Multi-dimensional Avikainen's estimates

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Abstract

Let X be a real-valued random variable with bounded density p_X with respect to Lebesgue measure. Then Avikainen proved [1] that for any real-valued random variable $\widehat{X}$, function of bounded variation $f : \mathbb{R} \to \mathbb{R}$ and $p, q \in [1, \infty)$, it holds that

$$\mathbb{E}\left[\left|f(X) - f(\widehat{X})\right|^q\right] \leq 3^{q+1} V(f)^q \left(\sup_{x\in\mathbb{R}} p_X(x)\right)^{\frac{p}{p+1}} \mathbb{E}\left[\left|X - \widehat{X}\right|^p\right]^{\frac{1}{p+1}}, \tag{1}$$

where $V(f)$ is the total variation of f. Here, there is no relationship between p and q. Note that this estimate is optimal, that is, there exist some random variables X, $\widehat{X}$ and function f such that the equality holds in (1) (see, Theorem 2.4 (ii) in [1]). Moreover, it can be applied to the numerical analysis on irregular functions of SDEs based on the Euler–Maruyama scheme and the multilevel Monte Carlo method [2]. The proof of this estimate is based on Skorokhod's "explicit" representation to embed the distribution of X in the probability space $([0,1], \mathscr{B}([0,1]), \mathrm{Leb})$ (see also, Proposition 5.3 in [3] for a simple proof). For multi-dimensional random variables, this representation is known as Skorokhod's embedding theorem. However, it might be difficult to apply it to the multi-dimensional case since it is not explicit.

In this talk, we will propose some versions of Avikainen's estimate (1) for multi-dimensional random variables. As mentioned above, it might be difficult to apply the approach in [1] for multi-dimensional random variables. Instead, we propose a new approach based on the Hardy–Littlewood maximal operator M for locally finite vector valued measures ν, which is defined by

$$M\nu(x) := \sup_{s>0} \fint_{B(x;s)} \mathrm{d}|\nu|(z), \ \fint_{B(x;s)} \mathrm{d}|\nu|(z) := \frac{|\nu|(B(x;s))}{\mathrm{Leb}(B(x;s))}, \ x \in \mathbb{R}^d,$$

where $|\nu|$ is the total variation of ν and $B(x;r)$ is the closed ball in $\mathbb{R}^d$ with center x and radius r. The operator M is well-studied in the fields of harmonic analysis. As an application of Vitali's covering lemma, it satisfies the following Hardy–Littlewood maximal weak type estimate

$$\mathrm{Leb}(\{x \in \mathbb{R}^d \ ; \ M\nu(x) > \lambda\}) \leq A_1 |\nu|(\mathbb{R}^d)\lambda^{-1}, \ \lambda > 0, \tag{2}$$

where the constant A_1 depends only on d (e.g. $A_1 = 5^d$). Using this estimate, we will prove that for any random variables $X, \widehat{X} : \Omega \to \mathbb{R}^d$ with density functions p_X and $p_{\widehat{X}}$ with respect to Lebesgue measure, respectively, and for any $f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $p \in (0, \infty)$ and $q \in [1, \infty)$, if p_X and $p_{\widehat{X}}$ are bounded, then it holds that

$$\mathbb{E}\left[\left|f(X) - f(\widehat{X})\right|^q\right] \leq (2\|f\|_\infty)^{q-1} \left\{4K_0^p \|f\|_\infty + A_1\{\|p_X\|_\infty + \|p_{\widehat{X}}\|_\infty\} \int_{\mathbb{R}^d} |Df|\right\} \mathbb{E}\left[\left|X - \widehat{X}\right|^p\right]^{\frac{1}{p+1}}.$$

Here, $BV(\mathbb{R}^d)$ is the class of functions f of bounded variation in $\mathbb{R}^d$, which is a subset of $L^1(\mathbb{R}^d)$ such that the total variation $|Df|(\mathbb{R}^d) = \int_{\mathbb{R}^d} |Df|$ of the Radon measure Df defined by

$$\int_{\mathbb{R}^d} |Df| := \sup\left\{ \int_{\mathbb{R}^d} f(x) \mathrm{div} g(x) \mathrm{d}x \ ; \ g \in C^1_{\mathrm{c}}(\mathbb{R}^d; \mathbb{R}^d) \text{ and } \sup_{x \in \mathbb{R}^d} |g(x)| \leq 1 \right\}$$

is finite, where the Radon measure Df is defined as the generalized derivative formulated by the integration by parts for functions of bounded variations. For example, a Sobolev space $W^{1,1}(\mathbb{R}^d)$ is subset of $BV(\mathbb{R}^d)$, and for a bounded subset E of $\mathbb{R}^d$ with C^2 boundary, $\mathbf{1}_E \in BV(\mathbb{R}^d) \setminus W^{1,1}(\mathbb{R}^d)$. The most important property of $f \in BV(\mathbb{R}^d)$ which we use in this talk is the following pointwise estimate

$$|f(x) - f(y)| \leq K_0 |x-y| \left\{ M_{2|x-y|}(Df)(x) + M_{2|x-y|}(Df)(y) \right\}, \text{ Leb-a.e. } x, y \in \mathbb{R}^d, \tag{3}$$

where for $R > 0$, $M_R \nu$ is the restricted Hardy–Littlewood maximal function defined by

$$M_R \nu(x) := \sup_{0 < s \leq R} \fint_{B(x;s)} \mathrm{d}|\nu|(z), \ x \in \mathbb{R}^d.$$

It is worth noting that Hajłasz [4, 5] characterized Sobolev spaces $W^{1,p}(\mathbb{R}^d)$, $1 \leq p < \infty$ by using the pointwise estimate (3), and defined Sobolev spaces on metric spaces using this pointwise estimate. Moreover, Lahti and Tuominen [6] generalized this characterization to $BV(\mathbb{R}^d)$ and defined BV on metric measure space (χ, d, μ) with "doubling" property : $\mu(B(x, 2r)) \leq C\mu(B(x,r))$. Note that the Hardy–Littlewood maximal weak type estimate (2) is also valid on separable metric spaces, and thus we can generalize Avikainens estimates on separable metric measure spaces with doubling property.

References

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