Optimal control for stochastic Volterra integral equations

Yushi Hamaguchi*
(Graduate School of Engineering Science, Osaka University)

This talk is based on our recent papers [1, 2]. We investigate infinite horizon optimal control problems of *stochastic Volterra integral equations* (SVIEs, for short). Suppose that the dynamics of the state process $X^u(\cdot)$ corresponding to a control process $u(\cdot)$ is described by the following controlled SVIE:

$$X^u(t) = \varphi(t) + \int_0^t b(t,s,X^u(s),u(s))\,\mathrm{d}s + \int_0^t \sigma(t,s,X^u(s),u(s))\,\mathrm{d}W(s),\ t \geq 0. \tag{1}$$

Here, $W(\cdot)$ is a one-dimensional Brownian motion, $\varphi(\cdot)$ is a given adapted process called the free term, and b and σ are vector-valued functions. The objective is to minimize a discounted cost functional

$$J_\lambda(u(\cdot)) := \mathbb{E}\Big[\int_0^\infty e^{-\lambda t} h(t,X^u(t),u(t))\,\mathrm{d}t\Big] \tag{2}$$

over all admissible control processes $u(\cdot)$, where h is a real-valued function called the running cost, and $\lambda > 0$ is a discount rate.

If the coefficients $b(t,s,x,u)$ and $\sigma(t,s,x,u)$ do not depend on the time-parameter t, and if the free term $\varphi(t) = x_0$ is a constant, then SVIE (1) is reduced to a controlled stochastic differential equation (SDE, for short)

$$\begin{cases} \mathrm{d}X^u(t) = b(t,X^u(t),u(t))\,\mathrm{d}t + \sigma(t,X^u(t),u(t))\,\mathrm{d}W(t),\ t \geq 0, \\ X^u(0) = x_0. \end{cases}$$

More importantly, SVIE (1) includes a class of controlled *fractional SDE* of the form

$$\begin{cases} {}^{\mathrm{C}}D^\alpha_{0+} X^u(t) = b(t,X^u(t),u(t)) + \sigma(t,X^u(t),u(t))\dfrac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \geq 0, \\ X^u(0) = x_0, \end{cases}$$

where ${}^{\mathrm{C}}D^\alpha_{0+}$ denotes the Caputo fractional derivative of order $\alpha \in (\frac{1}{2}, 1]$. Fractional differential systems are suitable tools to describe the dynamics of systems with memory effects and hereditary properties. There are many applications of fractional calculus in a variety of research fields including mathematical finance, physics, chemistry, biology, and other applied sciences. For detailed accounts of theory and applications of fractional calculus, see for example [3] and the references cited therein. The analysis of stochastic control problems of fractional SDEs is therefore an important topic, and this is a main motivation of our work.

For the discounted control problem (1)–(2), we provide both necessary and sufficient conditions for optimality by means of Pontryagin's maximum principle. The most important step in deriving

*E-mail: hmgch2950@gmail.com

maximum principles is the introduction of the *adjoint equation* which corresponds to the variation of the state equation via a duality relation. We show that the adjoint equation in our problem (1)–(2) becomes an *infinite horizon backward stochastic Volterra integral equation* (BSVIE, for short), which is a generalization of finite horizon BSVIEs introduced by Yong [4] to the infinite horizon framework. The optimal control is characterized by an infinite horizon BSVIE, together with an optimality condition.

As an example, we consider an infinite horizon linear–quadratic (LQ, for short) regulator problem for a fractional SDE with constant coefficients. Specifically, we treat the case where the controlled state dynamics is described by a (one-dimensional) linear fractional SDE

$$\begin{cases} {}^{\mathrm{C}}D^{\alpha}_{0+}X^u(t) = bX^u(t) + cu(t) + \sigma\dfrac{\mathrm{d}W(t)}{\mathrm{d}t},\ t \geq 0, \\ X^u(0) = x_0, \end{cases}$$

and the discounted cost functional is given by a quadratic functional of the state and control:

$$J_\lambda(u(\cdot)) = \mathbb{E}\Big[\int_0^\infty e^{-\lambda t}\Big\{|X^u(t)|^2 + \frac{1}{\gamma}|u(t)|^2\Big\}\,\mathrm{d}t\Big].$$

Based on the maximum principle, we show that there exists a unique optimal control for this problem. Moreover, we obtain an explicit formula for the optimal control process of the following form:

(optimal control) = (constant) × (current optimal state) + (Gaussian process),

which we call a *Gaussian state-feedback representation formula* for the optimal control. Here, the Gaussian process is a stochastic convolution of a deterministic function with respect to the Brownian motion $W(\cdot)$, and the function is determined via *linear Fredholm integral equations* depending only on the model parameters. The linear Fredoholm integral equations can be solved by using a Fredholm resolvent of the kernel, and we get the above Gaussian state-feedback representation formula.

References

[1] Y. Hamaguchi, Infinite horizon backward stochastic Volterra integral equations and discounted control problems, *ESAIM: Control Optim., Calc., Var.*, to appear, arXiv:2105.02438.

[2] Y. Hamaguchi, On the maximum principle for optimal control problems of stochastic Volterra integral equations with delay, *preprint*, arXiv:2109.06092.

[3] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications*. Gordon and Breach Science Publishers, Yverdon, Switzerland, 1987.

[4] J. Yong, Well-posedness and regularity of backward stochastic Volterra integral equations, *Probab. Theory Related Fields*, 142(1-2), 2–77, 2008.