NON RELATIVISTIC AND ULTRA RELATIVISTIC LIMITS IN 2D STOCHASTIC NONLINEAR DAMPED KLEIN-GORDON EQUATION

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This talk is based on [1], a joint work with Reika Fukuizumi (Tohoku University) and Takahisa Inui (Osaka University).

1. Introduction

We consider the following nonlinear damped Klein-Gordon equation on the two dimensional torus.

$$\begin{cases} \varepsilon^2\partial_t^2\Psi + 2\alpha\partial_t\Psi + (1-\Delta)\Psi + :|\Psi|^{2n}\Psi: = 2\sqrt{\mathrm{Re}(\alpha)}\xi, & t>0,\ x\in\mathbb{T}^2, \\ (\Psi,\varepsilon\partial_t\Psi)|_{t=0} = (\psi,\phi), & x\in\mathbb{T}^2, \end{cases} \tag{1}$$

where $\varepsilon > 0$, $\alpha \in \mathbb{C}$ is such that $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Im}(\alpha) \neq 0$, $n \in \mathbb{N}$, ξ is a complex-valued space-time white noise, i.e.

$$\mathbb{E}[\xi(t,x)] = 0, \qquad \mathbb{E}[\xi(t,x)\xi(s,y)] = 0, \qquad \mathbb{E}[\overline{\xi(t,x)}\xi(s,y)] = \delta(t-s)\delta(x-y),$$

$:|\Psi|^{2n}\Psi:$ denotes the renormalized nonlinearity, and (ψ,ϕ) is a pair of initial values. Eq. (1) describes the relativistic bosons at finite chemical potential. In the non relativistic limit ($\varepsilon \to 0$) and the ultra relativistic limit ($\mathrm{Im}(\alpha) \to 0$), the equation formally approaches to

$$2\alpha\partial_t\Psi + (1-\Delta)\Psi + :|\Psi|^{2n}\Psi: = 2\sqrt{\mathrm{Re}(\alpha)}\xi, \tag{2}$$

$$\varepsilon^2\partial_t^2\Psi + 2a\partial_t\Psi + (1-\Delta)\Psi + :|\Psi|^{2n}\Psi: = 2\sqrt{a}\xi, \qquad (a = \mathrm{Re}(\alpha)) \tag{3}$$

respectively. Eq. (1) formally interpolates Eq. (2) and Eq. (3). Eq. (2) is known as the stochastic Gross-Pitaevskii equation, describing Bose-Einstein condensates. Eq. (3) is the stochastic Goldstone model. In [2], the authors observed equilibrium properties of these models (in three dimensions) by numerical methods and found a statistical universality among them. We are motivated to justify rigorously the convergences (1)→(2) and (1)→(3), and the results in [2].

2. Main results

First we fix $\alpha \in \mathbb{C}$ and consider the non relativistic limit (1)→(2). The locall well-posedness is obtained from the standard Da Prato-Debussche trick and the following deterministic estimates.

Proposition 1. *Let $d \in \mathbb{N}$. For any $\varepsilon > 0$, let u_ε be the solution of the following linear equation with an exterior term $f(t,x)$.*

$$\begin{cases} \varepsilon^2\partial_t^2 u_\varepsilon + 2\alpha\partial_t u_\varepsilon + (1-\Delta)u_\varepsilon = f, & t>0,\ x\in\mathbb{T}^d, \\ (u_\varepsilon,\varepsilon\partial_t u_\varepsilon)|_{t=0} = (\psi,\phi), & x\in\mathbb{T}^d, \end{cases}$$

Then for any $\sigma \in \mathbb{R}$ and $T > 0$, one has the ε-uniform estimates

$$\|(u_\varepsilon,\varepsilon\partial_t u_\varepsilon)\|_{L^\infty(0,T;H^\sigma\times H^{\sigma-1})} \lesssim \|\psi\|_{H^\sigma} + \|\phi\|_{H^{\sigma-1}} + \|f\|_{L^2(0,T;H^{\sigma-1})}.$$

Moreover, for any $\theta \in [0,1]$ one has

$$\|u_\varepsilon - u_0\|_{L^\infty(0,T;H^\sigma)} \lesssim \varepsilon^\theta \big(\|\psi\|_{H^{\sigma+\theta}} + \|\phi\|_{H^{\sigma-1+\theta}} + \|f\|_{L^2(0,T;H^{\sigma-1+\theta})}\big),$$

where u_0 is the solution of

$$\begin{cases} 2\alpha\partial_t u_0 + (1-\Delta)u_0 = f, & t>0,\ x \in \mathbb{T}^d, \\ u_0|_{t=0} = \psi, & x \in \mathbb{T}^d, \end{cases}$$

To obtain the global well-posedness, we use the Gibbs measure

$$\rho(d\psi d\phi) = \frac{1}{\Gamma}\exp\left[-\frac{1}{2n+2}\int_{\mathbb{T}^2} :|\psi(x)|^{2n+2}: dx\right]\mu_0(d\psi)\mu_1(d\phi), \tag{4}$$

where μ_0 is the massive Gaussian free field (centered Gaussian measure with the covariance $(1-\Delta)^{-1}$), μ_1 is the distribution of spatial white noise, and Γ is a normalizing constant. Note that ρ is well-defined as a probability measure on $H^{-\delta} \times H^{-\delta-1}$ for any $\delta > 0$, and does not depend on ε or α.

We show the following result.

Theorem 2 ([1, Proposition 1 and Theorem 2])**.**

(1) *For any $\varepsilon > 0$, there exists a measurable set $\mathcal{O}_\varepsilon \subset H^{-\delta} \times H^{-\delta-1}$ such that $\rho(\mathcal{O}_\varepsilon) = 1$ and for any $(\psi,\phi) \in \mathcal{O}_\varepsilon$ the solution Ψ of (1) uniquely exists globally in time almost surely. Moreover, ρ is invariant under Ψ.*

(2) *Let $\varepsilon(j) = j^{-1}$ ($j \in \mathbb{N}$). Then there exists a measurable set $\mathcal{O} \subset H^{-\delta} \times H^{-\delta-1}$ such that $\rho(\mathcal{O}) = 1$ and for any $(\psi,\phi) \in \mathcal{O} \cap \bigcap_{j\in\mathbb{N}} \mathcal{O}_{\varepsilon(j)}$ the solution $\Psi_{\varepsilon(j)}$ of (1) with $\varepsilon = \varepsilon(j)$ converges to the solution Ψ of (2) as $j \to \infty$ in $C([0,\infty); H^{-\delta})$ almost surely.*

Next we fix $\varepsilon = 1$ and consider the dependence on α. By a similar argument, we have the following ultra relativistic limit.

Theorem 3 ([1, Corollary 2.2])**.** *Let $a \in (0,1)$ and $\alpha(j) = a + j^{-1}\sqrt{-1}$ ($j \in \mathbb{N}$). Then there exists a measurable set $\mathcal{B} \subset H^{-\delta} \times H^{-\delta-1}$ such that $\rho(\mathcal{B}) = 1$ and for any $(\psi,\phi) \in \mathcal{B}$ the solution $\Psi_{\alpha(j)}$ of (1) with $\alpha = \alpha(j)$ converges to the solution Ψ of (3) as $j \to \infty$ in $C([0,\infty); H^{-\delta})$ almost surely.*

References

[1] R. Fukuizumi, M. Hoshino, and T. Inui, *Non relativistic and ultra relativistic limits in 2d stochastic nonlinear damped Klein-Gordon equation*, `arXiv:2108.12183`.

[2] M. Kobayashi and L. Cugliandolo, *Quench dynamics of the three-dimensional $U(1)$ complex field theory: Geometric and scaling characterizations of the vortex tangle*, Phys. Rev. E **94** 062146 (2016).

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