LIOUVILLE THEOREM FOR V-HARMONIC MAPS UNDER NON-NEGATIVE (m, V)-RICCI CURVATURE FOR NON-POSITIVE m

Kazuhiro Kuwae (Fukuoka University)

1. Main Theorem

This is a joint work with Xiang-Dong Li (CAS AMSS), Songzi Li (Renmin Univ.) and Yohei Sakurai (Saitama Univ.).

Let (M, g, V) be a complete smooth Riemannian manifold with a smooth vector field V which is not necessarily to be the gradient $V = \nabla f$ for some $f \in C^2(M)$. Such a Riemannian manifold is equipped with canonical V-Laplacian and V-Ricci curvature. More precisely, the *non-symmetric V-Laplacian* is defined by

$$\Delta_V := \Delta - \langle V, \nabla \cdot \rangle$$

and the m-dimensional *V-Ricci curvature*, or the *(m, V)-Ricci curvature*, is defined as follows:

$$\mathrm{Ric}_V^m := \mathrm{Ric}_g + \frac{1}{2}\mathcal{L}_V g - \frac{V^* \otimes V^*}{m-n},$$

where $m \in [-\infty, n[\cup]n, +\infty]$, $\mathcal{L}_V g(X, Y) := \langle \nabla_X V, Y\rangle + \langle \nabla_Y V, X\rangle$ is the Lie derivative of g with respect to V, and V^* denotes its dual 1-form. To extend the definition of Ric_V^m for $m = n$, we make the convention that when $m = n$, we impose the constraint $V = 0$. We use the notation Ric_V^m and we make a convention that if $m = n$, then we always assume that V vanishes such that $\mathrm{Ric}_V^m = \mathrm{Ric}_g$. If $V = \nabla f$, we denote Ric_V^m by Ric_f^m. For $m_1 \in]-\infty, 1]$ and $m_2 \in [n, +\infty[$ we have the following order

$$\mathrm{Ric}_V^1 \geq \mathrm{Ric}_V^{m_1} \geq \mathrm{Ric}_V^{-\infty} = \mathrm{Ric}_V^{\infty} \geq \mathrm{Ric}_V^{m_2}. \tag{1}$$

So the condition $\mathrm{Ric}_V^m \geq Kg$ is weaker than $\mathrm{Ric}_V^\infty \geq Kg$ for $m \in]-\infty, 0]$ and $K \in \mathbb{R}$.

Let (N, h) be another complete smooth Riemaniann manifold. A smooth map $u : M \to N$ is said to be *V-harmonic* if $\tau_V(u) := \tau(u) - (\mathrm{d}u)(V) = 0$, where $\tau(u)$ is the tension field of u and $\mathrm{d}u : TM \to TN$ defined by $(\mathrm{d}u)_x : T_xM \to T_{u(x)}N$ is the differential of u (see [1, 2]). When $V = \nabla f$ for $f \in C^2(M)$, any V-harmonic map is called the *f-harmonic map*. The notion of V-harmonic map covers the various notions of harmonicity for maps, e.g., Hermitian harmonic maps, Weyl harmonic maps and affine harmonic maps (see [2] for these notions on harmonicity). In [1, Theorem 2], Chen-Jost-Qiu proved the bounded Liouville property for V-harmonic maps from complete Riemannian manifolds with $\mathrm{Ric}_V^\infty \geq 0$ into regular geodesic ball with $\mathrm{Sect}_N \leq \kappa$ for $\kappa \geq 0$ under the sublinear growth condition for V. Later, Qiu [8, Theorem 2] also proved the bounded Liouville property as in [1, Theorem 2] without assuming the sublinear growth condition for V. We will prove not only the bounded Liouville property for V-harmonic maps from complete Riemannian manifolds into regular geodesic ball but also Liouville property for sublinear growth V-harmonic map into Cartan-Hadamard manifold (N,h) under $\mathrm{Ric}_V^m \geq 0$ with $m \leq 0$. We always fix points $p \in M$ and $o \in N$, and also $m \in [-\infty, 0]$.

Definition 1.1. A function $u : M \to \mathbb{R}$ is said to be of *sublinear growth* if

$$\overline{\lim_{a\to\infty}}\, m_u(a)/a = 0,$$

where $m_u(a) := \sup_{r_p(x)<a} |u|$. A map $u : M \to N$ is said to be of *sublinear growth* if $d_N(u,o)$ is of sublinear growth for some/any $o \in N$.

Our first main result extends Cheng's Liouville theorem to sublinear growth V-harmonic maps on manifolds with $\mathrm{Ric}_V^m \geq 0$ and $\mathrm{Sect}_N \leq 0$.

Theorem 1.1. *Let (M,g) be a complete Riemaniann manifold with $\mathrm{Ric}_V^m \geq 0$ for some smooth vector field V and some constant $m \in [-\infty, 0]$, and (N,h) is a Cartan-Hadamard manifold, i.e., N is a complete connected and simply connected Riemannian manifold with $\mathrm{Sect}_N \leq 0$. Let $u : M \to N$ be a sublinear growth V-harmonic map. Then u is a constant map.*

A geodesic ball $B_R(o) \subset N$ with $\mathrm{Sect}_N \leq \kappa$ is said to be a *regular geodesic ball* if $B_R(o) \cap \mathrm{Cut}(o) = \emptyset$ and $R < \pi/2\sqrt{\kappa^+}$ with $\kappa^+ := \max\{\kappa, 0\}$. Our second main result extends Hildbrandt-Jost-Widemann and Choi's Liouville theorem as follows:

Theorem 1.2. *Let (M,g) be a complete Riemaniann manifold with $\mathrm{Ric}_V^m \geq 0$ for some smooth vector filed V and some constant $m \in [-\infty, 0]$, and N be a complete Riemannian manifold with $\mathrm{Sect}_N \leq \kappa$ with $\kappa > 0$. Let $u : M \to N$ be a V-harmonic map whose range is contained in a regular geodesic ball $B_R(o)$. Then u is a constant map.*

Theorem 1.3. *Let (M,g), (N,h) be two complete Riemannian manifolds. Suppose that the Δ_V-diffusion process $\mathbf{X}$ is recurrent and $\mathrm{Sect}_N \leq \kappa$ with $\kappa > 0$. Let $u : M \to N$ be a V-harmonic map into a regular geodesic ball $B_R(o)$. Then u is a constant map.*

References

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