特異なランダムシュレディンガー作用素について

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確率解析とその周辺

Abstract

In this talk, we consider a random Schrödinger operator

$$-\Delta - \xi, \tag{1}$$

where $\Delta = \sum_{k=1}^{d} \partial_k^2$ is the Laplacian on $\mathbb{R}^d$ and ξ is a random potential. This operator is often called *Anderson Hamiltonian*. We impose the minimum assumption on the regularity of ξ. Namely, the regularity is $-2+\delta$ for some $\delta \in (0,1)$ and in particular ξ does not need to be a function. Typical examples include the *white noise*, namely the centered Gaussian field with delta correlation, in d-dimensions with $d \in \{1,2,3\}$. Due to the low regularity of ξ, it is not obvious how to interpret the operator (1).

Mathematical study of Anderson Hamiltonians with singular potentials can date back to the work [3] by Fukushima and Nakao, where they considered the 1D white noise potential motivated by earlier results from physicists. They constructed the operator (1) with 1D white noise on $(-L,L)$ with Dirichlet boundary conditions as the self-adjoint operator associated to the closed symmetric form

$$H_0^1((-L,L)) \times H_0^1((-L,L)) \ni (u,v) \mapsto \int_{(-L,L)} \nabla u \cdot \nabla v - \int_{(-L,L)} \xi uv.$$

Essentially, their argument works if the regularity of the potential ξ is better than -1.

However, this approach does not work if the regularity of ξ is lower than -1. To treat such singular ξ, in fact, one had to wait the advent of the theory on *singular stochastic partial differential equations* (singular SPDEs), most notably the theory of regularity structures by Hairer [5] and the theory of paracontrolled distributions by Gubinelli, Imkeller and Perkowski [4]. Motivated by the latter theory, Allez and Chouk [1] constructed the operator (1) with 2D white noise on the torus as the limit of

$$-\Delta - \xi_\varepsilon + c_\varepsilon,$$

where ξ_ε is a regularized 2D white noise and c_ε is a suitably chosen constant. They obtained an explict domain of the operator and its action. Subsequently, many extensions, which will be discussed in the talk, followed.

A main result of the talk is to construct the operator (1) on bounded domains of $\mathbb{R}^d$ with Dirichlet boundary conditions under the most general setting. A precise statement is as follows.

Theorem. Let $\rho \in \mathcal{S}(\mathbb{R}^d)$ and set $\xi_\varepsilon := \xi \star (\varepsilon^{-d}\rho(\varepsilon^{-1}\cdot))$. Let U be a bounded domain of $\mathbb{R}^d$. If *the BPHZ models associated to the generalized parabolic Anderson model converge*, there exist constants c_ε, $\varepsilon > 0$, with the following property. If we write $\mathcal{H}^{\mathfrak{d}}_\varepsilon(U)$ for the self-adjoint operator $-\Delta - \xi_\varepsilon + c_\varepsilon$ on $L^2(U)$ with Dirichlet boundary conditions, then there exists a self-adjoint operator $\mathcal{H}^{\mathfrak{d}}(U)$ on $L^2(U)$ such that

$$\lim_{\varepsilon \downarrow 0} \|(\sqrt{-1} + \mathcal{H}^{\mathfrak{d}}(U))^{-1} - (\sqrt{-1} + \mathcal{H}^{\mathfrak{d}}_\varepsilon(U))^{-1}\|_{L^2(U) \to L^2(U)} = 0,$$

where the convergence is in probability. Furthermore, the eigenvalues of $\mathcal{H}^{\mathfrak{d}}_\varepsilon(U)$ converge to those of $\mathcal{H}^{\mathfrak{d}}(U)$ in probability.

We also construct, under a stronger assumption, the operator (1) on bounded Lipschitz domains with Neumann boundary conditions. This stronger assumption allows the 2D white noise but excludes the 3D white noise.

Once the operator (1) is constructed, a next natural question is to investigate spectral properties of the operator. One of the most studied objects in the theory of random Schrödinger operators is the *integrated density of states* (IDS), see [2, Chapter VI]. The IDS is a nonrandom, nondecreasing and right-continuous function on $\mathbb{R}$ and is often characterized as the vague limit of the normalized eigenvalue counting functions.

We construct the IDS of the operator (1) with singular potential and we relate its left tail to those of the principal eigenvalues. In particular, combined with the work [6] by Hsu and Labbé, we derive a precise tail behaviour of the IDS for the white noise in d dimensions ($d \leq 3$).

References

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