Analysis of L^p-Kato class measures for symmetric Markov processes

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This talk is based on [4, 5], which are joint works with Kazuhiro Kuwae (Fukuoka University).

Let $p \in [1, \infty)$. A Borel measure μ on $\mathbb{R}^d$ is said to be *of L^p-Kato class* K_d^p with respect to the Brownian motion on $\mathbb{R}^d$ if

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{\mu(\mathrm{d}y)}{|x-y|^{(d-2)p}} = 0 \quad \text{for } d \geq 3,$$

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} (\log |x-y|^{-1})^p \mu(\mathrm{d}y) = 0 \quad \text{for } d = 2,$$

$$\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \mu(\mathrm{d}y) < \infty \quad \text{for } d = 1.$$

We write K_d instead of K_d^1 for $p = 1$. The notion of (L^1-)Kato class K_d was introduced by T. Kato in order to solve the essential self-adjointness of the Schrödinger operator $-\Delta + V$ on $C_0^\infty(\mathbb{R}^d)$. The following theorem is proved by Aizenman and Simon [1] under $p = 1$ and noted by the speaker [7] for general $p \in (1, \infty)$ with $d - p(d-2) > 0$:

> *Let $p \in [1, \infty)$ with $d - p(d-2) > 0$. Then $\mu \in K_d^p$ if and only if $\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \big(\int_0^t p_s(x, y) \mathrm{d}s \big)^p \mu(\mathrm{d}y) \to 0$ as $t \to 0$,*

where $p_t(x, y) := \frac{1}{(2\pi t)^{d/2}} \exp\big(-\frac{|x-y|^2}{2t}\big)$ is the heat kernel of the Brownian motion. Kuwae and Takahashi [6] extend this result under $p = 1$ in a probabilistic way to symmetric Markov processes satisfying some heat kernel estimates. In the first half of this talk, we explain an extension of this for general $p \in [1, \infty)$ under the framework of [6].

In the latter half of the talk, we discuss the L^p-version of the notion of Green-tightness from two motivations. One is to give a Rellich-Kondrashov type compact embedding theorem for Dirichlet spaces. The classical Rellich-Kondrashov compact embedding theorem states that:

> *For a bounded domain $D \subset \mathbb{R}^d$ with smooth boundary and $p \in [1, \infty)$ with $d - p(d-2) > 0$, the inclusion $H^1(D) \hookrightarrow L^{2p}(D)$ is a compact operator.*

On the case $p = 1$, Takeda [8] extended such a compact embedding result to the framework of general Dirichlet spaces. The speaker [7] proposed the notion of L^p-Kato class to obtain the several probabilistic properties on the intersection measures, and he extended Takeda's result to $p \geq 1$ by additionally assuming that the measure is of L^p-Kato class with respect to the given Markov process. We will explain a natural extension of these results by introducing the notation of L^p-Green-tight measures.

The other motivation is more historical. The notion of Green-tightness for Kato class potential was introduced by Zhao [9] to consider the gaugeability for Feynman-Kac functionals and the subcriticality of Schrödinger operator $-\frac{1}{2}\Delta+V$ in the framework of d-dimensional Brownian motion with $d \geq 3$. However, the notion was not enough to develop such a theory for general symmetric Markov processes. To overcome this difficulty, Chen [2] gave a new notion of Green-tight smooth measures of Kato class. After that, Kim and Kuwae [3] showed that the both classes coincide provided the given symmetric Markov process possesses the resolvent strong Feller property. We will explain a natural extension of their result to general $p \geq 1$. We also characterize the class of L^p-Green-tight measures of L^p-Kato class when the given Markov process has the heat kernel estimates set in the first half of the talk.

References

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