied Dunkl heat equation $(\Delta_k - \partial_t)u, \ u(\cdot, 0) = f \in C_b(\mathbb{R}^d; \mathbb{R})$ where the Dunkl Laplacian defined by $\Delta_k f(x) := \sum_{i=1}^d T_i^2$ and has the following explicit form

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha \in R_+} k \left\{ \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} + \frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x\rangle^2} \right\}.$$

Rösler and Voit [9] introduced Dunkl processes Y which are càdlàg Markov processes with infinitesimal generator $\Delta_k/2$ and is martingale with the scaling property. On the other hand, a radian Dunkl process $X = (X(t))_{t\geq 0}$ is a continuous Markov process with infinitesimal generator $L_k^W/2$ defined by

$$\frac{L_k^W f(x)}{2} := \frac{\Delta f(x)}{2} + \sum_{\alpha \in R_+} k \frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle},$$

and is a W-radial part of the Dunkl process Y, that is, for the canonical projection $\pi : \mathbb{R}^d \to \mathbb{R}^d/W$, $X = \pi(Y)$, as identifying the space $\mathbb{R}^d/W$ to the (fundamental) Weyl chamber $\mathbb{W} := \{x \in$

$\mathbb{R}^d$; $\langle \alpha, x \rangle > 0,\ \alpha \in R_+\}$ of the root system R. Schapira [10] and Demini [3] proved that a radial Dunkl process X satisfies the following $\mathbb{W}$-valued stochastic differential equation (SDE)

$$\mathrm{d}X(t) = \mathrm{d}B(t) + \sum_{\alpha \in R_+} k \frac{\alpha}{\langle \alpha, X(t) \rangle} \mathrm{d}t,\ X(0) = x(0) \in \mathbb{W}, \tag{1}$$

where $B = (B(t))_{t \geq 0}$ is a d-dimensional standard Brownian motion. For example, if $R := \{\pm 1\}$ then X is a Bessel process, and for a type A_{d-1} root system, that is, $R := \{e_i - e_j \in \mathbb{R}^d \ ; \ i \neq j\} \subset \{x \in \mathbb{R}^d ; \sum_{i=1}^d x_i = 0\}$, then X is a Dyson's Brownian motion.

In this talk, inspired by [1, 4, 6, 7], we introduce a $\mathbb{W}$-valuded numerical scheme for a class of radial Dunkl processes (1) corresponding to arbitrary (reduced) root systems. It is worth noting that *the numerical scheme can be implemented on a computer*. We also study its rate of convergence in L^p-norm. The key idea of the proof is to use the change of measure based on Girsanov theorem for radial Dunkl processes, which was proved in [2] for general radial Dunkl processes, and [11] for the Bessel case.

References

[1] Alfonsi, A. Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process. *Statist. Probab. Lett.* **83**(2) 602–607 (2013).

[2] Chybiryakov, O. Processus de Dunkl et relation de Lamperti. PhD thesis, University Paris 6 (2006).

[3] Demni, N. Radial Dunkl processes: existence, uniqueness and hitting time. *C. R. Math. Acad. Sci. Paris, Ser. I* **347** 1125–1128 (2009).

[4] Dereich, S., Neuenkirch, A. and Szpruch, L. An Euler-type method for the strong approximation for the Cox-Ingersoll-Ross process. *Proc. R. Soc. A* **468** 1105–1115 (2012).

[5] Dunkl, C. Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* **311**(1) 167–183 (1989).

[6] Neuenkirch, A. and Szpruch, L. First order strong approximations of scalar SDEs defined in a domain. *Numer. Math.* **128**, 103–136 (2014).

[7] Ngo, H.-L. and Taguchi, D. Semi-implicit Euler–Maruyama approximation for non-colliding particle systems. *Ann. Appl. Probab.* **30**(2) 673–705 (2020).

[8] Rösler, M. Generalized Hermite polynomials and the heat equation for Dunkl operators. *Commun. Math. Phys* **192** 519–541 (1998).

[9] Rösler, M. and Voit, M. Markov processes related with Dunkl operators. *Adv, Appl. Math.* **21**(4) 575–643 (1998).

[10] Schapira, B. The Heckman–Opdam Markov processes. *Probab. Theory Related Fields* **138**(3–4) 495–519 (2007).

[11] Yor, M. Loi de l'indice du lacet brownien, et distribution de Hartman-Watson. *Z. Wahrsch. Verw. Gebiete* **53**(1) 71–95 (1980).